Testing Linear Factor Pricing Models with Large Cross-Sections: A Distribution-Free Approach

Sermin Gungor
Bank of Canada

Richard Luger†
Georgia State University

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Abstract

We develop a finite-sample distribution-free procedure to test the beta-pricing representation of linear factor pricing models. In sharp contrast to extant finite-sample tests, our framework allows for unknown forms of non-normalities, heteroskedasticity, and time-varying covariances. The power of the proposed test procedure increases as the time series lengthens and/or the cross-section becomes larger. So the criticism sometimes heard that non-parametric tests lack power does not apply here, since the number of test assets is chosen by the user. This also stands in contrast to the usual tests that lose power or may not even be computable if the number of test assets is too large.

JEL classification: C12; C14; C33; G11; G12

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†Corresponding author. *By mail:* Department of Risk Management and Insurance, Georgia State University, P.O. Box 4036, Atlanta, GA 30302-4036, United States. Tel.: 404-413-7489. Fax: 404-413-7499. *By e-mail:* rluger@gsu.edu (R. Luger).

1 Introduction

Many asset pricing models predict that expected returns depend linearly on "beta" coefficients relative to one or more portfolios or factors. The beta is the regression coefficient of the asset return on the factor. In the capital asset pricing model (CAPM) of Sharpe (1964) and Lintner (1965), the single beta measures the systematic risk or co-movement with the returns on the market portfolio. Accordingly, assets with higher betas should offer in equilibrium higher expected returns. The Arbitrage Pricing Theory (APT) of Ross (1976), developed on the basis of arbitrage arguments, can be more general than the CAPM in that it relates expected returns with multiple beta coefficients. Merton (1973) and Breeden (1979) develop models based on investor optimization and equilibrium arguments that also lead to multiple-beta pricing.

Empirical tests of the validity of beta pricing relationships are often conducted within the context of multivariate linear factor models. When the factors are traded portfolios and a riskfree asset is available, exact factor pricing implies that the vector of asset return intercepts will be zero. These tests are interpreted as tests of the mean-variance efficiency of a benchmark portfolio in the single-beta model or that some combination of the factor portfolios is mean-variance efficient in multiple-beta models. In this context, standard asymptotic theory provides a poor approximation to the finite-sample distribution of the usual Wald and likelihood ratio (LR) test statistics, even with fairly large samples. Shanken (1996), Campbell, Lo, and MacKinlay (1997), and Dufour and Khalaf (2002) document severe size distortions for those tests, with overrejections growing quickly as the number of equations in the multivariate model increases. The simulation evidence in Ferson and Foerster (1994) and Gungor and Luger (2009) shows that tests based on the Generalized Method of Moments (GMM) à la MacKinlay and Richardson (1991) suffer from the same problem. As a result, empirical tests of beta-pricing representations can be severely affected and can lead to spurious rejections of their validity.

The assumptions underlying standard asymptotic arguments can be questionable when dealing with financial asset returns data. In the context of the consumption CAPM for example, Kocherlakota (1997) shows that the model disturbances are so heavy-tailed that they do not satisfy the Central Limit Theorem. In such an environment, standard methods of inference can lead to spurious rejections even asymptotically and Kocherlakota instead relies on jackknifing to devise a method of testing the consumption CAPM. Similarly, Affleck-Graves and McDonald (1989) and Chou and Zhou (2006) suggest the use of bootstrap techniques to provide more robust and reliable asset pricing tests.

There are very few methods that provide truly exact, finite-sample tests.[1] The most prominent one is probably the F-test of Gibbons, Ross, and Shanken (1989) (GRS). The exact distribution theory for that test rests on the assumption that the vectors of model disturbances are independent and identically distributed (i.i.d.) each period according to a multivariate normal distribution. As we already mentioned, there is ample evidence that financial returns exhibit non-normalities; for more evidence, see Fama (1965), Blattberg and Gonedes (1974), Hsu (1982), Affleck-Graves and McDonald (1989), and Zhou (1993). Beaulieu, Dufour, and Khalaf (2007) generalize the GRS approach for testing mean-variance efficiency. Their simulation-based approach does not necessarily assume normality but it does nevertheless require that the disturbance distribution be parametrically specified, at least up to a finite number of unknown nuisance parameters. Gungor and Luger (2009) propose exact tests of the mean-variance efficiency of a single reference portfolio, whose exactness does not depend on any parametric assumptions.

In this paper we extend the idea of Gungor and Luger (2009) to obtain tests of multiple-beta pricing representations that relax three assumptions of the GRS test: (i) the assumption of identically distributed disturbances, (ii) the assumption of normally distributed disturbances, and (iii) the restriction on the number of test assets. The proposed test procedure is based on finite-sample pivots that are valid without any assumptions about the specific distribution of the disturbances in the factor model. We propose an adaptive approach based on a split-sample technique to obtain a single portfolio representation judiciously formed to avoid power losses that can occur in

[1] A number of Bayesian approaches have also been proposed. These include Shanken (1987), Harvey and Zhou (1990), and Kandel, McCulloch, and Stambaugh (1995).

naive portfolio groupings. For other examples of split-sample techniques, see Dufour and Taamouti (2005), Jouneau-Sion and Torrès (2006), and Dufour and Taamouti (2010).

A very attractive feature of our approach is that it is applicable even if the number of test assets is greater than the length of the time series. This stands in sharp contrast to the GRS test or any other approach based on usual estimates of the disturbance covariance matrix. In order to avoid singularities and be computable, those approaches require the size of the cross-section be less than that of the time series. In fact, great care must be taken when applying the GRS test since its power does not increase monotonically with the number of test assets and all the power may be lost if too many are included. This problem is related to the fact that the number of covariances that need to be estimated grows rapidly with the number of included test assets. As a result, the precision with which this increasing number of parameters can be estimated deteriorates given a fixed time-series.[2]

Our proposed test procedure then exploits results from Coudin and Dufour (2009) on median regressions to construct sign-based statistics, one of which is the sign analogue of the usual F-test. The motivation for using signs comes from an impossibility result due to Lehmann and Stein (1949) that shows that the only tests which yield reliable inference under sufficiently general distributional assumptions, allowing non-normal, possibly heteroskedastic, independent observations are based on sign statistics. This means that all other methods, including the standard heteroskedasticity and autocorrelation-corrected (HAC) methods developed by White (1980) and Newey and West (1987) among others, which are not based on signs, cannot be proved to be valid and reliable for any sample size.

The paper is organized as follows. Section 2 presents the linear factor model used to describe the asset returns, the null hypothesis to be tested, and the benchmark GRS test. We provide an illustration of the effects of increasing the number of test assets on the power of the GRS test. In Section 3 we develop the new test procedure. We begin that section by presenting the statistical

[2]The notorious noisiness of unrestricted sample covariances is a well-known problem in the portfolio management literature; see Michaud (1989), Jagannathan and Ma (2003), and Ledoit and Wolf (2003, 2004), among others.

framework and then proceed to describe each step of the procedure. Section 4 contains the results of simulation experiments designed to compare the performance of the proposed test procedure with several of the standard tests. In Section 5 we apply the procedure to test the Sharpe-Lintner version of the CAPM and the well-known Fama-French three factor model. Section 6 concludes.

2 Factor model

Suppose there exists a riskless asset for each period of time and define $\mathbf{r}_t$ as an $N \times 1$ vector of time-t returns on N assets in excess of riskless rate of return. Suppose further that those excess returns are described by the linear K-factor model

$$\mathbf{r}_t = \mathbf{a} + \mathbf{B}\mathbf{f}_t + \boldsymbol{\varepsilon}_t, \tag{1}$$

where $\mathbf{f}_t$ is a $K \times 1$ vector of common factor portfolio excess returns, $\mathbf{B}$ is the $N \times K$ matrix of betas (or factor loadings), and $\mathbf{a}$ and $\boldsymbol{\varepsilon}_t$ are $N \times 1$ vectors of factor model intercepts and disturbances, respectively. Although not required for the proposed procedure, the vector $\boldsymbol{\varepsilon}_t$ is usually assumed to have well-defined first and second moments satisfying $E[\boldsymbol{\varepsilon}_t|\mathbf{f}_t] = \mathbf{0}$ and $E[\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t'|\mathbf{f}_t] = \boldsymbol{\Sigma}$, a finite $N \times N$ matrix.

Exact factor pricing implies that expected returns depend linearly on the betas associated to the factor portfolio returns:

$$E_t[\mathbf{r}_t] = \mathbf{B}\boldsymbol{\lambda}, \tag{2}$$

where $\boldsymbol{\lambda}$ is a $K \times 1$ vector of expected excess returns associated with $\mathbf{f}_t$, which represent market-wide risk premiums since they apply to all traded securities. The beta-pricing representation in (2) is a generalization of the CAPM of Sharpe (1964) and Lintner (1965), which asserts that the expected excess return on an asset is linearly related to its single beta. This beta measures the asset's systematic risk or co-movement with the excess return on the market portfolio—the portfolio of all invested wealth. Equivalently, the CAPM says that the market portfolio is mean-variance efficient in the investment universe comprising all possible assets.[3] The pricing relationship in (2) is more

[3] A benchmark portfolio with excess returns r_p is said to be mean-variance efficient with respect to a given set of

general since it says that a combination (portfolio) of the factor portfolios is mean-variance efficient; see Jobson (1982), Jobson and Korkie (1982, 1985), Grinblatt and Titman (1987), Shanken (1987), and Huberman, Kandel, and Stambaugh (1987) for more on the relation between factor models and mean-variance efficiency.

The beta-pricing representation in (2) is a restriction on expected returns which can be assessed by testing the hypothesis

$$H_0 : \mathbf{a} = \mathbf{0} \text{ against } H_1 : \mathbf{a} \neq \mathbf{0}, \tag{3}$$

under the maintained factor structure specification in (1). If the pricing errors, $\mathbf{a}$, are in fact different from zero, then (2) does not hold meaning that there is no way to combine the factor portfolios to obtain one that is mean-variance efficient.

GRS propose a multivariate F-test of (3) that all the pricing errors are jointly equal to zero. Their test assumes that the vectors of disturbance terms $\boldsymbol{\varepsilon}_t$, $t = 1, ..., T$, in (1) are independent and normally distributed around zero with non-singular cross-sectional covariance matrix each period, conditional on the $T \times K$ collection of factors $\mathbf{F} = [\boldsymbol{f}'_1, ..., \boldsymbol{f}'_T]'$. Under normality, the methods of maximum likelihood and ordinary least squares (OLS) yield the same unconstrained estimates of $\mathbf{a}$ and $\mathbf{B}$:

$$\hat{\mathbf{a}} = \bar{\mathbf{r}} - \hat{\mathbf{B}}\bar{\mathbf{f}}, \tag{4}$$

$$\hat{\mathbf{B}} = \left[\sum_{t=1}^{T}(\mathbf{r}_t - \bar{\mathbf{r}})(\mathbf{f}_t - \bar{\mathbf{f}})'\right]\left[\sum_{t=1}^{T}(\mathbf{f}_t - \bar{\mathbf{f}})(\mathbf{f}_t - \bar{\mathbf{f}})'\right]^{-1}, \tag{5}$$

where $\bar{\mathbf{r}} = T^{-1}\sum_{t=1}^{T}\mathbf{r}_t$ and $\bar{\mathbf{f}} = T^{-1}\sum_{t=1}^{T}\mathbf{f}_t$, and the estimate of the disturbance covariance matrix is

$$\hat{\boldsymbol{\Sigma}} = \frac{1}{T}\sum_{t=1}^{T}(\mathbf{r}_t - \hat{\mathbf{a}} - \hat{\mathbf{B}}\mathbf{f}_t)(\mathbf{r}_t - \hat{\mathbf{a}} - \hat{\mathbf{B}}\mathbf{f}_t)'. \tag{6}$$

The GRS test statistic is

$$J_1 = \frac{T - N - K}{N}\left[1 + \bar{\mathbf{f}}'\hat{\boldsymbol{\Omega}}^{-1}\bar{\mathbf{f}}\right]^{-1}\hat{\mathbf{a}}'\hat{\boldsymbol{\Sigma}}^{-1}\hat{\mathbf{a}}, \tag{7}$$

N test assets with excess returns $\mathbf{r}_t$ if it is not possible to form another portfolio of those N assets and the benchmark portfolio with the same variance as r_p but a higher expected return.

where $\hat{\boldsymbol{\Omega}}$ is given by

$$\hat{\boldsymbol{\Omega}} = \frac{1}{T}\sum_{t=1}^{T}(\mathbf{f}_t - \bar{\mathbf{f}})(\mathbf{f}_t - \bar{\mathbf{f}})'.$$

Under the null hypothesis H_0, the statistic J_1 follows a central F distribution with N degrees of freedom in the numerator and $(T - N - K)$ degrees of freedom in the denominator.

In practical applications of the GRS test, one needs to decide the appropriate number N of test assets to include. It might seem natural to try to use as many test assets as possible in order to increase the probability of rejecting H_0 when it is false. As the test asset universe expands it becomes more likely that non-zero pricing errors will be detected, if indeed there are any. However, the choice of N is restricted by T in order to keep the estimate of the disturbance covariance matrix in (6) from becoming singular, and the choice of T itself is often restricted owing to concerns about parameter stability. For instance, it is quite common to see studies where $T = 60$ monthly returns and N is between 10 and 30. The effects of increasing the number of test assets on test power is discussed in GRS, Campbell, Lo, and MacKinlay (1997, p. 206) and Sentana (2009). When N increases, three effects come into play: (i) the increase in the value of J_1's non-centrality parameter, which increases power, (ii) the increase in the number of degrees of freedom of the numerator, which decreases power, and (iii) the decrease in the number of degrees of freedom of the denominator due to the additional parameters that need to be estimated, which also decreases power.

