Linkages across Sovereign Debt Markets*

Cristina Arellano — Federal Reserve Bank of Minneapolis

Yan Bai — University of Rochester

PRELIMINARY

October 19, 2012

Abstract

This paper studies linkages across sovereign debt markets when countries choose to default and renegotiate their debt. Countries are linked to one another by borrowing and renegotiating from common lenders who have concave payoffs. Countries are strategic large players and understand that their choices for loans, defaults, and renegotiations affect debt prices and recoveries as well as future choices for other countries. Defaults and non-renegotiations in one country lower lenders' payoffs which increase the cost of funds for other countries and lead to more defaults and non-renegotiations. The simultaneity in defaults induces correlation in interest rates across countries. In the model, renegotiations with one country can have positive spillovers to other countries by reducing their risk of default. The model can rationalize some of the recent events in Europe.

---

*The views expressed herein are those of the authors and not necessarily those of the Federal Reserve Bank of Minneapolis or the Federal Reserve System. E-mails: arellano.cristina@gmail.com; yanbai06@gmail.com

1 Introduction

Sovereign debt crises tend to happen in bunches. During the 1980s almost all Latin American countries defaulted and subsequently renegotiated their sovereign debt. During the recent European debt crises, Greece, Ireland, Italy, Portugal, and Spain are struggling with their sovereign debt and Greece already initiated the restructuring process. During these crises, interest rates increase simultaneously for multiple countries. Figure 1 illustrates the co-movement in country interest rates in Europe since 2009. The discussions about orderly renegotiations and bailouts to Greece are often cited as essential to prevent further contagion in the region. Despite sovereign debt crises happening in tandem, theoretical work on sovereign default has often been restricted to study countries in isolation.

This paper studies linkages across sovereign debt markets when countries choose to default and renegotiate their debt. Countries are linked to one another by borrowing and renegotiating from common lenders who have concave payoffs. Countries are strategic large players and understand that their choices for loans, defaults, and renegotiations affect debt prices and recoveries as well as future choices for other countries. Defaults and non-renegotiations in one country lower lenders' payoffs which increase the cost of funds for other countries and lead to more defaults and non-renegotiations. The simultaneity in defaults induces correlation in interest rates across countries. The model also provides a theory where renegotiations with one country can have positive spillovers to other countries by reducing the risk of default.

The model economy consists of two symmetric countries who borrow from a continuum of competitive lenders. The borrowing countries can default on their debt. Default entails costs in terms of access to financial markets and direct output costs. Countries in default choose to renegotiate the debt and pay the recovery. After renegotiation, sanctions are lifted for defaulters and they regain access to financial markets. Lenders are competitive and their kernel depends on the choices of loans, defaults and renegotiations of the borrowing countries. The recovery of defaulted debt is determined through Nash bargaining during renegotiation.

Countries are linked because the prices of debt and the recovery are determined jointly and depend on the choices of default, borrowing, and renegotiation of all countries. Importantly, borrowing countries understand that their choices impact all debt prices and recoveries and consider these effects when optimizing. The price of debt reflects the risk-adjusted compensation for the loss lenders face in case of default and incorporates three main elements: the lenders' cost of funds, the risk-adjusted default probability, and the risk-adjusted recovery rate. When one country decides to default (or to not renegotiate) the price of debt for the other country worsens because lenders' marginal valuation rises which increases the cost of funds as well as the risk-adjusted default and recovery rate.

Renegotiations also respond to other countries choices and states. Recovery is determined such that the marginal cost for the borrower from paying the recovery equals the marginal benefit of lenders from receiving the recovery. When other countries default (or fail to renegotiate), the marginal benefit for lenders increases which raises the recovery. Higher recovery lowers the likelihood of renegotiation. However, renegotiations can also deter other countries from defaulting. Hence, when other countries are close to default, recoveries are actually lower because the marginal benefit to the lender also includes preventing the costs associated with a second default.

We solve the model numerically and to focus on our mechanisms, we study the case of uncorrelated income shocks across countries. Our model predicts that country interest rates co-move as in the data because defaults happen together. We find that the conditional spread of the home country is about 3% higher when the foreign country has spreads above its median relative to below its median. Our model also predicts that recoveries in the home country are linked to the credit conditions of the foreign country. Recovery rates are about 8% lower at home when the foreign country has spreads above its median relative to below its median. When the foreign country defaults, the recovery rates jump about 20 percentage points. Such increase in recovery rates arising from a foreign default reduces the incidence of renegotiation to almost zero.

The model in this paper builds on the work of Aguiar and Gopinath (2006) and Arellano (2008), who model equilibrium default with incomplete markets, as in the seminal paper on sovereign debt by Eaton and Gersovitz (1981). These papers analyze the case of risk neutral lenders, abstract from recovery, and focus on default experiences of single countries. Borri and Verdelhan (2009), Presno and Puozo (2011) and Lizarazo (2010a) study the case of risk averse lenders. They show that risk aversion allows the model to generate spreads larger than default probabilities, which is a feature of the data. Borri and Verdelhan also show empirically that a common factor drives a substantial portion of the variation observed. Lizarazo (2010b) studies contagion in a model similar to ours where multiple borrowers trade with a risk averse lenders. Her model can generate co-movement in spreads across borrowing countries however she abstracts from any debt renegotiation and strategic interactions. Yue (2010), D'Erasmo (2011), and Benjamin and Wright (2009) study debt renegotiation in a model with risk neutral lenders. They find that debt renegotiation allows the model to match better the default frequencies and the debt to output ratios.

2 Model

Consider an economy where two symmetric countries, Home and Foreign, borrow from a continuum of foreign lenders. Debt contracts are unenforceable and countries can choose to default on their debt whenever they want. Countries that default get a bad credit standing, are excluded from borrowing, and suffer a direct output cost. Countries in default can renegotiate their debt. During renegotiation the defaulting country and the lender bargain over the recovery. After renegotiation is complete, countries regain good credit standing.

We consider an economy where each borrowing country i for $i = \{1, 2\}$ receives a stochastic endowment y_i each period. Let $y = \{y_1, y_2\}$ be the vector of endowments for each country in a period. These shocks follow a Markov process with transition matrix $\pi_y(y', y)$. We assume lenders face no additional shocks.

The timing of events in this economy is as follows. Each country starts each period with a level of debt b_i and a credit standing h_i. Countries with good credit standing have $h_i = 0$ and decide whether to default or repay their debts with the indicator function d_i. If they repay they set $d_i = 0$, maintain their good standing for next period and choose new debt choices b_i'. If they default $d_i = 1$, they don't pay their debt and start next period with bad credit standing $h_i = 1$. Countries with bad credit standing decide whether to renegotiate or not with indicator function z_i. If country i renegotiates $z_i = 1$ and if it doesn't renegotiate $z_i = 0$. If they renegotiate, then they bargain with the lenders over the recovery ϕ_i to be paid. Countries that renegotiate start the next period with good credit standing and zero debt. If they don't renegotiate, then the maintain their bad credit standing. The endogenous aggregate states consist of the vector of debt holdings $b = \{b_1, b_2\}$ and their credit standing $h = \{h_1, h_2\}$. The economy wide state s incorporates the endogenous and exogenous states: $s = \{b, h, y\}$.

2.1 Borrowing Countries

The representative household in each borrowing country i receives utility from consumption c_{it} and has preferences given by

$$E \sum_{t=0}^{\infty} \beta^t u(c_{it}), \tag{1}$$

where $0 < \beta < 1$ is the time discount factor and $u(\cdot)$ is increasing and concave.

The government of the borrowing country is benevolent and its objective is to maximize the utility of households. While in good credit standing, the government trades one period discount bonds with foreign lenders. The government also decides whether to repay or default on its debt. While in default, the government is in bad credit standing and it decides whether to renegotiate or not. If the government renegotiates then it bargain with the lenders over the recovery to be repaid. The government rebates back to households all the proceedings from its credit operations in a lump sum fashion.