To illustrate the net effect of increasing N on the power of the GRS test, we simulated model (1) with $K = 1$, where the returns on the single factor are random draws from the standard normal distribution. The elements of the independent disturbance vector were also drawn from the standard normal distribution thereby ensuring the exactness of the GRS test. We set $T = 60$ and considered $a_i = 0.05$, 0.10, and 0.15 for $i = 1, ..., N$ and we let the number of test assets N range from 1 to 58. The chosen values for a_i are well within the range of what we find with monthly stock returns. Figure 1 shows the power of the GRS test as a function of N, where for any given N the higher power is associated with higher pricing errors. In line with the discussion in GRS, this figure clearly shows the power of the test given this specification rising as N increases up to about one half of T and then decreasing beyond that. The results in Table 5.2 of Campbell, Lo, and MacKinlay (1997) show several other alternatives against which the power of the GRS test declines as N increases. Furthermore, there are no general results about how to devise an optimal multivariate test. So great care must somehow be taken when choosing the number of test assets since power does not increase monotonically with N and if the cross-section is too large, then the GRS test may lose all its power or may not even be computable. In fact, any procedure that relies on standard unrestricted estimates of the covariance matrix of regression disturbances will have this singularity problem when N exceeds T.

3 Test procedure

In this section we develop a procedure to test H_0 in (3) that relaxes three assumptions of the GRS test: (i) the assumption of identically distributed disturbances, (ii) the assumption of normally distributed disturbances, and (iii) the restriction on the number of test assets.

To motivate our approach, consider the problem of testing the following hypothesis:

$H_0^{(0)}$: $u_1, ..., u_T$ are independent random variables each with a distribution symmetric around zero.

Note that $H_0^{(0)}$ allows for the presence of heteroskedasticity of unknown form. The following characterization of heteroskedasticity-robust tests, taken from Dufour (2003), is due to Lehmann and Stein (1949); see also Pratt and Gibbons (1981, p. 218) and Dufour and Hallin (1991).

Theorem (Lehmann and Stein 1949). *If a test has level α for $H_0^{(0)}$, where $0 < \alpha < 1$, then it must satisfy the condition*

$$\Pr\left[\textit{Rejecting } H_0^{(0)} \mid |u_1|, ..., |u_T|\right] \leq \alpha \textit{ under } H_0^{(0)}.$$

In words, this theorem from classical non-parametric statistics states that the only tests which yield valid inference under sufficiently general distributional assumptions, allowing non-normal, possibly heteroskedastic, independent observations are ones that are conditional on the absolute values of the observations; i.e., they must be based on sign statistics. Conversely, if a test procedure does not satisfy the condition in the above theorem for all levels $0 < \alpha < 1$, then its true size is 1 irrespective of its nominal size (Dufour, 2003). With this characterization in mind, we next present the statistical framework and then proceed to describe each step of the proposed inference procedure.

3.1 Statistical framework

As in the GRS framework, we assume that the disturbance vectors $\boldsymbol{\varepsilon}_t$ in (1) are independently distributed over time, conditional on $\mathbf{F}$. We do not require the disturbance vectors to be identically distributed, but we do assume that they satisfy a multivariate symmetry condition each period. In what follows the symbol $\stackrel{d}{=}$ stands for the equality in distribution.

Assumption 1. *The cross-sectional disturbance vectors $\boldsymbol{\varepsilon}_t$, $t = 1, ..., T$, are mutually independent, continuous, and reflectively symmetric so that $\boldsymbol{\varepsilon}_t \stackrel{d}{=} -\boldsymbol{\varepsilon}_t$, conditional on $\mathbf{F}$.*

The reflective symmetry condition in Assumption 1 can be equivalently expressed in terms of the density function, if it exists, as $f_t(\boldsymbol{\varepsilon}_t) = f_t(-\boldsymbol{\varepsilon}_t)$. Recall that a random variable ε_t is symmetric around zero if and only if $\varepsilon_t \stackrel{d}{=} -\varepsilon_t$, so the symmetry assumption made here represents the most direct non-parametric extension of univariate symmetry; see Serfling (2006) for more concepts of multivariate symmetry.

The class of distributions encompassed by Assumption 1 includes elliptically symmetric distributions, which play a very important role in mean-variance analysis because they guarantee full compatibility with expected utility maximization regardless of investor preferences; see Chamberlain (1983), Owen and Rabinovitch (1983), and Berk (1997). The random vector $\boldsymbol{\varepsilon}_t$ is said to be elliptically symmetric if there exists a non-singular $N \times N$ matrix A_t such that the unit vector $A_t\boldsymbol{\varepsilon}_t/\|A_t\boldsymbol{\varepsilon}_t\|$ is distributed uniformly on $\mathbb{S}^N$, the unit spherical shell of $\mathbb{R}^N$, and $A_t\boldsymbol{\varepsilon}_t/\|A_t\boldsymbol{\varepsilon}_t\|$ is independent of $\|A_t\boldsymbol{\varepsilon}_t\|$. Here $\|\cdot\|$ stands for the Euclidean norm. The class of elliptically symmetric distributions includes the well-known multivariate normal and Student-t distributions, among many others. From these characterizations, it is easy to see that the following relation holds among the classes of distributions:

$$\text{elliptical symmetry} \subset \text{reflective symmetry};$$

see also Neuhaus and Zhu (1999). So obviously the reflective symmetry condition in Assumption 1 is less stringent than elliptical symmetry. For instance, a mixture (finite or not) of distributions each one elliptically symmetric around the origin is not necessarily elliptically symmetric but it is reflectively symmetric. Note also that the distribution of $\mathbf{f}_t$ in (1) may be skewed thereby inducing asymmetry in the unconditional distribution of $\mathbf{r}_t$.

Assumption 1 does not require the vectors $\boldsymbol{\varepsilon}_t$ to be identically distributed nor does it restrict their degree of heterogeneity. This is a very attractive feature since it is well known that financial returns often depart quite dramatically from Gaussian conditions; see Fama (1965), Blattberg and Gonedes (1974), and Hsu (1982), among many others. In particular, the distribution of asset returns appears to have much heavier tails and is more peaked than a normal distribution. The following quote from Fama and MacBeth (1973, p. 619) emphasizes the importance of recognizing non-normalities:

> In interpreting [these] t-statistics one should keep in mind the evidence of Fama (1965) and Blume (1970) which suggests that distributions of common stock returns are "thick-tailed" relative to the normal distribution and probably conform better to nonnormal symmetric stable distributions than to the normal. From Fama and Babiak (1968), this evidence means that when one interprets large t-statistics under the assumption that the underlying variables are normal, the probability or significance levels obtained are likely to be overestimates.

The present framework leaves open not only the possibility of unknown forms of non-normality, but also heteroskedasticity and time-varying covariances among the $\boldsymbol{\varepsilon}_t$'s. For example, when $(\mathbf{r}_t, \boldsymbol{f}_t)$ are elliptically distributed but non-normal, the conditional covariance matrix of $\boldsymbol{\varepsilon}_t$ depends on the contemporaneous $\boldsymbol{f}_t$; see MacKinlay and Richardson (1991) and Zhou (1993). Here the covariance structure of the disturbance terms could be any function of the common factors (contemporaneous or not). The simulation study presented below includes a contemporaneous heteroskedasticity specification.

3.2 Portfolio formation

A usual practice in the application of the GRS test is to base it on portfolio groupings in order to have N much less than T. As Shanken (1996) notes, this has the potential effect of reducing the residual variances and increasing the precision with which $\mathbf{a} = (a_1, ..., a_N)'$ is estimated. On the other hand, as Roll (1979) emphasizes, individual stock expected return deviations under the alternative can cancel out in portfolios, which would reduce the power of the GRS test unless the portfolios are combined in proportion to their weighting in the tangency portfolio. So ideally, all the pricing errors that make up the vector $\mathbf{a}$ in (1) would be of the same sign to avoid power losses when forming an equally-weighted portfolio of the test assets. In the spirit of weighted portfolio groupings, our approach here is an adaptive one based on a split-sample technique, where the first subsample is used to obtain an estimate of $\mathbf{a}$. That estimate is then used to form a single portfolio that judiciously avoids power losses. Finally, a conditional test of H_0 is performed using only the returns on that portfolio observed over the second subsample. It is important to note that in the present framework this approach does not introduce any of the data-snooping size distortions (i.e. the appearance of statistical significance when the null hypothesis is true) discussed in Lo and MacKinlay (1990), since the estimation results are conditionally (on the factors) independent of the second subsample test outcomes.

Let $T = T_1 + T_2$. In matrix form, the first T_1 returns on asset i can be represented by

$$\mathbf{r}_i^{(1)} = a_i \boldsymbol{\iota} + \mathbf{F}^{(1)} \mathbf{b}_i + \boldsymbol{\varepsilon}_i^{(1)}, \tag{8}$$

where $\mathbf{r}_i^{(1)} = [r_{i1}, ..., r_{iT_1}]'$ and $\mathbf{F}^{(1)} = [\boldsymbol{f}_1', ..., \boldsymbol{f}_{T_1}']'$ collect the time series of T_1 returns on asset i and the factors, respectively, $\boldsymbol{\iota}$ is a conformable vector of ones, $\mathbf{b}_i'$ is the i^{th} row of $\mathbf{B}$ in (1), and

$\boldsymbol{\varepsilon}_i^1 = [\varepsilon_{i1}, ..., \varepsilon_{iT_1}]'$.

Assumption 2. *Only the first T_1 observations on $\mathbf{r}_t$ and $\mathbf{f}_t$ are used to compute the subsample estimates $\hat{a}_1, ..., \hat{a}_N$.*

This assumption does not restrict the choice of estimation method, so the subsample estimates $\hat{a}_1, ..., \hat{a}_N$ could be obtained by OLS, quasi-maximum likelihood, or any other method.[4] A well-known problem with OLS is that it is very sensitive to the presence of large disturbances and outliers; see Section 5.4 below for evidence. An alternative estimation method is to minimize the sum of the absolute deviations in computing the regression lines (Bassett and Koenker 1978). The resulting least absolute deviations (LAD) estimator may be more efficient than OLS in heavy-tailed samples where extreme observations are more likely to occur. For more on the efficiency of LAD versus OLS, see Glahe and Hunt (1970), Hunt, Dowling, and Glahe (1974), Pfaffenberger and Dinkel (1978), Rosenberg and Carlson (1977), and Mitra (1987). The results reported below in the simulation study and the empirical application are based on LAD.

With the estimates $\hat{\boldsymbol{a}} = (\hat{a}_1, ..., \hat{a}_N)'$ in hand, a vector of statistically motivated portfolio weights $\hat{\boldsymbol{\omega}} = (\hat{\omega}_1, \hat{\omega}_2, ..., \hat{\omega}_N)'$ is computed according to:

$$\hat{\omega}_i = sign(\hat{a}_i)\frac{1}{N}, \tag{9}$$

for $i = 1, ..., N$, and these weights are then used to find the T_2 returns of a portfolio computed as $y_t = \sum_{i=1}^N \hat{\omega}_i r_{it}$, $t = T_1 + 1, ..., T$. We shall first provide a distributional result for y_t that holds under H_0 and then explain in what sense the weights in (9) will maximize the power of the proposed distribution-free tests. In the following, $\delta = \sum_{i=1}^N \hat{\omega}_i a_i$ is the sum of the weighted a_i's over the second subsample.

Proposition 1. *Under H_0 and when Assumptions 1 and 2 hold, y_t is represented by the single equation*

$$y_t = \delta + \mathbf{f}_t'\boldsymbol{\beta} + u_t, \ \textit{for } t = T_1 + 1, ..., T, \tag{10}$$

[4]Of course, T_1 must at least be enough to obtain the estimates $\hat{a}_1, ..., \hat{a}_N$ by the chosen method.

where $\delta = 0$ *and* $(u_{T_1+1}, ..., u_T) \overset{d}{=} (\pm u_{T_1+1}, ..., \pm u_T)$, *conditional on* $\mathbf{F}$.

Proof. The conditional location part of (10) follows from the common factor structure in (1), and the fact that δ is zero under H_0 is obvious. The independence of the disturbance vectors maintained in Assumption 1 implies that the T_2 vectors $\boldsymbol{\varepsilon}_t$, $t = T_1 + 1, ..., T$, are conditionally independent of the vector of weights $(\hat{\omega}_1, \hat{\omega}_2, ..., \hat{\omega}_N)$ given $\mathbf{F}$, since under Assumption 2 those weights are based only on the first T_1 observations of $\mathbf{r}_t$ and $\mathbf{f}_t$. Thus we see that given $\mathbf{F}$ and $\hat{\boldsymbol{\omega}}$,

$$\big(\hat{\omega}_1\varepsilon_{1t}, \hat{\omega}_2\varepsilon_{2t}, ..., \hat{\omega}_N\varepsilon_{Nt}\big) \overset{d}{=} \big(-\hat{\omega}_1\varepsilon_{1t}, -\hat{\omega}_2\varepsilon_{2t}, ..., -\hat{\omega}_N\varepsilon_{Nt}\big), \tag{11}$$

for $t = T_1 + 1, ..., T$. Let $u_t = \sum_{i=1}^{N} \hat{\omega}_i\varepsilon_{it}$. For a given t, (11) implies that $u_t \overset{d}{=} -u_t$, since any linear combination of the elements of a reflectively symmetric vector is itself symmetric (Behboodian 1990, Theorem 2). Moreover, this fact applies to each of the T_2 conditionally independent random variables $u_{T_1+1}, ..., u_T$. So, given $\mathbf{F}$, the 2^{T_2} possible T_2 vectors

$$\big(\pm |u_{T_1+1}|, \pm|u_{T_1+2}|, ..., \pm|u_T|\big)$$

are equally likely values for $(u_{T_1+1}, ..., u_T)$, where $\pm|u_t|$ means that $|u_t|$ is assigned either a positive or negative sign with probability 1/2. □

The construction of a test based on a single portfolio grouping is reminiscent of a mean-variance efficiency test proposed in Bossaerts and Hillion (1995) based on $\boldsymbol{\iota}'\hat{\boldsymbol{a}}$ and another one proposed in Gungor and Luger (2009) that implicitly exploits $\boldsymbol{\iota}'\boldsymbol{a}$. Those approaches can suffer power losses depending on whether the a_i's tend to cancel out when summed. Splitting the sample and applying the weights in (9) when forming the portfolio offsets that problem.[5]

The portfolio weights in (9) are in fact optimal in a certain sense. To see how, suppose for an instant that $\boldsymbol{\varepsilon}_t$ has well-defined first and second moments satisfying $E[\boldsymbol{\varepsilon}_t|\mathbf{f}_t] = \mathbf{0}$ and $E[\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t'|\mathbf{f}_t] = \boldsymbol{\Sigma}_t$. Assumption 1 then implies that the mean and median (point of symmetry) coincide at zero for any

[5] Of course if one believes *a priori* that the a_i's don't tend to cancel out, then there is no need to split the sample and the test can proceed simply with $\omega_i = 1/N$.

component of $\boldsymbol{\varepsilon}_t$. The power of our test procedure depends on $E[\boldsymbol{\omega}'\mathbf{r}_t - \boldsymbol{\omega}'\mathbf{B}\mathbf{f}_t|\mathbf{f}_t] = \boldsymbol{\omega}'\mathbf{a}$. The next section shows how we deal with the presence of the nuisance parameters comprising $\boldsymbol{\omega}'\mathbf{B}$. As in the usual mean-variance portfolio selection problem, choosing $\boldsymbol{\omega}$ to increase $\boldsymbol{\omega}'\mathbf{a}$ also entails an increase in the portfolio's variance, $\boldsymbol{\omega}'\boldsymbol{\Sigma}_t\boldsymbol{\omega}$, which decreases test power. So the problem we face is to maximize $\boldsymbol{\omega}'\mathbf{a}$ subject to a target value for the variance. Here we set the target to $\boldsymbol{\iota}'\boldsymbol{\Sigma}_t\boldsymbol{\iota}/N^2$, the variance of the naive equally-weighted portfolio which simply allocates equally across the N assets.[6] It is easy to see that $\boldsymbol{\omega} = sign(\boldsymbol{a})/N$ will maximize power while keeping the resulting portfolio variance as close as possible to the target. Of course $sign(\boldsymbol{a})$ is unknown, so (9) uses $sign(\hat{\boldsymbol{a}})$. The possible discrepancy between the achieved and target variance values is given by $\frac{2}{N^2}\sum_{i=1}^{N}\sum_{j=i+1}^{N}\big(sign(\hat{a}_i)sign(\hat{a}_j) - 1\big)\sigma_{ij,t}$, which depends on the off-diagonal (covariance) terms of $\boldsymbol{\Sigma}_t$ but not on any of its diagonal (variance) terms. Note that in an approximate APT factor model, those off-diagonal terms tend to zero as $N \to \infty$.