The price for bonds is a function $q_i(o_i, o_{-i}, s)$ that depends on the choices of country i to default and new loans $o_i = \{d_i, b_i'\}$, the choices of the other country,

o_{-i}, which include default, new loans, and renegotiation, $o_{-i} = \{d_{-i}, b'_{-i}, z_{-i}\}$ and all aggregate states s. The bond price compensates the lender for the risk adjusted loss in case of default and it depends on the choices of country 1 and 2 and the aggregate states, (o_1, o_2, s), because the lender's kernel, as well as default, renegotiation, and recovery depend on all these variables. Below we specify how the bond price functions are determined.

When the government is in good credit standing $h_i = 0$ and chooses to repay its debts $d_i = 0$, the resource constraint for borrowing country i is the following

$$c_i = y_i - b_i + q_i(o_i, o_{-i}, s)b'_i \tag{2}$$

If the government with $h_i = 0$ defaults by setting $d_i = 1$, the government doesn't pay its outstanding debt b_i, it is excluded from trading international bonds, and it incurs output costs y^d_{it}. Consumption equal output during these periods.

$$c_i = y^d_i. \tag{3}$$

Following Arellano (2008) we assume that borrowers lose a fraction λ of output if output is above a threshold: $y^d_t = \begin{cases} y_t & \text{if } y_t \leq (1-\lambda)\bar{y} \\ (1-\lambda)\bar{y} & \text{if } y_t > (1-\lambda)\bar{y} \end{cases}$, where $\bar{y}$ is the mean level of output.

Default changes the credit standing of the country to $h_i = 1$. Every period a government with $h_i = 1$ chooses to renegotiate its debts or not. The indicator function $z_i = 0$ if it doesn't renegotiate and $z_i = 1$ if it renegotiates. In periods when the government doesn't renegotiate, consumption equals output $c_{it} = y^d_{it}$.

If the government renegotiates, then it bargains with the lenders over the recovery $\phi_i(o_{-i}, s)$. The recovery is the amount that the government pays back lenders to regain its good credit standing. The recovery depends on the other country's choices o'_{-i} as well as the aggregate state s because the bargaining outcome depend on these. When $z_i = 1$ the resource constraint for the economy is

$$c_{it} = y_{it} - \phi_i(o_{-i}, s) \tag{4}$$

After renegotiating, the country starts next period with zero debt due $b_i' = 0$ and good credit standing $h_i = 0$.

We represent the borrowing country's problem as a recursive dynamic programming problem. Let $v_i(s)$ be the value function of the borrowing country i that has good credit standing $h_i = 0$. Borrowing country i decides whether to default or not after endowment shocks are realized

$$v_i(s) = \max_{d_i=\{0,1\}} \{d_i v_i^{nd}(s) + (1 - d_i) v_i^d(s)\} \tag{5}$$

where $v_i^{nd}(s)$ is the value to the country conditional on not defaulting and $v_i^d(s)$ is the value of default. $d_i(s) = 1$ if the country chooses default and zero otherwise.

If the country repays the debt $h_i' = 0$, and the country chooses optimal consumption and savings

$$v_i^{nd}(s) = \max_{c_i, b_i'} \{u(c_i) + \beta \sum_{y'} \pi(y', y) v_i(s')\} \tag{6}$$

subject to (2), taking as given the other country's choices $o_{-i} = \{d_{-i}, b_{-i}', z_{-i}, c_{-i}\}$ with

$$o_{-i} = O(s) \tag{7}$$

The aggregate state the following period $s' = \{b', h', y'\}$ is completely determined by both countries choices.

If the country defaults, it is does not pay the debt, cannot borrow, consumes output y^d and $h_i' = 1$

$$v_i^d(s) = \{u(y_i^d) + \beta \sum_{y'} \pi(y', y) w_i(s')\} \tag{8}$$

subject to (7).

After default the country has bad credit standing $h_i = 1$ and maintains the level of the defaulted debt b_i. Let w_i be the value function associated with being in bad credit standing. Once $h_i = 1$ the country decides whether to renegotiate or not

$$w_i(s) = \max_{z_i=\{0,1\}} \{z_i w_i^r(s) + (1 - z_i) w_i^{nr}(s)\}. \tag{9}$$

$z_i(s) = 1$ if the country chooses renegotiate and zero otherwise. The length of renegotiation is endogenous and equals to the time that the borrower takes to choose to renegotiate. Let $w_i^r(s)$ be the value associated with renegotiation and $w_i^{nr}(s)$ the value of not renegotiating the debt.

If the country renegotiates, then and the country repays the recovery $\phi_i(o_{-i}, s)$. Recovery is a function that will be derived below. This function depends on the current states s as well as the choices of the other country o'_{-i} because the payoffs from renegotiation depend on these variables .Renegotiation allows the country to avoid the output cost as well as access to international borrowing. Following renegotiation the country starts with zero debt $b'_i = 0$, and with good credit standing $h'_i = 0$

$$w_i^r(s) = \{u(y_i - \phi_i(o_{-i}, s)) + \beta \sum_{y'} \pi(y', y) v_i(s')\} \tag{10}$$

subject to (7) where s' incorporates that country's i state for debt, $b'_i = 0$, and credit standing $h'_i = 0$. If the country does not renegotiate, then it remains excluded from financial markets and consuming y_i^d with $h'_i = 1$

$$w_i^{nr}(s) = \{u(y_i^d) + \beta \sum_{y'} \pi(y', y) w_i(s')\}$$

subject to (7) where s' incorporates that country's i state for debt, $b'_i = b_i$, and credit standing $h'_i = 1$. Note that $w_i^{nr}(s) = v_i^d(s)$.

This problem delivers value functions $v_i(s)$ and $w_i(s)$ and decision rules for debt $B'_i(s)$, default $D_i(s)$, and repayment $Z_i(s)$ which we label with capital letters.

2.2 Lenders

There is continuum of foreign lenders whose objective is to maximize the present discounted value of dividends d_{Lt}

$$E\sum_{t=0}^{\infty}\delta^t g(d_{Lt}), \tag{11}$$

where $0 < \delta < 1$ is the lender's time discount factor and $g(\cdot)$ is an increasing and concave function. We assume that $\beta < \delta$.

Every period lenders receives a constant payoff from the net operations of other loans $r_L L$ and deposits $r_d D$ which we summarize by $y_L = r_L L - r_d D$. Lenders trade bonds with the two borrowing countries. We assume that lenders honor all financial contracts. To make it explicit that lenders are competitive, we will denote the lenders' holding of the countries' bonds by ℓ_i (which in equilibrium equal b_i) and the equilibrium prices of bonds and recovery rates with $Q_i(s)$ and $\Phi_i(s)$.

Lenders choose optimal dividends d_L and loans to the borrowing countries ℓ_1', and ℓ_2', taking as given the prices of bonds $Q_i(s)$ and the recovery rate of bonds $\Phi_i(s)$. The value function for the lender is given by

$$v^L(\ell_1,\ell_2,s) = \max_{c_L,\left\{\ell_i' \text{ if } h_i=0 \text{ and } d_i(s)=0\right\}_{i=1,2}} \left\{g(d_L) + \delta\sum_{y'}\pi(y',y)v^L(\ell_1',\ell_2',s')\right\} \tag{12}$$

They maximize their value subject to their budget constraint that depends on the credit standing of each borrowing country and whether they default or renegotiate

$$d_L(s) = y_L + \sum_{i=1,2}(1-h_i)\left[1-d_i(s)\right](\ell_i - Q_i(s)\ell_i') + \sum_{i=1,2}h_i z_i(s)\frac{\Phi_i(s)\ell_i}{b_i} \tag{13}$$

$$b_i^R(s)\frac{\ell_i}{b_i}$$

the evolution of the endogenous states in states where they don't trade with each country