The weights in (9) are quite intuitive and represent the optimal choice in our distribution-free context where possible forms of distribution heterogeneity (e.g. time-varying variances and covariances) are left completely unspecified. Note that optimal weights in a strict mean-variance sense cannot be used here since finding those requires an estimate of the (possibly time-varying) covariance structure and that is precisely what we are trying to avoid.

3.3 Test statistics

The model in (10) can be represented in matrix form as $\mathbf{y} = \delta\boldsymbol{\iota} + \mathbf{F}^{(2)}\boldsymbol{\beta} + \mathbf{u}$, where $\mathbf{F}^{(2)} = [\boldsymbol{f}'_{T_1+1}, ..., \boldsymbol{f}'_T]'$ and the elements of $\mathbf{u}$ follow what Coudin and Dufour (2009) call a "mediangale" which is similar to the usual martingale difference concept except that the median takes the place of the expectation. The following result is an immediate consequence of the strict conditional mediangale property in Proposition 2.1 of Coudin and Dufour. Here we define $s[x] = 1$, if $x \geq 0$,

[6] It is interesting to note that DeMiguel, Garlappi, and Uppal (2009) find that the asset allocation errors of the suboptimal (from the mean-variance perspective) $1/N$ strategy are smaller than those of optimizing models in the presence of parameter uncertainty.

and $s[x] = -1$, if $x < 0$.

Proposition 2. *Under Assumptions 1 and 2, the T_2 disturbance sign vector*

$$s(\mathbf{y} - \delta\boldsymbol{\iota} - \mathbf{F}^{(2)}\boldsymbol{\beta}) = \left(s[y_{T_1+1} - \delta\boldsymbol{\iota} - \boldsymbol{f}'_{T_1+1}\boldsymbol{\beta}], ..., s[y_T - \delta\boldsymbol{\iota} - \boldsymbol{f}'_T\boldsymbol{\beta}]\right)$$

follows a distribution free of nuisance parameters, conditional on $\mathbf{F}^{(2)}$. Its exact distribution can be simulated to any degree of accuracy simply by repeatedly drawing $\tilde{S}_{T_2} = (\tilde{s}_1, ..., \tilde{s}_{T_2})'$, whose elements are independent Bernoulli variables such that $\Pr[\tilde{s}_t = 1] = \Pr[\tilde{s}_t = -1] = 1/2$.

A corollary of this proposition is that any function of the disturbance sign vector and the factors, say $\Psi = \Psi(s(\mathbf{y} - \delta\boldsymbol{\iota} - \mathbf{F}^{(2)}\boldsymbol{\beta}); \mathbf{F}^{(2)})$, is also pivotal, conditional on $\mathbf{F}^{(2)}$. To see the usefulness of this result, consider the problem of testing

$$H_0(\delta_0, \boldsymbol{\beta}_0) : \delta = \delta_0,\ \boldsymbol{\beta} = \boldsymbol{\beta}_0, \tag{12}$$

where δ_0 and $\boldsymbol{\beta}_0$ are specified values. Following Coudin and Dufour, we consider two test statistics for (12) given by the quadratic forms

$$SX(\delta_0, \boldsymbol{\beta}_0) = s(\mathbf{y} - \delta_0\boldsymbol{\iota} - \mathbf{F}^{(2)}\boldsymbol{\beta}_0)'\mathbf{X}\mathbf{X}'s(\mathbf{y} - \delta_0\boldsymbol{\iota} - \mathbf{F}^{(2)}\boldsymbol{\beta}_0), \tag{13}$$

$$SP(\delta_0, \boldsymbol{\beta}_0) = s(\mathbf{y} - \delta_0\boldsymbol{\iota} - \mathbf{F}^{(2)}\boldsymbol{\beta}_0)'\mathbf{P}(\mathbf{X})s(\mathbf{y} - \delta_0\boldsymbol{\iota} - \mathbf{F}^{(2)}\boldsymbol{\beta}_0), \tag{14}$$

where $\mathbf{X} = [\boldsymbol{\iota}, \mathbf{F}^{(2)}]$ and $\mathbf{P}(\mathbf{X}) = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$ projects orthogonally onto the subspace spanned by the columns of $\mathbf{X}$. Boldin, Simonova, and Tyurin (1997) show that these statistics can be associated with locally most powerful tests in the case of i.i.d. disturbances under some regularity conditions and Coudin and Dufour extend that proof to more general disturbances that are not necessarily i.i.d., but only satisfy the mediangale property. It is interesting to note that (14) can be interpreted as a sign analogue of the usual F-test for testing the hypothesis that all the coefficients in a regression of $s(\mathbf{y} - \delta_0\boldsymbol{\iota} - \mathbf{F}^{(2)}\boldsymbol{\beta}_0)$ on $\mathbf{X}$ are zero.

Under $H_0(\delta_0, \boldsymbol{\beta}_0)$ and conditional on $\mathbf{F}^{(2)}$, the statistics in (13) and (14) are distributed like $\widetilde{SX} = \tilde{S}'_{T_2}\mathbf{X}\mathbf{X}'\tilde{S}_{T_2}$ and $\widetilde{SP} = \tilde{S}'_{T_2}\mathbf{P}(\mathbf{X})\tilde{S}_{T_2}$, respectively. This means that appropriate critical values from the conditional distributions may be found to obtain finite-sample tests of $H_0(\delta_0, \boldsymbol{\beta}_0)$. For instance, consider the statistic in (14). The decision rule is then to reject $H_0(\delta_0, \boldsymbol{\beta}_0)$ at level α if $SP(\delta_0, \boldsymbol{\beta}_0)$ is greater than the $(1-\alpha)$-quantile of the distribution obtained by simulating $\widetilde{SP}$, say c_α^{SP}. A critical value c_α^{SX} can be found in similar fashion by simulating values $\widetilde{SX}$. When (13) and (14) are evaluated at the true parameter values $(\delta, \boldsymbol{\beta})$, Proposition 2 implies that $\Pr[SX(\delta, \boldsymbol{\beta}) > c_\alpha^{SX}] = \alpha$ and $\Pr[SP(\delta, \boldsymbol{\beta}) > c_\alpha^{SP}] = \alpha$ as well. So for all $0 < \alpha < 1$, the critical regions $\{SX(\delta_0, \boldsymbol{\beta}_0) > c_\alpha^{SX}\}$ and $\{SP(\delta_0, \boldsymbol{\beta}_0) > c_\alpha^{SP}\}$ each have size α.[7] Note also that the critical values c_α^{SX} and c_α^{SP} only need to be computed once, since they do not depend on δ_0, $\boldsymbol{\beta}_0$.

Here the value of interest is $\delta_0 = 0$ which means that we are dealing with point null hypotheses of the form

$$H_0(\boldsymbol{\beta}_0) : \delta = 0,\ \boldsymbol{\beta} = \boldsymbol{\beta}_0, \tag{15}$$

where $\beta_0 \in \mathcal{B}$, an admissible parameter space. The null hypothesis implied by (3) that we wish to test is

$$H_0 : \bigcup_{\beta_0 \in \mathcal{B}} H_0(\boldsymbol{\beta}_0), \tag{16}$$

the union of (15) taken over $\mathcal{B}$. The expression in (16) makes clear that the elements of $\boldsymbol{\beta}$ are nuisance parameters in this setting, since they are not constrained to a single value under the null hypothesis. In order to test such a hypothesis, Dufour (2006) suggests a minimax argument as in Savin (1984) which may be stated as "reject the null whenever for all admissible values of the nuisance parameters under the null, the corresponding point null hypothesis is rejected;" see also Jouneau-Sion and Torrès (2006) for an application of this argument in a different context. In general, Dufour's rule in practice consists of maximizing the p-value of the sample test statistic over the set of nuisance parameters. Given the present framework, this rule amounts to minimizing the values of the SX and SP statistics over $\mathcal{B}$. To see why, define

$$SX_L = \inf_{\beta_0 \in \mathcal{B}} SX(0, \boldsymbol{\beta}_0) \text{ and } SP_L = \inf_{\beta_0 \in \mathcal{B}} SP(0, \boldsymbol{\beta}_0), \tag{17}$$

[7] Following the terminology in Lehmann and Romano (2005, Chapter 3), we say that a test of H_0 has *size* α if $\Pr[\text{Rejecting } H_0 \mid H_0 \text{ true}] = \alpha$, and that it has *level* α if $\Pr[\text{Rejecting } H_0 \mid H_0 \text{ true}] \leq \alpha$.

and observe that under H_0 in (16) we have

$$0 \leq SP_L \leq SP(0, \boldsymbol{\beta}),$$

which shows that SP_L is boundedly pivotal. This property further implies under H_0 that

$$\Pr[SP_L > c_\alpha^{SP}] \leq \Pr[SP(0, \boldsymbol{\beta}) > c_\alpha^{SP}] = \alpha.$$

In other words, the test that rejects the null hypothesis H_0 whenever $SP_L > c_\alpha^{SP}$ has level α. The same argument applies to (13) to get the critical region $SX_L > c_\alpha^{SX}$.

Here we compute SX_L and SP_L in (17) by searching over a grid $\mathcal{B}(\hat{\boldsymbol{\beta}})$ specified around LAD point estimates $\hat{\boldsymbol{\beta}}$, which are computed in the restricted (i.e. no intercept) median regression model $\mathbf{y} = \mathbf{F}^{(2)}\boldsymbol{\beta} + \mathbf{u}$. Of course, more sophisticated optimization methods such as simulated annealing could be used to find SX_L and SP_L. The advantage of the naive grid search is that it is completely reliable and feasible when the dimension of $\boldsymbol{\beta}$ is not too large. An important remark is that the search for SX_L and SP_L can be stopped and the null hypothesis can no longer be rejected at level α as soon as a grid point yields a non-rejection. At that point, values of $\boldsymbol{\beta}$ have been found that make the model compatible with the data, meaning the model should not be rejected. For instance, if $SP(0, \hat{\boldsymbol{\beta}}) \leq c_\alpha^{SP}$ then SP_L does not reject either and H_0 in (16) is not significant at level α.

3.4 Summary of test procedure

Suppose that one wishes to use the SP_L statistic in (17). In a preliminary step, the reference distribution for that statistic is simulated to the desired degree of accuracy by generating a large number, say M, of simulated i.i.d. values $\widetilde{SP}_1, ..., \widetilde{SP}_M$ and the α-level critical value c_α^{SP} is determined from the simulated distribution. The rest of the test procedure then proceeds according to the following steps:

1. Compute the estimates $\hat{a}_i$ of a_i, for $i = 1, ..., N$, using the first T_1 observations on $\mathbf{r}_t$ and $\mathbf{f}_t$.

2. Compute the weights $\hat{\omega}_i$, for $i = 1, ..., N$, according to:

$$\hat{\omega}_i = sign(\hat{a}_i)\frac{1}{N},$$

and then compute T_2 portfolio returns as $y_t = \sum_{i=1}^{N} \hat{\omega}_i r_{it}$, $t = T_1 + 1, ..., T$.

3. Find $SP_L = \inf_{\beta_0 \in \mathcal{B}} SP(0, \boldsymbol{\beta}_0)$.

4. Reject the null hypothesis $H_0 : \mathbf{a} = \mathbf{0}$ at level α if $SP_L > c_\alpha^{SP}$, or equivalently in terms of the p-value if $\hat{p}(SP_L) \leq \alpha$. Here the p-value can be computed as

$$\hat{p}(SP_L) = \frac{1}{M} \sum_{j=1}^{M} \mathbb{I}[\widetilde{SP}_j > SP_L],$$

where $\mathbb{I}[A]$ is the indicator function of event A.

This procedure yields a finite-sample and distribution-free test of H_0 in (3) over the class of all disturbance distributions satisfying Assumption 1.

4 Simulation evidence

We present the results of some simulation experiments to compare the performance of the proposed test procedure with several standard tests. The first of the benchmarks for comparison purposes is the GRS test in (7). The other benchmarks are the usual LR test, an adjusted LR test, and a test based on GMM. The latter is a particularly important benchmark here, since in principle it is "robust" to non-normality and heteroskedasticity of returns.

The LR test is based on a comparison of the constrained and unconstrained log-likelihood functions evaluated at the maximum likelihood estimates. The unconstrained estimates are given in (4), (5), and (6). For the constrained case, the maximum likelihood estimates are

$$\begin{aligned}
\hat{\mathbf{B}}^* &= \left[\sum_{t=1}^{T} \mathbf{r}_t \mathbf{f}_t'\right] \left[\sum_{t=1}^{T} \mathbf{f}_t \mathbf{f}_t'\right]^{-1}, \\
\hat{\boldsymbol{\Sigma}}^* &= \frac{1}{T} \sum_{t=1}^{T} \left(\mathbf{r}_t - \hat{\mathbf{B}}^* \mathbf{f}_t\right) \left(\mathbf{r}_t - \hat{\mathbf{B}}^* \mathbf{f}_t\right)'.
\end{aligned}$$

The LR test statistic, J_2, is then given by

$$J_2 = T \left[\log |\hat{\boldsymbol{\Sigma}}^*| - \log |\hat{\boldsymbol{\Sigma}}|\right],$$

which, under the null hypothesis, follows an asymptotic chi-square distribution with N degrees of freedom, χ^2_N. As we shall see, the finite sample behavior of J_2 can differ vastly from what asymptotic theory predicts. Jobson and Korkie (1982) suggest an adjustment to J_2 in order to improve its finite-sample size properties when used with critical values from the χ^2_N distribution. The adjusted LR statistic is

$$J_3 = \frac{T - (N/2) - K - 1}{T} J_2,$$

which also follows the asymptotic χ^2_N distribution, under H_0.

MacKinlay and Richardson (1991) develop tests of mean-variance efficiency in a GMM framework. For the asset pricing model in (1), the GMM tests are based on the moments of the following $(K+1)N \times 1$ vector:

$$\mathbf{g}_t(\boldsymbol{\theta}) = \begin{pmatrix} 1 \\ \mathbf{f}_t \end{pmatrix} \otimes \boldsymbol{\varepsilon}_t(\boldsymbol{\theta}), \tag{18}$$

where $\boldsymbol{\varepsilon}_t(\boldsymbol{\theta}) = \mathbf{r}_t - \mathbf{a} - \mathbf{B}\mathbf{f}_t$. The symbol $\otimes$ refers to the Kronecker product. Here $\boldsymbol{\theta} = (\mathbf{a}', vec(\mathbf{B})')'$, where $vec(\mathbf{B})$ is an $NK \times 1$ vector obtained by stacking the columns of $\mathbf{B}$, one below the other, with the columns ordered from left to right. The model specification in (1) implies the moment conditions $E(\mathbf{g}_t(\boldsymbol{\theta}_0)) = \mathbf{0}$, where $\boldsymbol{\theta}_0$ is the true parameter vector. The system in (18) is exactly identified which implies that the GMM procedure yields the same estimates of $\boldsymbol{\theta}$ as does OLS applied equation by equation. The covariance matrix of the GMM estimator $\hat{\boldsymbol{\theta}}$ is given by $\mathbf{V} = [\mathbf{D}_0' \mathbf{S}_0^{-1} \mathbf{D}_0]^{-1}$, where $\mathbf{D}_0 = E[\partial \mathbf{g}_T(\boldsymbol{\theta})/\partial \boldsymbol{\theta}']$ with $\mathbf{g}_T(\boldsymbol{\theta}) = T^{-1} \sum_{t=1}^{T} \mathbf{g}_t(\boldsymbol{\theta})$ and $\mathbf{S}_0 = \sum_{s=-\infty}^{+\infty} E[\mathbf{g}_t(\boldsymbol{\theta}) \mathbf{g}_{t-s}(\boldsymbol{\theta})']$; see Campbell, Lo, and MacKinlay (1997, Chapter 5). The GMM-based Wald test statistic is

$$J_4 = T\hat{\mathbf{a}}' \left[\mathbf{R} \left(\hat{\mathbf{D}}^{-1} \hat{\mathbf{S}} (\hat{\mathbf{D}}')^{-1} \right) \mathbf{R}' \right]^{-1} \hat{\mathbf{a}}, \tag{19}$$

where $\hat{\mathbf{D}}$ and $\hat{\mathbf{S}}$ are consistent estimators of $\mathbf{D}_0$ and $\mathbf{S}_0$, respectively, and $\mathbf{R} = (1, \mathbf{0}_K) \otimes \mathbf{I}_N$, with $\mathbf{0}_K$ denoting a row vector of K zeros and $\mathbf{I}_N$ as the $N \times N$ identity matrix. Under H_0, the statistic J_4 follows the χ^2_N distribution asymptotically.

We examine cases with $K = 1$ and $K = 3$ in model (1). For convenience, the three-factor specification we consider is given again here as

$$r_{it} = a_i + b_{i1} f_{1t} + b_{i2} f_{2t} + b_{i3} f_{3t} + \varepsilon_{it}, \quad t = 1, ..., T, \ \ i = 1, ..., N, \tag{20}$$

with common factor returns following independent stochastic volatility processes of the form:

$$f_{jt} = \exp(h_{jt}/2)\epsilon_{jt} \text{ with } h_{jt} = \lambda_j h_{j,t-1} + \xi_{jt},$$

where the independent terms ϵ_{jt} and ξ_{jt} are both i.i.d. according to a standard normal distribution and the persistence parameters λ_j are set to 0.5. For $j = 1, .., 3$, the b_{ij}'s are randomly drawn from a uniform distribution between 0.5 and 1.5. All the tests are conducted at the nominal 5% level and critical values for SX_L and SP_L are determined using 10,000 simulations. In the experiments we choose mispricing values a and set half the intercept values as $a_i = a$ and the other half as $a_i = -a$. We denote this in the tables as $|a_i| = a$. The estimates of a_i, $i = 1, ..., N$, in Step 1 are found via LAD. Finally, there are 1000 replications in each experiment.

Consider first the single-factor model so b_{i2} and b_{i3} in (20) are zero, in which case the null hypothesis is a test of the mean-variance efficiency of the given factor portfolio. In the application of the test procedure, a choice needs to be made about where to split the sample. While this choice has no effect on the level of the tests, it obviously matters for their power. We do not have analytical results on how to split the sample, so we resort to simulations. Table 1 shows the power of the test procedure applied with the SX_L and SP_L statistics for various values of T_1/T, where $|a_i| = 0.20$, 0.15, and 0.10. These values are well within the range found in our empirical application, where the intercepts estimated with monthly stock returns range in values from -0.5 to 1.5. Here we set $T = 60$ and $N = 100$ and the disturbance terms ε_{it} are drawn randomly from the standard normal distribution. As expected, the results show that for any given value of T_1/T the power increases as $|a_i|$ increases. Overall, the results suggest that no less that 30% and no more than 50% of the time-series observations should be used as the first subsample in order to maximize power. Accordingly, the testing strategy represented by $T_1 = 0.4T$ is pursued in the remaining comparative experiments.

We also include in our comparisons two distribution-free tests proposed by Gungor and Luger (2009) that are applicable in the single-factor case. The building block of those tests is

$$z_{it} = \left(\frac{r_{i,t+m}}{f_{1,t+m}} - \frac{r_{it}}{f_{1t}} \right) \times \frac{(f_{1t} - f_{1,t+m})}{f_{1t} f_{1,t+m}}, \tag{21}$$

defined for $t = 1, ..., m$, where $m = T/2$ is assumed to be an integer. The first test is based on the sign statistic

$$S_i = \frac{\sum_{t=1}^{m} 0.5(s[z_{it}] + 1) - m/2}{\sqrt{m/4}} \tag{22}$$

and the second one is based the Wilcoxon signed rank statistic

$$W_i = \frac{\sum_{t=1}^{m} 0.5(s[z_{it}] + 1)\text{Rank}(|z_{it}|) - m(m+1)/4}{\sqrt{m(m+1)(2m+1)/24}}, \tag{23}$$

where $\text{Rank}(|z_{it}|)$ is the rank of $|z_{it}|$ when $|z_{i1}|, ..., |z_{im}|$ are placed in ascending order of magnitude. Gungor and Luger (2009) show that a time-series symmetry condition ensures that both (22) and (23) have limiting (as $m \to \infty$) standard normal distributions. Under the further assumption that the disturbance terms are cross-sectionally independent, conditional on $(f_{11}, ..., f_{1T})'$, the sum-type statistics

$$SD = \sum_{i=1}^{N} S_i^2 \text{ and } WD = \sum_{i=1}^{N} W_i^2 \tag{24}$$

follow an asymptotic chi-square distribution with N degrees of freedom. Simulation results show that this approximation works extremely well and just like the test procedure proposed here, the SD and WD test statistics can be calculated even if N is large.

Tables 2–5 show the empirical size (Panel A) and power (Panel B) of the considered tests when $|a_i| = 0.15$ and $T = 60, 120$ and N takes on values between 10 and 500. When they don't respect the nominal level constraint, the power results for the J tests are based on size-corrected critical values. It is important to emphasize that size-corrected tests are not feasible in practice, especially under the very general symmetry condition in Assumption 1. They are merely used here as theoretical benchmarks for the truly distribution-free tests. In particular, we wish to see how the power of the new tests compares to these benchmarks as T and N vary.

The results in Table 2 correspond to the single-factor model where the disturbance terms ε_{it} are i.i.d. in both the time-series and the cross-section according to a standard normal distribution. Here the parametric J_1 behaves according to its distributional theory and the distribution-free SD and WD tests also behave well under the null with empirical rejection rates close to the nominal level. From Table 2, the conservative SX_L and SP_L tests are also seen to satisfy the level constraint in the sense that the probability of a Type I error remains bounded by the nominal level of significance. The J_2, J_3, and J_4 tests, however, suffer massive size distortions as the number of equations increases.[8] When $T = 120$ and $N = 100$, the LR test (J_2) rejects the true null with an empirical probability of 100% and in the case of the adjusted LR test (J_3) that probability is still above 50%. Notice also that the J tests are not computable when N exceeds T. Such cases are indicated with "-" in the tables.

In Panel B of Table 2, we see the same phenomenon as in Figure 1: for a fixed T, the power of the GRS J_1 test rises and then eventually drops as N keeps on increasing. Note that J_1, J_2, and J_3 have identical size-corrected powers, since they are all related via monotonic transformations (Campbell, Lo, and MacKinlay 1997, Ch. 5). On the contrary, the power of the SD and WD tests and that of the new SX_L and SP_L tests increases monotonically with N.

The next specification we consider resembles a stochastic volatility model and introduces dependence between the conditional covariance matrix and f_{1t}. Specifically, we let $\varepsilon_{it} = \exp(\lambda_i f_{1t}/2)\eta_{it}$, where the innovations η_{it} are standard normal and the λ_i's are randomly drawn from a uniform distribution between 1.5 and 2.5. It should be noted that such a contemporaneous heteroskedastic specification finds empirical support in Duffee (1995, 2001) and it is easy to see that it generates ε_{it}'s with time-varying excess kurtosis—a well-known feature of asset returns. Panel A of Table 3 reveals that all the parametric tests have massive size distortions in this case, and these over-rejections worsen as N increases for a given T.[9] When $T = 120$, the J tests all have empirical sizes

[8] This overrejection problem with standard asymptotic tests in multivariate regression models is also documented in Stambaugh (1982), Jobson and Korkie (1982), Amsler and Schmidt (1985), MacKinlay (1987), Stewart (1997), and Dufour and Khalaf (2002).

[9] The sensitivity of the GRS test to contemporaneous heteroskedasticity is also documented in MacKinlay and

around 20%. The probability of a Type I error for all those tests exceeds 65% when N is increased to 50. In sharp contrast, the four distribution-free tests satisfy the nominal 5% level constraint, no matter T and N. As in the first example, Panel B shows the power of the distribution-free tests increasing with both T and N in this heteroskedastic case.

At this point, one may wonder what is the advantage of the new SX_L and SP_L tests over the SD and WD tests of Gungor and Luger (2009) since the latter display better power in Panel B of Tables 2 and 3. Those tests achieve higher power because they eliminate the b_{i1}'s from the inference problem through the long differences in (21), whereas the new tests proceed through a minimization of the test statistics over the intervening nuisance parameter space. A limitation of the SD and WD tests, however, is that they are valid only under the assumption that the model disturbances are cross-sectionally independent. Table 4 reports the empirical size of the those tests when the cross-sectional disturbances inherit a common factor structure. Namely, we let $\varepsilon_{it} = \gamma_i w_t + e_{it}$, where the common factor w_t and the idiosyncratic term e_{it} are independent and both i.i.d. according to a standard normal distribution. The factor loadings γ_i are drawn from a uniform distribution over the interval $[0, U_{max}]$ and U_{max} is varied between 0.5 and 2.0. The nominal level is 0.05 and we consider $T = 60$, 120 and $N = 10$, 100. We see from Table 4 that the SD and WD tests are fairly robust to mild cross-sectional correlation, but start over-rejecting as the cross-sectional dependence increases and this problem is further exacerbated when N increases. As expected, Table 4 shows that SX_L and SP_L are not affected by cross-sectional dependence.

Table 5 reports the results for the three-factor model, as given in (20). The disturbance terms are governed by $\varepsilon_{it} = \exp(\lambda_i f_t^*/2)\eta_{it}$ like in the case of Table 3, except that now $f_t^* = (f_{1t} + f_{2t} + f_{3t})/3$ so all three factors contribute (equally) to the variance heterogeneity. Note that the SD and WD statistics in (24) are not computable in the presence of multiple factors. Table 5 echoes the previous findings for the J tests about their size distortions and decreasing power as N increases. What's new in Table 5 is that the SX_L and SP_L tests appear to be more conservative in this case, so N needs to be increased in order to attain the power levels seen in Tables 2 and 3. In the empirical

Richardson (1991), Zhou (1993), and Gungor and Luger (2009).

illustration that follows, we apply the new tests with $N = 10$, 100, and 503 test assets.

5 Empirical illustration

In this section we illustrate the new tests with two empirical applications. First, we examine the Sharpe-Lintner version of the CAPM. This single-factor model uses the excess returns of a value-weighted stock market index of all stocks listed on the NYSE, AMEX, and NASDAQ. Second, we test the more general three-factor model of Fama and French (1993), which adds two factors to the CAPM specification: (i) the average returns on three small market capitalization portfolios minus the average return on three big market capitalization portfolios, and (ii) the average return on two value portfolios minus the average return on two growth portfolios. Note that the CAPM is nested within the Fama-French model. This means that if there was no sampling uncertainty, finding that the market portfolio is mean-variance efficient would trivially imply the validity of the three-factor model. Conversely, if the three-factor model does not hold, then the CAPM is also rejected.