$$\ell_i^{'} = \left\{ \begin{array}{l} \ell_i' \text{ if } \ (h_i = 0 \text{ and } d_i(s) = 1) \ \text{ or } \ (h_i = 1 \text{ and } z_i(s) = 0) \\ 0 \ \text{ if } (h_i = 1 \text{ and } z_i(s) = 1) \end{array} \right\} \text{ for } i = \{1, 2\} \tag{14}$$

and the law motion of the state

$$s' = S(s) \tag{15}$$

When lenders are trading with the country i the first order conditions for ℓ_i' is

$$g'(c_L)Q_i(s) = \sum_{s'} \pi(y', y) v_{\ell_i}^L(\ell_1', \ell_2', s'; (h_i' = 0))$$

The envelope conditions for this problem depend on the state s. It is useful to separate these envelope conditions in states where the borrowing country is in good or bad credit standing

$$v_{\ell_i}^L(\ell_1, \ell_2, s; (h_i = 0)) = [1 - d_i(s)]\, g'(c_L) + d_i(s)\delta \sum_{y'} \pi(y', y) v_{\ell_i}^L(\ell_1, \ell_2', s'; (h_i' = 1)) \tag{16}$$

$$v_{\ell_i}^L(\ell_1, \ell_2, s; (h_i = 1)) = z_i(s) g'(c_L) \frac{\Phi_i(s)}{b_i} + [1 - z_i(s)]\delta \sum_{y'} \pi(y', y) v_{\ell_i}^L(\ell_1, \ell_2', s'; (h_i' = 1)) \tag{17}$$

Combining the first order condition and envelope conditions we get that the bond price $Q_i(s)$ satisfies

$$g'(d_L)Q_i(s) = \delta \sum_{y'} \pi(y', y) \left\{ g'(d_L')\, [1 - d_i(s')] + d_i(s')\delta \sum_{y'} \pi(y'', y') v_{\ell_i}^L(\ell_1'', \ell_2'', s''; (h_i'' = 1)) \right\} \tag{18}$$

where d_L is defined by (13) and $v_{\ell_i}^L(\ell_1'', \ell_2'', s''; (h_i'' = 1))$ is defined recursively by (17).

The price of bonds can be written in a more intuitive manner by defining lenders' pricing kernel $m(s', s)$ as the marginal rate of substitution for the representative lender across periods

$$m(s', s) = \frac{\delta \pi(y', y) g'(s')}{g'(s)}. \tag{19}$$

and defining the present value of recovery $\zeta_i(s)$ as $\zeta_i(s) = \delta \sum_{y'} \pi(y', y) \frac{g'(s')}{g(s)} v^L_{\ell_i}(\ell_1, \ell_2, s; (h_i = 1))$.

The price of bonds $Q_i(s)$ can now be compactly written as

$$Q_i(s) = \sum_{s'} [m(s', s)(1 - d_i(s')) + d_i(s') m(s', s) \zeta_i(s'))] = \sum_{s'} [m(s', s)(1 - d_i(s')(1 - \zeta_i(s'))] \tag{20}$$

where the present value of recovery is defined recursively by

$$\zeta_i(s) = \sum_{s'} \left[m(s', s) z_i(s') \frac{\Phi_i(s')}{b'_i} + (1 - z_i(s')) \zeta_i(s') \right] \tag{21}$$

The bond price (20) and the value of recovery (21) are easily interpretable. The bond price contains two elements: the payoff in non-default states $d_i(s') = 0$ and in default states $d_i(s') = 1$. The lender discounts cash flows by the pricing kernel $m(s', s)$ and hence states are weighted by $m(s', s)$. For every unit of loan ℓ'_i, the lender gets 1 unit in the non-default states and the value of recovery $\zeta_i(s')$ in default states.

The recovery value is the expected payoff from defaulted debt the following period. It also contains two pieces. If the country renegotiates next period $z_i(s') = 1$, and the value for recovery for every unit of loan is $\frac{\Phi_i(s')}{b'_i}$. If the country doesn't renegotiate, $z_i(s') = 0$ and the present value of recovery is given by the discounted value of future recovery given by $\zeta_i(s')$. These future recovery values are weighted by the pricing kernel $m(s', s)$ which implies that recovery values are weighted more heavily for states s' that feature a higher pricing kernel.