We test both specifications with three sets of test assets comprising the stocks traded on the NYSE, AMEX, and NASDAQ markets for the 38-year period from January 1973 to December 2010 (456 months). The first two data sets are the monthly returns on 10 portfolios formed on size, and 100 portfolios formed on both size and the book-to-market ratio. Those two data sets are available in Ken French's online data library. The third data set we make use of comprises the returns on 503 individual stocks traded on the markets mentioned above. These represent all the stocks for which there is data in the Center for Research in Security Prices (CRSP) monthly stock files for the entire 38-year sample period. The complete list of company names is given in the appendix. Finally, we use the one-month U.S. Treasury bill as the risk-free asset.

Figure 2 plots the excess returns of the stock market index over the full sample period. That figure shows that this typical return series contains several extreme observations. For instance, the returns seen during the stock market crash of October 1987, the financial crisis of 2008, and at some other points in time as well are obviously not representative of normal market activity; we discuss the effects of extreme observations in Section 5.4. It is also quite common in the empirical finance literature to perform asset pricing tests over subperiods out of concerns about parameter stability. So in addition to the entire 38-year period, we also examine six 5-year, one 8-year, and three 10-year subperiods. For other examples of this practice, see Campbell, Lo, and MacKinlay (1997), Gungor and Luger (2009), and Ray, Savin, and Tiwari (2009).

5.1 10 size portfolios

Table 6 reports the CAPM test results based on the ten size portfolios. The numbers reported in parenthesis are p-values and the entries in bold represent cases of significance at the 5% level. We see here that the parametric J tests reject the null hypothesis over the entire sample period with p-values no more than 5%. The non-parametric SD and WD tests also indicate strong rejections. In contrast, the SX_L and SP_L tests clearly do not reject the mean-variance efficiency of the market index.

Looking at the subperiods, we see that the only rejection by the new tests occurs with SP_L in the 10-year subperiod 1/73–12/82. In the 5-year subperiod 1/98–12/02, the J_2 and J_4 tests reject the CAPM specification. The results for the J tests during the last 10-year subperiod from 1/93 to 12/02 agree with the rejection findings for the entire sample period. Besides the obvious differences between the parametric and non-parametric inference results, Table 6 also reveals some differences between the SD and WD tests and the proposed SX_L and SP_L tests. One possible reason for the disagreement across these non-parametric tests could be the presence of cross-sectional disturbance correlations. As we saw in Table 4, the SD and WD tests are not robust to such correlations, whereas the new tests allow for cross-sectional dependencies just like the GRS test.

Table 7 shows the results for the Fama-French model. For the entire 38-year sample period, the results in Table 7 are in line with those for the single-factor model in Table 6. The standard J tests reject the null with very low p-values, whereas the distribution-free SX_L and SP_L tests are not significant. In the 5-year subperiods, we see some disagreements among the parametric tests. For instance, during 1/98–12/02 the J_1 and J_3 indicate non-rejections, while J_2 and J_4 point toward rejections of the null. The results for the last two 10-year subperiods resemble those for the entire sample period and the J tests depict a more consistent picture.

Table 7 shows that the SX_L and SP_L tests never reject the three-factor specification. Taken at face value, these results would suggest that the excess returns of the 10 size portfolios are well explained by the three Fama-French factors. This is entirely consistent with the non-rejections seen in Table 6 and it suggests that the size and the book-to-market factors play no role; i.e., the CAPM factor alone can price the 10 size portfolios.

Upon observing that the Fama-French model is never rejected by the non-parametric SX_L and SP_L tests with 10 test assets, one may be concerned about the ability of the new procedure to reject the null, when the alternative is true. In order to boost power, we proceed next with a tenfold increase in the number of test assets.

5.2 100 size and book-to-market portfolios

Tables 8 and 9 show the test results for the two considered factor specifications using return data on 100 portfolios formed on size and book-to-market. Note that with $N = 100$, none of the parametric tests are computable in the 5-year subperiods where $T = 60$.

Using the entire sample, the J tests decisively reject both factor specifications. In the single-factor case (Table 8), the inference results based on the SD and WD tests are in agreement with the parametric ones. Note again that the SD and WD tests cannot be computed in the three-factor specification (Table 9). The interesting result in Tables 8 and 9 is that the SP_L also indicates rejections. Compared to Tables 6 and 7, it would seem that increasing N from 10 to 100 delivered more power, at least for the entire sample period. Rejections are also seen for SP_L in the last 10-year subperiod.

The rejections by the SP_L test seen in Tables 8 and 9 with $N = 100$ beg the question: what would happen if N was increased even further? To answer that question, we turn next to the individual stock data.

5.3 503 individual stocks

Tables 10 and 11 report the test results using the returns on 503 individual stocks. Here the J tests cannot be computed, since $N > T$. The most striking result is that now for the entire sample period ($T = 456$) the preferred SP_L test no longer indicates a rejection of either the CAPM nor the Fama-French three-factor model. This stands in sharp contrast to the results seen in Tables 8 and 9 with 100 portfolio returns. The results in Tables 8 and 10 from the SD and WD tests also agree with the non-rejection of the CAPM when moving from portfolios to individual stocks.

These results suggest that the excess returns on individual stocks are well explained by the CAPM, which in turn suggests that the size and the book-to-market factors play no role in pricing this collection of assets. It also appears that creating portfolios on the basis of size and book-to-market biases the test outcomes toward a rejection of the model's validity. This finding with the newly proposed SP_L test is entirely consistent with the analysis in Lo and MacKinlay (2000), who show that sorting stocks into groups based on variables that are correlated with returns is a questionable practice since it favors a rejection of the asset pricing model under consideration; see also Berk (2000) for related theoretical analysis. Finally, it is interesting to note that this conclusion about the validity of the CAPM is also reached by Zhou (1993), Vorkink (2003), Gungor and Luger (2009), and Ray, Savin, and Tiwari (2009).

5.4 Extreme observations

Looking back upon the results in Tables 6 and 7 with 10 size portfolios, one might think that the differences between the parametric J tests and the SX_L and SP_L tests is due to a lack of power by the latter when N is small. However, another plausible reason for these differences is the adverse effect that a small number of extreme observations can have on the OLS estimates used to compute the J tests; see Vorkink (2003). To investigate that possibility we recompute the parametric tests with winsorized data. This procedure has the effect of decreasing the magnitude of extreme observations but leaves them as important points in the sample.

Table 12 shows the results of the J tests with the 10 size portfolios when the full-sample returns are winsorized at the 0.3%, 0.5%, 0.7%, and 1% levels. In the single-factor case (Panel A), the J tests cease to be significant at the 5% level with returns winsorized at 0.3%. For the three-factor model (Panel B), the same pattern of increasing p-values occurs across winsorization levels. These results clearly show that OLS-based inference can be very sensitive to the presence of even just a few extreme observations.

6 Conclusion

The beta-pricing representation of linear factor pricing models is typically assessed with tests based on OLS or GMM. In this context, standard asymptotic theory is known to provide a poor approximation to the finite-sample distribution of those test statistics, even with fairly large samples. In particular, the asymptotic tests tend to over-reject the null hypothesis when in fact it is true, and these size distortions grow quickly as the number of included test assets increases. So the conclusions of empirical studies that adopt such procedures can lead one to spuriously reject the validity of the asset pricing model.

Exact finite-sample methods that avoid the spurious rejection problem usually rely on strong distributional assumptions about the model's disturbance terms. A prominent example is the GRS test that assumes that the disturbance terms are identically distributed each period according to a multivariate normal distribution. Yet it is known from the empirical finance literature that financial asset returns are non-normal, exhibiting time-varying covariance structures and excess kurtosis. These stylized facts would put into question the reliability of any inference method that assumes that the cross-sectional distribution of disturbance terms is homogenous over time.

Another serious problem with standard inference methods has to do with the choice of how many tests assets to include. Indeed, if too many are included relative to the number of available time-series observations, the GRS test may lose all its power or may not even be computable. In fact, any procedure that relies on unrestricted estimates of the covariance matrix of regression disturbances will no longer be computable owing to the singularity that occurs when the size of the cross-section exceeds the length of the time series.

In this paper we have proposed a finite-sample test procedure that overcomes these problems. Specifically, our statistical framework makes no parametric assumptions about the distribution of the disturbance terms in the factor model. The only requirement is that the cross-section disturbance vectors be independent over time, conditional on the factors, and reflectively symmetric each period. The class of reflectively symmetric distributions includes elliptically symmetric ones, which are theoretically consistent with mean-variance analysis. Our non-parametric framework leaves open the possibility of unknown forms of time-varying non-normalities and many other distribution heterogeneities, such as time-varying covariance structures, time-varying kurtosis, etc.

The procedure is an adaptive one that first splits the sample to combine the assets into a single portfolio using weights based on the sign of estimated regression intercepts from a subsample. This solves the problem of too many assets and could of course be used in conjunction with an assumed parametric form of the multivariate disturbance distribution. The Lehmann and Stein (1949) impossibility theorem, however, shows that if we wish to remain completely agnostic about heteroskedasticity, then the only valid tests are those based on sign statistics. Even though some studies such as Affleck-Graves and McDonald (1989) report evidence showing the GRS test to be fairly robust to (some specified) deviations from normality, we find it hard to have faith in a parametric procedure whose assumptions are so obviously at odds with the empirical evidence. Moreover our results show that the power of the new sign-based test procedure increases as either the time-series lengthens and/or the cross-section becomes larger. So the truly robust inference procedure developed here offers a very compelling way to assess linear factor pricing models, especially with a large number of test assets.

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Table 1

Empirical power comparisons for various sample splits

T_1/T	0.2	0.3	0.4	0.5	0.6	0.7	0.8
$\|a_i\| = 0.20$							
SX_L	77.9	84.0	88.4	81.2	73.7	52.9	9.5
SP_L	89.5	94.2	97.4	96.8	94.6	81.8	34.4
$\|a_i\| = 0.15$							
SX_L	31.3	36.2	38.0	34.3	32.0	17.9	2.6
SP_L	48.4	54.2	58.7	55.7	53.5	34.4	9.4
$\|a_i\| = 0.10$							
SX_L	4.6	5.0	6.0	5.1	3.5	2.1	0.2
SP_L	9.4	10.2	12.3	10.0	8.2	4.3	1.7

Notes: This table reports the empirical power (in percentages) of the proposed test procedure based on the SX_L and SP_L statistics in (17) for various sample splits, T_1/T. The sample size is $T = 60$ and the number of test assets is $N = 100$. The returns are generated according to a single-factor model with i.i.d. disturbances following a standard normal distribution. The notation $|a_i| = a$ means that $N/2$ pricing errors are set as $a_i = -a$ and the other half are set as $a_i = a$. The nominal level is 0.05 and the results are based on 1000 replications.

Table 2

Empirical size and power: 1-factor model with homoskedastic disturbances

T	N	J_1	J_2	J_3	J_4	SD	WD	SX_L	SP_L
Panel A: Size									
60	10	5.4	10.0	5.4	9.0	4.8	4.9	0.1	0.2
	25	5.2	31.7	6.9	15.6	4.7	3.8	0.1	0.4
	50	4.1	98.7	40.7	36.6	3.9	3.6	0.3	0.5
	100	-	-	-	-	5.0	4.6	0.2	0.3
	125	-	-	-	-	4.4	2.9	0.1	0.3
120	10	4.2	5.9	4.2	5.8	5.0	4.5	0.2	0.5
	25	4.1	11.9	4.5	7.2	4.9	4.0	0.7	1.2
	50	5.3	43.0	8.2	16.1	4.1	4.3	0.3	0.8
	100	4.7	100.0	54.1	34.3	4.3	5.0	0.4	0.6
	125	-	-	-	-	4.5	4.8	0.5	0.9
Panel B: Size-corrected power									
60	10	59.5	59.5	59.5	57.8	19.4	22.3	3.3	6.2
	25	75.4	75.4	75.4	74.1	29.6	33.2	5.3	12.3
	50	43.6	43.6	43.6	42.1	43.3	52.1	16.7	28.1
	100	-	-	-	-	70.6	79.0	38.0	58.7
	125	-	-	-	-	77.2	85.5	48.9	71.1
120	10	95.8	95.8	95.8	95.7	39.0	42.9	15.9	25.2
	25	100.0	100.0	100.0	100.0	64.6	73.1	42.1	59.5
	50	100.0	100.0	100.0	100.0	87.6	93.8	74.4	87.6
	100	98.3	98.3	98.3	98.6	98.2	99.6	97.5	99.6
	125	-	-	-	-	99.5	99.8	99.4	100.0

Notes: This table reports the empirical size (Panel A) and size-corrected power (Panel B) of the GRS test (J_1), the LR test (J_2), an adjusted LR test (J_3), a GMM-based test (J_4), a sign test (SD), a Wilcoxon signed rank test (SD), and the proposed SX_L and SP_L tests. The returns are generated according to a single-factor model with i.i.d. disturbances following a standard normal distribution. The pricing errors are zero under H_0, whereas $N/2$ pricing errors are set equal to -0.15 and the other half are set to 0.15 under H_1. The nominal level is 0.05 and entries are percentage rates. The results are based on 1000 replications and the symbol "-" is used whenever a test is not computable.

Table 3
Empirical size and power: 1-factor model with contemporaneous heteroskedasticity

T	N	J_1	J_2	J_3	J_4	SD	WD	SX_L	SP_L
Panel A: Size									
60	10	22.7	33.7	22.9	27.1	5.0	5.1	0.7	1.0
	25	43.9	79.9	47.7	60.2	5.1	4.5	0.6	1.4
	50	46.4	95.3	86.2	82.7	4.9	4.7	0.3	0.6
	100	-	-	-	-	4.8	4.3	0.3	1.1
	200	-	-	-	-	5.1	3.6	0.5	1.2
120	10	18.9	22.7	18.9	18.5	5.0	4.2	0.7	1.4
	25	38.6	56.4	39.3	43.5	3.7	3.7	0.5	1.9
	50	65.2	93.3	68.9	75.7	4.1	4.3	0.5	1.6
	100	67.6	94.5	89.8	81.7	5.1	4.8	1.4	1.3
	200	-	-	-	-	4.0	3.9	1.5	1.7
Panel B: Size-corrected power									
60	10	18.7	18.7	18.7	21.4	14.2	17.2	1.5	3.3
	25	31.7	31.7	31.7	30.8	21.7	23.9	2.2	5.0
	50	34.6	34.6	34.6	22.7	31.4	34.7	5.0	9.0
	100	-	-	-	-	50.8	58.5	10.3	15.4
	200	-	-	-	-	74.4	82.6	19.8	28.2
120	10	24.1	24.1	24.1	24.7	29.7	30.9	6.2	10.8
	25	50.7	50.7	50.7	52.9	47.8	53.1	14.1	21.3
	50	78.9	78.9	78.9	74.9	71.2	77.3	30.4	38.3
	100	73.8	73.8	73.8	65.3	89.6	95.3	52.3	60.1
	200	-	-	-	-	99.0	99.6	78.6	84.1

Notes: This table reports the empirical size (Panel A) and size-corrected power (Panel B) of the GRS test (J_1), the LR test (J_2), an adjusted LR test (J_3), a GMM-based test (J_4), a sign test (SD), a Wilcoxon signed rank test (SD), and the proposed SX_L and SP_L tests. The returns are generated according to a single-factor model with contemporaneous heteroskedastic disturbances. The pricing errors are zero under H_0, whereas $N/2$ pricing errors are set equal to -0.15 and the other half are set to 0.15 under H_1. The nominal level is 0.05 and entries are percentage rates. The results are based on 1000 replications and the symbol "-" is used whenever a test is not computable.

Table 4

Empirical size under cross-sectional disturbance factor structure

	$N = 10$				$N = 100$			
U_{max}	0.5	1.0	1.5	2.0	0.5	1.0	1.5	2.0
$T = 60$								
SD	5.9	6.2	7.6	9.0	7.6	13.7	18.4	20.3
WD	5.7	7.7	8.3	9.7	8.5	15.9	18.8	20.7
SX_L	0.0	0.2	0.3	0.1	0.2	0.2	0.1	0.3
SP_L	0.4	0.5	0.6	0.9	0.4	0.3	0.5	0.2
$T = 120$								
SD	6.7	6.6	8.6	9.3	6.4	12.0	15.2	17.7
WD	6.1	7.1	8.7	8.9	6.5	13.2	15.7	17.8
SX_L	0.2	0.1	0.2	0.4	0.2	0.2	0.3	0.1
SP_L	0.7	0.5	0.5	0.8	0.2	0.9	0.4	0.7

Notes: This table reports the empirical size of a sign test (SD), a Wilcoxon signed rank test (WD), and the proposed SX_L and SP_L tests when the 1-factor model cross-sectional disturbances have a factor structure; the factor loadings are uniformly distributed over $[0, U_{max}]$. The nominal level is 0.05 and entries are percentage rates. The results are based on 1000 replications

Table 5

Empirical size and power: 3-factor model with contemporaneous heteroskedasticity

T	N	J_1	J_2	J_3	J_4	SD	WD	SX_L	SP_L
Panel A: Size									
60	25	30.2	66.9	34.3	57.2	-	-	0.0	0.0
	50	29.5	99.8	86.5	92.4	-	-	0.0	0.1
	100	-	-	-	-	-	-	0.0	0.0
	200	-	-	-	-	-	-	0.0	0.0
	500	-	-	-	-	-	-	0.0	0.1
120	25	25.6	41.5	26.6	36.9	-	-	0.0	0.3
	50	49.4	89.3	55.8	72.7	-	-	0.0	0.3
	100	60.4	100.0	97.8	95.6	-	-	0.2	0.8
	200	-	-	-	-	-	-	0.4	0.6
	500	-	-	-	-	-	-	1.8	2.4
Panel B: Size-corrected power									
60	25	35.2	35.2	35.2	33.6	-	-	0.0	0.2
	50	24.1	24.1	24.1	23.8	-	-	0.0	1.4
	100	-	-	-	-	-	-	0.3	3.9
	200	-	-	-	-	-	-	0.6	12.3
	500	-	-	-	-	-	-	5.9	43.7
120	25	72.7	72.7	72.7	80.2	-	-	1.0	7.9
	50	90.5	90.5	90.5	90.1	-	-	7.2	19.4
	100	81.9	81.9	81.9	69.0	-	-	28.2	53.7
	200	-	-	-	-	-	-	65.8	88.6
	500	-	-	-	-	-	-	95.8	99.2

Notes: This table reports the empirical size (Panel A) and size-corrected power (Panel B) of the GRS test (J_1), the LR test (J_2), an adjusted LR test (J_3), a GMM-based test (J_4), a sign test (SD), a Wilcoxon signed rank test (SD), and the proposed SX_L and SP_L tests. The returns are generated according to a 3-factor model with contemporaneous heteroskedastic disturbances. The pricing errors are zero under H_0, whereas $N/2$ pricing errors are set equal to -0.15 and the other half are set to 0.15 under H_1. The nominal level is 0.05 and entries are percentage rates. The results are based on 1000 replications and the symbol "-" is used whenever a test is not computable.

Table 6
Tests of the CAPM with 10 size portfolios

Time period	J_1	J_2	J_3	J_4	SD	WD	SX_L	SP_L
38-year period								
1/73–12/10	**1.83**	**18.43**	**18.14**	**18.48**	**38.45**	**35.25**	208.77	0.06
	(0.05)	(0.05)	(0.05)	(0.04)	(0.00)	(0.00)	(0.92)	(0.97)
5-year subperiods and an 8-year subperiod								
1/73–12/77	0.56	6.46	5.71	5.24	16.80	15.09	197.79	4.02
	(0.83)	(0.77)	(0.84)	(0.87)	(0.08)	(0.12)	(0.64)	(0.14)
1/78–12/82	1.12	12.38	10.93	10.90	**54.93**	**42.28**	204.06	3.77
	(0.36)	(0.26)	(0.36)	(0.37)	(0.00)	(0.00)	(0.69)	(0.16)
1/83–12/87	0.81	9.20	8.12	8.75	5.46	5.14	128.48	1.24
	(0.62)	(0.51)	(0.61)	(0.55)	(0.85)	(0.88)	(0.81)	(0.55)
1/88–12/92	0.79	9.01	7.96	8.13	**21.60**	10.83	16.98	0.22
	(0.63)	(0.53)	(0.63)	(0.62)	(0.01)	(0.37)	(0.94)	(0.89)
1/93–12/97	1.08	12.00	10.60	12.64	2.26	1.85	37.58	0.72
	(0.39)	(0.29)	(0.38)	(0.24)	(0.99)	(0.99)	(0.86)	(0.69)
1/98–12/02	1.87	**19.43**	17.16	**18.66**	4.93	2.85	100.96	1.00
	(0.07)	(0.03)	(0.07)	(0.04)	(0.89)	(0.98)	(0.82)	(0.63)
1/03–12/10	1.73	17.84	16.54	17.45	10.16	7.37	45.42	0.03
	(0.09)	(0.06)	(0.09)	(0.06)	(0.42)	(0.68)	(0.93)	(0.98)
10-year subperiods								
1/73–12/82	0.73	7.85	7.39	7.49	**46.00**	**60.16**	1298.92	**15.01**
	(0.69)	(0.64)	(0.68)	(0.67)	(0.00)	(0.00)	(0.38)	(0.00)
1/83–12/92	1.48	15.29	14.40	14.59	11.00	8.30	112.25	0.54
	(0.15)	(0.12)	(0.15)	(0.14)	(0.36)	(0.60)	(0.87)	(0.77)
1/93–12/02	**2.07**	**20.93**	**19.71**	**20.17**	4.60	3.84	146.37	1.06
	(0.03)	(0.02)	(0.03)	(0.02)	(0.92)	(0.95)	(0.85)	(0.58)

Notes: The results are based on value-weighted returns of 10 portfolios formed on size. The market portfolio is the value-weighted return on all NYSE, AMEX, and NASDAQ stocks and the risk-free rate is the 1-month Treasury bill rate. The numbers in parentheses are the p-values and entries in bold represent cases of significance at the 5% level.

Table 7
Tests of the Fama-French model with 10 size portfolios

Time period	J_1	J_2	J_3	J_4	SD	WD	SX_L	SP_L
38-year period								
1/73–12/10	**2.04**	**20.49**	**20.18**	**20.21**	-	-	3480.59	3.57
	(0.03)	(0.02)	(0.03)	(0.02)			(0.82)	(0.47)
5-year subperiods and an 8-year subperiod								
1/73–12/77	0.39	4.62	4.08	5.37	-	-	475.03	4.01
	(0.94)	(0.91)	(0.94)	(0.86)			(0.75)	(0.41)
1/78–12/82	0.77	8.75	7.73	9.29	-	-	114.95	0.41
	(0.65)	(0.55)	(0.65)	(0.50)			(0.97)	(0.98)
1/83–12/87	1.10	12.24	10.82	10.29	-	-	128.47	2.04
	(0.37)	(0.26)	(0.37)	(0.41)			(0.97)	(0.78)
1/88–12/92	1.18	13.03	11.51	12.32	-	-	70.82	0.77
	(0.32)	(0.22)	(0.32)	(0.26)			(0.98)	(0.94)
1/93–12/97	**2.25**	**22.74**	**20.08**	**33.30**	-	-	143.55	5.58
	(0.03)	(0.01)	(0.02)	(0.00)			(0.93)	(0.24)
1/98–12/02	1.70	**17.92**	15.83	**18.93**	-	-	556.46	0.98
	(0.10)	(0.05)	(0.10)	(0.04)			(0.92)	(0.91)
1/03–12/10	1.65	17.01	15.76	**18.02**	-	-	257.68	0.42
	(0.10)	(0.07)	(0.10)	(0.05)			(0.95)	(0.98)
10-year subperiods								
1/73–12/82	0.19	2.09	1.97	2.14	-	-	255.08	1.86
	(0.99)	(0.99)	(0.99)	(0.99)			(0.96)	(0.77)
1/83–12/92	**1.98**	**20.11**	**18.94**	**19.92**	-	-	33.03	0.26
	(0.04)	(0.02)	(0.04)	(0.03)			(0.99)	(0.99)
1/93–12/02	**2.00**	**20.21**	**19.03**	**21.26**	-	-	79.26	0.03
	(0.04)	(0.02)	(0.03)	(0.01)			(0.99)	(0.99)

Notes: The results are based on value-weighted returns of 10 portfolios formed on size, the returns on three Fama-French factors, and the 1-month Treasury bill rate as the risk-free rate. The numbers in parentheses are the p-values and entries in bold represent cases of significance at the 5% level. The symbol "-" is used whenever a test is not computable.

Table 8
Tests of the CAPM with 100 size and book-to-market portfolios

Time period	J_1	J_2	J_3	J_4	SD	WD	SX_L	SP_L
38-year period								
1/73–12/10	**2.33**	**230.61**	**204.31**	**234.70**	**335.36**	**411.62**	1891.92	**6.44**
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.58)	(0.04)
5-year subperiods and an 8-year subperiod								
1/73–12/77	-	-	-	-	113.46	**173.09**	118.76	1.13
					(0.16)	(0.00)	(0.74)	(0.56)
1/78–12/82	-	-	-	-	**277.46**	**254.71**	474.82	4.79
					(0.00)	(0.00)	(0.51)	(0.09)
1/83–12/87	-	-	-	-	**165.86**	**182.26**	210.42	1.69
					(0.00)	(0.00)	(0.74)	(0.45)
1/88–12/92	-	-	-	-	**150.93**	119.90	10.10	0.03
					(0.00)	(0.08)	(0.99)	(0.98)
1/93–12/97	-	-	-	-	94.00	116.30	149.19	4.65
					(0.65)	(0.12)	(0.69)	(0.09)
1/98–12/02	-	-	-	-	87.86	68.85	181.72	1.94
					(0.80)	(0.99)	(0.73)	(0.37)
1/03–12/10	-	-	-	-	76.41	85.34	79.36	0.11
					(0.96)	(0.85)	(0.89)	(0.93)
10-year subperiods								
1/73–12/82	1.11	**230.87**	**130.83**	111.03	**241.06**	**337.84**	197.98	2.72
	(0.41)	(0.00)	(0.02)	(0.21)	(0.00)	(0.00)	(0.79)	(0.26)
1/83–12/92	1.54	**265.16**	**150.26**	**171.58**	**146.53**	**195.12**	196.67	1.89
	(0.14)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.80)	(0.39)
1/93–12/02	1.76	**279.61**	**158.44**	**182.19**	**161.06**	**188.85**	909.54	**9.44**
	(0.08)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.54)	(0.01)

Notes: The results are based on value-weighted returns of 100 portfolios formed on size and book-to-market. The market portfolio is the value-weighted return on all NYSE, AMEX, and NASDAQ stocks and the risk-free rate is the 1-month Treasury bill rate. The numbers in parentheses are the p-values and entries in bold represent cases of significance at the 5% level. The symbol "-" is used whenever a test is not computable.

Table 9
Tests of the Fama-French model with 100 size and book-to-market portfolios

Time period	J_1	J_2	J_3	J_4	SD	WD	SX_L	SP_L
38-year period								
1/73–12/10	**2.10**	**212.21**	**188.01**	**223.26**	-	-	4936.47	**16.36**
	(0.00)	(0.00)	(0.00)	(0.00)			(0.73)	(0.00)
5-year subperiods and an 8-year subperiod								
1/73–12/77	-	-	-	-	-	-	247.97	5.35
							(0.89)	(0.25)
1/78–12/82	-	-	-	-	-	-	24.67	0.09
							(0.99)	(0.99)
1/83–12/87	-	-	-	-	-	-	180.22	1.98
							(0.95)	(0.79)
1/88–12/92	-	-	-	-	-	-	200.97	1.35
							(0.91)	(0.86)
1/93–12/97	-	-	-	-	-	-	26.93	0.10
							(0.99)	(0.99)
1/98–12/02	-	-	-	-	-	-	101.62	0.42
							(0.99)	(0.98)
1/03–12/10	-	-	-	-	-	-	352.05	0.68
							(0.92)	(0.95)
10-year subperiods								
1/73–12/82	1.15	**234.83**	**133.07**	**175.55**	-	-	163.86	1.58
	(0.38)	(0.00)	(0.02)	(0.00)			(0.98)	(0.82)
1/83–12/92	1.64	**272.18**	**154.23**	**255.23**	-	-	267.53	3.20
	(0.12)	(0.00)	(0.00)	(0.00)			(0.96)	(0.53)
1/93–12/02	**2.17**	**302.54**	**171.44**	**288.46**	-	-	1822.39	**13.79**
	(0.03)	(0.00)	(0.00)	(0.00)			(0.78)	(0.00)

Notes: The results are based on value-weighted returns of 100 portfolios formed on size and book-to-market, the returns on three Fama-French factors, and the 1-month Treasury bill rate as the risk-free rate. The numbers in parentheses are the p-values and entries in bold represent cases of significance at the 5% level. The symbol "-" is used whenever a test is not computable.