The bond price compensates the lender for any covariation between its kernel and the bond payoffs. If default happens in states when $m(s', s)$ is high, the price contains a positive risk premia for the default event. Moreover, if the value of recovery is low when $m(s', s)$ is high, the price also contains positive risk premia for the covariation of recovery.

2.3 Renegotiation protocol

In this two country environment, debt renegotiation happens for one country or two countries simultaneously. The renegotiation protocol we consider is one where every period all the parties renegotiating bargain over the recoveries together.

Let's first consider the case when only one defaulter country chooses to renegotiate. When a country sets $z_i(s) = 1$ it bargains immediately with a committee of all lenders on the recovery that will be repaid. The renegotiation process follows Nash bargaining. Consider a candidate recovery value $\hat{\phi}_i$. The payoff for lenders from renegotiating and receiving recovery $\hat{\phi}_i$ equals the value of the representative lender evaluated at the aggregate debt values, $V^L(s; \hat{\phi}_i) \equiv v^L(b_1, b_2, s; \hat{\phi}_i)$. The payoff for the borrower from renegotiation is $w_i^r(s; \hat{\phi}_i)$ for this candidate value of recovery $\hat{\phi}_i$.If the two parties do not reach an agreement, the defaulter country is in permanent financial autarky and $y_i = y_i^d$ and gets a threat value equal to

$$v_i^{aut}(y_i) = \{u(y_i^d) + \beta \sum_{y_i'} \pi_y(y_i', y_i) v_i^{aut}(y')\}$$

and all lenders receives zero of the debt and will be permanently in financial autarky with the defaulter country. Lenders will still have access to financial trading with the other non-defaulting country. Let $v^{L,fail}(b_{-i}, s_{-i})$ be the value to all lenders from trading with only one borrowing country, which is specified below.

The recovery ϕ_i maximizes the weighted surplus for the defaulter country and the lenders. The bargaining power for the borrower is θ and that for lenders is $(1 - \theta)$. Recovery ϕ_i solves

$$\max_{\phi_i \epsilon [0,1]} \left[w_i^r(s; \phi_i) - v_i^{aut}(y_i)\right]^\theta \left[V^L(s; \phi_i) - V^{L,fail}(s_{-i})\right]^{1-\theta}$$

subject to both parties receiving a non-negative surplus from the renegotiation:

$w^r(s;\phi) - v^{aut}(y_i) \geq 0$, and $V^L(s;\phi_i) - V^{L,fail}(s_{-i}) \geq 0$, and law of motion $s' = S(s)$.

Now consider the case when two defaulter countries choose to renegotiate by setting $z_i(s) = z_{-i}(s) = 1$. Here the two countries bargain together with the committee of all lenders. If the parties do not reach an agreement, all parties remain in financial autarky thereafter. The recoveries $\{\phi_i, \phi_{-i}\}$ solve

$$\max_{\{\phi_i,\phi_{-i}\}\epsilon[0,1]} \{\left[w_i^r(s;\phi_i) - v_i^{aut}(y_i)\right]\left[w_{-i}^r(s;\phi_{-i}) - v_{-i}^{aut}(y_{-i})\right]\}^{\theta}\left[V^L(s;\phi_i,\phi_{-i}) - V^{L,aut}\right]^{1-\theta}$$

subject to all parties receiving a non-negative surplus from the renegotiationand law of motion $s' = S(s)$. The outside option for the lenders in this case is $V^{L,aut} = \frac{g(y_L)}{1-\delta}$.