Table 10

Tests of the CAPM with 503 individual stocks

Time period	J_1	J_2	J_3	J_4	SD	WD	SX_L	SP_L
38-year period								
1/73–12/10	-	-	-	-	478.26	483.68	696.34	2.03
					(0.78)	(0.72)	(0.79)	(0.36)
5-year subperiods and an 8-year subperiod								
1/73–12/77	-	-	-	-	466.13	436.72	242.51	**6.22**
					(0.87)	(0.98)	(0.59)	(0.04)
1/78–12/82	-	-	-	-	**566.26**	522.36	106.39	0.66
					(0.02)	(0.26)	(0.80)	(0.72)
1/83–12/87	-	-	-	-	492.53	504.74	20.79	0.31
					(0.62)	(0.46)	(0.95)	(0.86)
1/88–12/92	-	-	-	-	496.40	500.58	35.88	0.05
					(0.57)	(0.52)	(0.90)	(0.97)
1/93–12/97	-	-	-	-	418.13	432.12	10.76	0.11
					(0.99)	(0.99)	(0.95)	(0.95)
1/98–12/02	-	-	-	-	429.46	353.28	205.36	4.74
					(0.99)	(1.00)	(0.71)	(0.09)
1/03–12/10	-	-	-	-	497.16	484.88	200.58	1.85
					(0.56)	(0.71)	(0.78)	(0.40)
10-year subperiods								
1/73–12/82	-	-	-	-	**594.33**	**600.21**	201.15	2.55
					(0.00)	(0.00)	(0.78)	(0.28)
1/83–12/92	-	-	-	-	511.40	503.47	28.39	0.24
					(0.38)	(0.48)	(0.96)	(0.88)
1/93–12/02	-	-	-	-	538.13	481.90	199.84	2.72
					(0.13)	(0.74)	(0.81)	(0.26)

Notes: The results are based on returns of 503 individual stocks traded on the NYSE, AMEX, and NASDAQ. The market portfolio is the value-weighted return on all NYSE, AMEX, and NASDAQ stocks and the risk-free rate is the 1-month Treasury bill rate. The numbers in parentheses are the p-values and entries in bold represent cases of significance at the 5% level. The symbol “-” is used whenever a test is not computable.

Table 11

Tests of the Fama-French model with 503 individual stocks

Time period	J_1	J_2	J_3	J_4	SD	WD	SX_L	SP_L
38-year period								
1/73–12/10	-	-	-	-	-	-	2387.42	6.71
							(0.89)	(0.15)
5-year subperiods and an 8-year subperiod								
1/73–12/77	-	-	-	-	-	-	182.15	0.29
							(0.93)	(0.99)
1/78–12/82	-	-	-	-	-	-	463.17	2.49
							(0.81)	(0.67)
1/83–12/87	-	-	-	-	-	-	191.72	1.87
							(0.95)	(0.81)
1/88–12/92	-	-	-	-	-	-	246.14	1.12
							(0.88)	(0.90)
1/93–12/97	-	-	-	-	-	-	90.24	0.52
							(0.96)	(0.97)
1/98–12/02	-	-	-	-	-	-	642.42	2.61
							(0.91)	(0.65)
1/03–12/10	-	-	-	-	-	-	149.46	0.15
							(0.98)	(0.99)
10-year subperiods								
1/73–12/82	-	-	-	-	-	-	483.76	0.87
							(0.89)	(0.93)
1/83–12/92	-	-	-	-	-	-	1165.84	0.77
							(0.68)	(0.94)
1/93–12/02	-	-	-	-	-	-	775.66	0.73
							(0.93)	(0.95)

Notes: The results are based on returns of 503 individual stocks traded on the NYSE, AMEX, and NASDAQ, the returns on three Fama-French factors, and the 1-month Treasury bill rate as the risk-free rate. The numbers in parentheses are the p-values and entries in bold represent cases of significance at the 5% level. The symbol "-" is used whenever a test is not computable.

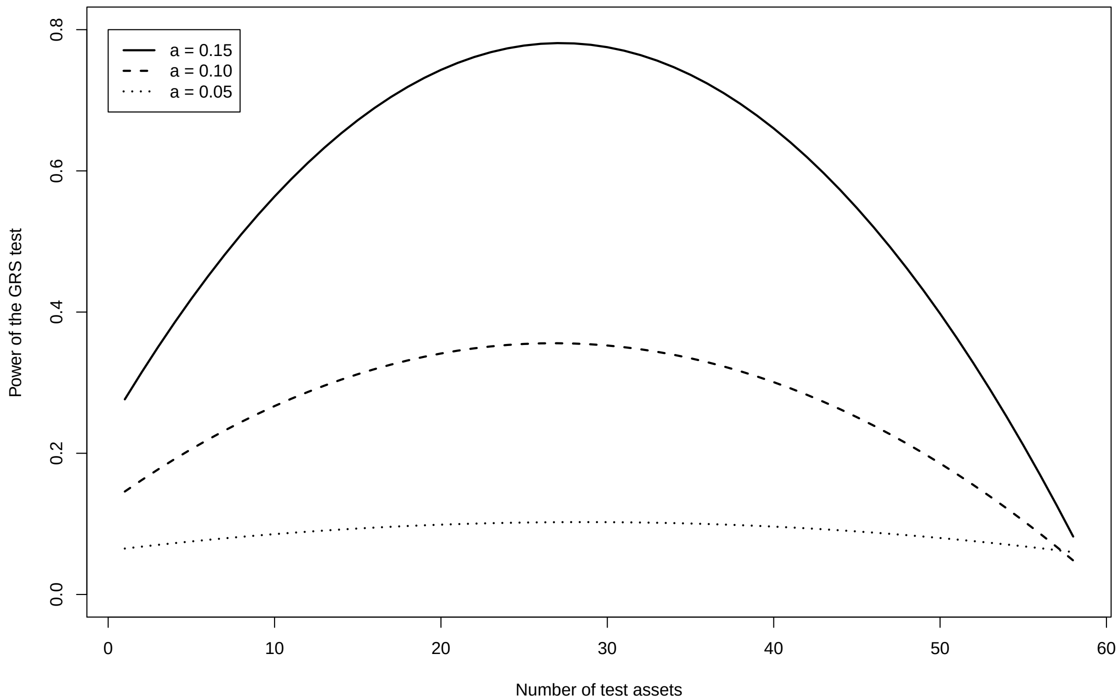
Table 12

Sensitivity of parametric tests to extreme observations

	0%	0.3%	0.5%	0.7%	1.0%
Panel A: CAPM					
J_1	**1.83**	1.54	0.89	0.64	0.59
	(0.05)	(0.12)	(0.54)	(0.77)	(0.81)
J_2	**18.43**	15.55	9.05	6.54	6.07
	(0.05)	(0.11)	(0.52)	(0.76)	(0.80)
J_3	**18.14**	15.31	8.91	6.44	5.97
	(0.05)	(0.12)	(0.54)	(0.77)	(0.81)
J_4	**18.48**	14.94	8.12	6.11	5.82
	(0.04)	(0.13)	(0.61)	(0.80)	(0.83)
Panel B: Fama-French model					
J_1	**2.04**	1.70	0.87	0.59	0.55
	(0.03)	(0.07)	(0.55)	(0.81)	(0.85)
J_2	**20.49**	17.13	8.91	6.05	5.63
	(0.02)	(0.07)	(0.54)	(0.81)	(0.84)
J_3	**20.18**	16.86	8.77	5.96	5.54
	(0.03)	(0.07)	(0.55)	(0.82)	(0.85)
J_4	**20.21**	16.31	8.16	5.63	5.10
	(0.02)	(0.09)	(0.61)	(0.84)	(0.88)

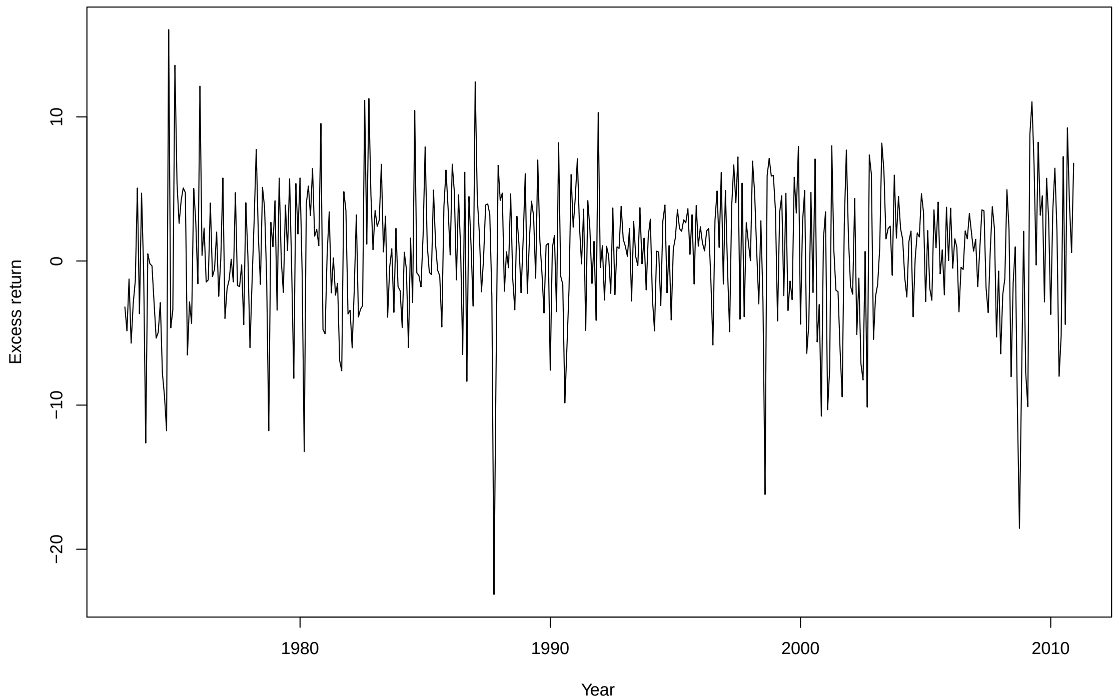
Notes: This table shows the results of the parametric tests with the 10 size portfolios when the returns for the full sample period from January 1973 to December 2010 are winsorized at various small levels. Panels A and B correspond to the 1- and 3-factor models, respectively. The numbers in parenthesis are p-values and bold entries represent cases of significance at the 5% level.

Figure 1



Notes: This figure plots the power of the GRS test as a function of the number of included test assets. The returns are generated from a 1-factor model with normally distributed disturbances. The sample size is $T = 60$ and the number of test assets N ranges from 1 to 58. The test is performed at a nominal 0.05 level. The higher power curves are associated with greater pricing errors.

Figure 2



Notes: This figure plots the monthly excess returns (in percentage) of a value-weighted stock market index of all stocks listed on the NYSE, AMEX, and NASDAQ for the period from January 1973 to December 2010.

Appendix: Table A

List of 503 Stocks Traded on NYSE, AMEX, and NASDAQ.

Company Name	Company Name	Company Name
ALCOA INC	BP PLC	DOW CHEMICAL CO
ABM INDUSTRIES INC	POPULAR INC	DPL INC
ABBOTT LABORATORIES	BRISTOW GROUP INC	DTE ENERGY CO
ACETO CORP	BASSETT FURNITURE INDUSTRIES INC	DUKE ENERGY CORP NEW
ALBERTO CULVER CO NEW	BRUSH ENGINEERED MATERIALS INC	ENNIS INC
ANALOG DEVICES INC	BALDWIN & LYONS INC	ECOLAB INC
ARCHER DANIELS MIDLAND CO	BOWL AMERICA INC	CONSOLIDATED EDISON INC
AUTOMATIC DATA PROCESSING INC	BROWN SHOE CO INC NEW	EMPIRE DISTRICT ELEC CO
ADAMS EXPRESS CO	CONAGRA INC	EDUCATIONAL DEVELOPMENT CORP
AMEREN CORP	CANON INC	EQUIFAX INC
AMERICAN ELECTRIC POWER CO INC	CASTLE A M & CO	EDISON INTERNATIONAL
AGL RESOURCES INC	CASCADE CORP	EASTMAN KODAK CO
AGILYSYS INC	CATERPILLAR INC	EMERSON ELECTRIC CO
AMERICAN INTERNATIONAL GROUP INC	CHUBB CORP	EQT CORP
HADERA PAPER LTD	CINCINNATI BELL INC NEW	ESTERLINE TECHNOLOGIES CORP
ALICO INC	COOPER INDUSTRIES PLC	EATON CORP
ALLETE INC	COMMERCE BANCSHARES INC	ENTERGY CORP NEW
ALEXANDER & BALDWIN INC	CABOT CORP	EATON VANCE CORP
ALASKA AIRGROUP INC	CROWN HOLDINGS INC	EXELON CORP
ALEXANDERS INC	CONSTELLATION ENERGY GROUP INC	FORD MOTOR CO DEL
AMERICAN GREETINGS CORP	CENTRAL SECURITIES CORP	FIRST ACCEPTANCE CORP
APPLIED MATERIALS INC	CH ENERGY GROUP INC	FARMER BROTHERS CO
ADVANCED MICRO DEVICES INC	CHARMING SHOPPES INC	FAMILY DOLLAR STORES INC
AMR CORP DEL	CINCINNATI FINANCIAL CORP	FIRSTENERGY CORP
AMERICAN NATIONAL INS CO	COLGATE PALMOLIVE CO	FRANKLIN ELECTRIC INC
ANAREN INC	CLARCOR INC	FROZEN FOOD EXPRESS INDS INC
AMPCO PITTSBURGH CORP	CLIFFS NATURAL RESOURCES INC	FIRST HORIZON NATIONAL CORP
APACHE CORP	CORELOGIC INC	FOOT LOCKER INC
AIR PRODUCTS & CHEMICALS INC	CLOROX CO	FLOWSERVE CORP
APOGEE ENTERPRISES INC	COMERICA INC	FLEXSTEEL INDUSTRIES INC
ARDEN GROUP INC	COMMERCIAL METALS CO	FMC CORP
ARTS WAY MANUFACTURING INC	CUMMINS INC	FORTUNE BRANDS INC
ASA LIMITED	CMS ENERGY CORP	FERRO CORP
ASHLAND INC NEW	CENTERPOINT ENERGY INC	FURMANITE CORP
ALLEGHENY TECHNOLOGIES	CON WAY INC	FEDERAL SIGNAL CORP
ATRION CORP	CONOCOPHILLIPS	FOREST OIL CORP
ATWOOD OCEANICS INC	CAMPBELL SOUP CO	FRONTIER OIL CORP
CROSS A T CO	CHESAPEAKE UTILITIES CORP	FRONTIER COMMUNICATIONS CORP
AVISTA CORP	CRANE CO	FULLER H B CO
AVON PRODUCTS INC	COURIER CORP	GENERAL AMERICAN INVESTORS INC
AVNET INC	CARPENTER TECHNOLOGY CORP	NICOR INC
AVERY DENNISON CORP	COMPUTER SCIENCES CORP	GANNETT INC
AMERICAN STATES WATER CO	CARLISLE COMPANIES	GENESCO INC
AMERICAN EXPRESS CO	COOPER TIRE & RUBBER CO	GENERAL DYNAMICS CORP
AMREP CORP	CONSOLIDATED TOMOKA LAND CO	GENERAL ELECTRIC CO
ALLEGHENY ENERGY INC	CTS CORP	GRIFFON CORP
BOEING CO	CUBIC CORP	GRACO INC
BANK OF AMERICA CORP	CONTINENTAL MATERIALS CORP	GRAHAM CORP
BAXTER INTERNATIONAL INC	COUSINS PROPERTIES INC	GENERAL MILLS INC
BB & T CORP	CENTRAL VERMONT PUB SVC CORP	G & K SERVICES INC
BRUNSWICK CORP	CVS CAREMARK CORP	GOLDEN ENTERPRISES INC
BRINKS CO	CHEVRON CORP NEW	CORNING INC
BARD C R INC	CURTISS WRIGHT CORP	GATX CORP
BANCROFT FUND LTD	CALIFORNIA WATER SERVICE GROUP	GREAT NORTHERN IRON ORE PPTYS
RIVUS BOND FUND	DIEBOLD INC	GENUINE PARTS CO
BECTON DICKINSON & CO	DONALDSON INC	GP STRATEGIES CORP
BROWN FORMAN CORP	DUCOMMUN INC DE	GOODRICH CORP
BRIGGS & STRATTON CORP	DU PONT E I DE NEMOURS & CO	GERBER SCIENTIFIC INC
BIGLARI HOLDINGS INC	DILLARDS INC	GORMAN RUPP CO
BHP LTD	DEERE & CO	GLAXOSMITHKLINE PLC
BANK OF NEW YORK MELLON CORP	DANAHER CORP	GOODYEAR TIRE & RUBR CO
BLACK HILLS CORP	DIODES INC	GETTY REALTY CORP NEW
BEMIS CO INC	DISNEY WALT CO	GRAINGER W W INC
BRISTOL MYERS SQUIBB CO	DELUXE CORP	GREAT PLAINS ENERGY INC
BANK OF HAWAII CORP	DOVER CORP	GENCORP INC

Table A (contd.)
List of 503 Stocks Traded on NYSE, AMEX, and NASDAQ.

Company Name	Company Name	Company Name
GYRODYNE COMPANY AMERICA INC	LEGGETT & PLATT INC	NATIONAL WESTERN LIFE INS CO
HALLIBURTON COMPANY	LENNAR CORP	NEXEN INC
HASBRO INC	LACLEDE GROUP INC	NEW YORK TIMES CO
HUNTINGTON BANCSHARES INC	LGL GROUP INC	OGE ENERGY CORP
HAWAIIAN ELECTRIC INDUSTRIES INC	LILLY ELI & CO	ONEOK INC NEW
HEICO CORP NEW	LOCKHEED MARTIN CORP	OLIN CORP
HESS CORP	LINCOLN NATIONAL CORP IN	OMNICOM GROUP INC
HITACHI LIMITED	SNYDERS LANCE INC	OWENS & MINOR INC NEW
HECLA MINING CO	ALLIANT ENERGY CORP	OFFICEMAX INC NEW
HONDA MOTOR LTD	LOWES COMPANIES INC	OLD REPUBLIC INTERNATIONAL CORP
HMG COURTLAND PROPERTIES LTD	LOUISIANA PACIFIC CORP	OVERSEAS SHIPHOLDING GROUP INC
HNI CORP	SOUTHWEST AIRLINES CO	OTTER TAIL CORP
HEINZ HJ CO	LUBRIZOL CORP	OXFORD INDUSTRIES INC
HOLLY CORP	LA Z BOY INC	OCCIDENTAL PETROLEUM CORP
HONEYWELL INTERNATIONAL INC	MASCO CORP	PITNEY BOWES INC
HELMERICH & PAYNE INC	MCDONALDS CORP	PEP BOYS MANNY MOE & JACK
HEWLETT PACKARD CO	MOODYS CORP	PANASONIC CORP
BLOCK H & R INC	MARCUS CORP	PACCAR INC
HARBINGER GROUP INC	MEREDITH CORP	PG & E CORP
HORMEL FOODS CORP	MEDTRONIC INC	POTLATCH CORP NEW
HARRIS CORP	MDU RESOURCES GROUP INC	PRESIDENTIAL REALTY CORP NEW
HARSCO CORP	MASSEY ENERGY CO	PUBLIC SERVICE ENTERPRISE GP INC
HOST HOTELS & RESORTS INC	MEDIA GENERAL INC	PETROLEUM & RESOURCES CORP
HERSHEY CO	MGE ENERGY INC	PEPSICO INC
HUBBELL INC	MCGRAW HILL COS INC	PFIZER INC
HUMANA INC	MCCORMICK & CO INC	PROCTER & GAMBLE CO
HEXCEL CORP NEW	MILLER HERMAN INC	PROGRESS ENERGY INC
INTERNATIONAL BUSINESS MACHS COR	MARSH & MCLENNAN COS INC	PROGRESSIVE CORP OH
IDACORP INC	3M CO	PARKER HANNIFIN CORP
INTERNATIONAL FLAVORS & FRAG INC	ALTRIA GROUP INC	PULTE GROUP INC
INTRICON CORP	MOOG INC	PARKER DRILLING CO
IMPERIAL OIL LTD	MOLEX INC	PERKINELMER INC
INTEL CORP	MOTOROLA INC	PARK OHIO HOLDINGS CORP
INTERNATIONAL PAPER CO	MET PRO CORP	PROTECTIVE LIFE CORP
INTERPUBLIC GROUP COS INC	MERCK & CO INC NEW	PLAYBOY ENTERPRISES INC
INGERSOLL RAND PLC	MARATHON OIL CORP	PALL CORP
INTERNATIONAL RECTIFIER CORP	MINE SAFETY APPLIANCES CO	PMFG INC
ITT CORP	MESABI TRUST	PNC FINANCIAL SERVICES GRP INC
J ALEXANDERS CORP	MIDDLESEX WATER CO	PNM RESOURCES INC
JOHNSON CONTROLS INC	M & T BANK CORP	PENTAIR INC
PENNEY JC CO INC	MTS SYSTEMS CORP	PINNACLE WEST CAPITAL CORP
JACOBS ENGINEERING GROUP INC	MANITOWOC CO INC	PIEDMONT NATURAL GAS INC
HANCOCK JOHN INVS TR	MASTEC INC	PEPCO HOLDINGS INC
JOHNSON & JOHNSON	MURPHY OIL CORP	PPG INDUSTRIES INC
JPMORGAN CHASE & CO	MEADWESTVACO CORP	PPL CORP
NORDSTROM INC	MYERS INDUSTRIES INC	PENN VIRGINIA CORP
KELLOGG CO	NOBLE ENERGY INC	PHILLIPS VAN HEUSEN CORP
KANSAS CITY LIFE INS CO	NACCO INDUSTRIES INC	RYDER SYSTEMS INC
KELLY SERVICES INC	NEXTERA ENERGY INC	RITE AID CORP
KEWAUNEE SCIENTIFIC CORP	NEWMONT MINING CORP	RAND CAPITAL CORP
KEYCORP NEW	NEWMARKET CORP	RAVEN INDUSTRIES INC
KULICKE & SOFFA INDS INC	NATIONAL FUEL GAS CO NJ	ROBBINS & MYERS INC
KIMBERLY CLARK CORP	NISOURCE INC	ROWAN COMPANIES INC
KENNAMETAL INC	NEW JERSEY RES	REGIONS FINANCIAL CORP NEW
COCA COLA CO	NL INDUSTRIES INC	STURM RUGER & CO INC
KROGER COMPANY	NOBILITY HOMES INC	RLI CORP
KANSAS CITY SOUTHERN	NORTHROP GRUMMAN CORP	ROLLINS INC
QUAKER CHEMICAL CORP	NATIONAL PRESTO INDS INC	RPM INTERNATIONAL INC
LOEWS CORP	NATIONAL SECURITY GROUP INC	DONNELLEY RR & SONS CO
LANCASTER COLONY CORP	NATIONAL SEMICONDUCTOR CORP	RADIOSHACK CORP
LAWSON PRODUCTS INC	NORTHERN TRUST CORP	RUBY TUESDAY INC
LA BARGE INC	NORTHEAST UTILITIES	RAYTHEON CO
LASERCARD CORPORATION	NUCOR CORP	SPRINT NEXTEL CORP
LANDAUER INC	NV ENERGY INC	SCANA CORP NEW
LEE ENTERPRISES INC	NEWELL RUBBERMAID INC	SERVICE CORP INTERNATIONAL

Table A (contd.)
List of 503 Stocks Traded on NYSE, AMEX, and NASDAQ.

Company Name	Company Name
STEPAN CO	TASTY BAKING CO
STARRETT LS CO	TORO COMPANY
SAFEGUARD SCIENTIFICS INC	TWIN DISC INC
SUPERIOR UNIFORM GROUP INC	TEXAS INDUSTRIES INC
SCHULMAN A INC	TEXAS INSTRUMENTS INC
SHERWIN WILLIAMS CO	TEXTRON INC
SOUTH JERSEY INDS INC	TRI CONTINENTAL CORP
SKYLINE CORP	TYCO INTERNATIONAL LTD SWTZLND
SCHLUMBERGER LTD	URSTADT BIDDLE PROPERTIES INC
SARA LEE CORP	UNITED FIRE & CAS CO
STANDARD MOTOR PRODUCTS INC	UNIFI INC
SEMTECH CORP	UGI CORP NEW
SNAP ON INC	UIL HOLDINGS CORP
SONY CORP	UNISYS CORP
SOUTHERN CO	UMB FINANCIAL CORP
SONOCO PRODUCTS CO	UNILEVER NV
SPARTON CORP	UNION PACIFIC CORP
SPHERIX INC	UNISOURCE ENERGY CORP
SPX CORP	URS CORP NEW
STANDARD REGISTER CO	US BANCORP DEL
SUNLINK HEALTH SYSTEMS INC	UNITED TECHNOLOGIES CORP
STEWART INFORMATION SVCS CORP	UNIVERSAL CORPORATION
STERLING BANCORP	VARIAN MEDICAL SYSTEMS INC
QUESTAR CORP	INVESCO VAN KAMPEN BOND FUND
STATE STREET CORP	VF CORP
SUNOCO INC	VIRCO MFG CORP
SUPERVALU INC	VULCAN MATERIALS CO
STANLEY BLACK & DECKER INC	VALMONT INDUSTRIES INC
SKYWORKS SOLUTIONS INC	VORNADO REALTY TRUST
SOUTHWESTERN ENERGY CO	VALPEY FISHER CORP
SOUTHWEST GAS CORP	VECTREN CORP
STANDEX INTERNATIONAL CORP	VIAD CORP
SENSIENT TECHNOLOGIES CORP	WALGREEN CO
SYNALLOY CORP	WISCONSIN ENERGY CORP
SYSCO CORP	WEYCO GROUP INC
TIDEWATER INC	WELLS FARGO & CO NEW
TECO ENERGY INC	WELLS GARDNER ELECTRS CORP
TECUMSEH PRODUCTS CO	WGL HOLDINGS INC
INTEGRYS ENERGY GROUP INC	WINNEBAGO INDUSTRIES INC
TERADYNE INC	WHIRLPOOL CORP
TELEFLEX INC	WILMINGTON TRUST CORP
TARGET CORP	WILLIAMS COS
TENET HEALTHCARE CORP	WEIS MARKETS INC
TJX COMPANIES INC NEW	WAL MART STORES INC
TIMKEN COMPANY	WORTHINGTON INDUSTRIES INC
TOYOTA MOTOR CORP	WASHINGTON POST CO
THERMO FISHER SCIENTIFIC INC	WESTAR ENERGY INC
THOMAS & BETTS CORP	WSI INDUSTRIES INC
TENNANT CO	AQUA AMERICA INC
TODD SHIPYARDS CORP	WOLVERINE WORLD WIDE INC
TUTOR PERINI CORP	WEYERHAEUSER CO
TEXAS PACIFIC LAND TRUST	XCEL ENERGY INC
TOOTSIE ROLL INDS INC	EXXON MOBIL CORP
TRINITY INDUSTRIES INC	XEROX CORP
TRAVELERS COMPANIES INC	YRC WORLDWIDE INC
TYSON FOODS INC	ZIONS BANCORP
TESORO CORP	