When only one country is renegotiating, the outside option for lenders $V^{L,fail}(s_{-i})$ is the value to the lenders of trading only with country $-i$ when debt is ℓ_{-i}, $v^{L,fail}(\ell_{-i}, s_{-i})$ This value is similar to the problem specified in (12) except that it only trades with one country

$$v^{L,fail}(\ell_{-i}, s_{-i}) = \max_{d_L,\ell'_{-i}}\{u(d_L) + \delta\sum_{y'}\pi(y',y)v^{L,fail}(\ell'_{-i}, s'_{-i})\} \tag{22}$$

subject to its budget constraint

$$d_L = y_L + (1-h_{-i})\left[1 - d_{-i}(s)\right]\left(\ell_{-i} - Q_{-i}^{fail}(s)\ell'_{-i}\right) + h_{-i}z_{-i}(s)\Phi_{-i}^{fail}(s)\ell_{-i}$$

the evolution of the endogenous states similar to equation (14) and a law of motion of aggregate states for the case that country $-i$ is dealing alone with lenders $s'_{-i} = S^{fail}(s_{-i})$.

The problems for country $-i$ in the case when it trades alone with the lenders are similar to (5) and (9) with three main differences. First, its aggregate states are only $s_{-i} = \{b_{-i}, h_{-i}, y_{-i}\}$. Second, the price $q_{-i}^{fail}(o_{-i}, s_{-i})$ and recovery $\phi_{-i}^{fail}(o_{-i}, s_{-i})$ depend only on its own states and its own choices. Third, the evolution of the other countries choices and states are irrelevant because the other country is assumed to be in autarky forever. The decision rules for this problem are labeled as $B_i^{fail\prime}(s_i)$

for borrowing $D_i^{fail}(s_i)$ for default, and $Z_i^{fail}(s_i)$ for repayment. These decisions in turn determine the evolution of the aggregate state $s'_{-i} = S^{fail}(s_{-i})$.

2.4 Functions for Bond Prices and Recovery

In our model, competitive lenders trade bonds with the two borrowing countries, who are big players. Borrowing countries internalize the effects their choices of default, borrowing and renegotiation $\{d_1, b'_1, z_1, d_2, b'_2, z_2\}$ have on bond prices and engage in Cournot competition with one another.

Consider first the case when both countries are in good credit standing, $h_1 = h_2 = 0$ and they are choosing default and new loans. Countries understand that for every choice $o = \{d_1, b'_1, d_2, b'_2\}$, bond prices $\{q_1, q_2\}$ have to satisfy the demand system determined by lenders' first order conditions:

$$\begin{aligned} q_1 &= \sum_{s'} m(s', s; q_1, q_2, o) \left[1 - D_1(s')(1 - \zeta_1(s'))\right] \qquad (23) \\ q_2 &= \sum_{s'} m(s^{'}, s; q_1, q_2, o) \left[1 - D_2(s')(1 - \zeta_2(s'))\right] \end{aligned}$$

where the state tomorrow s' depends countries' choices $o = \{d_1, b'_1, d_2, b'_2\}$, i.e. the states for debt b' and credit standing h' in s' depend on today's choice of debt and default, and where the lender's kernel $m(s', s; q_1, q_2, o)$ is itself a function of the equilibrium prices, countries choices, and current and future states.

To make explicit that borrowing countries internalize the demand system, it is informative to expand lender's kernel $m(s', s; q_1, q_2, o)$ by considering a candidate choice $\hat{o} = \{\hat{d}_1, \hat{b}'_1, \hat{d}_2, \hat{b}'_2\}$ which induce a state tomorrow $\hat{s}'$. The lenders kernel equals to the marginal utility of dividends tomorrow relative to today. Dividends today and tomorrow are given by

$$d_L(s, q_1, q_2, \hat{o}) = y_L + \sum_{i=1,2} \left[1 - \hat{d}_i\right] \left(b_i - q_i \hat{b}'_i\right),$$

$$
\begin{aligned}
d_L'(\hat{s}') &= y_L + \sum_{i=1,2} (1-\hat{h}_i')\left[1-D_i(\hat{s}')\right]\left(\hat{b}_i' - Q_i(\hat{s}')B_i''(\hat{s}')\right) \\
&\quad + \sum_{i=1,2} \hat{h}_i Z_i(\hat{s}')\Phi_i(\hat{s}')
\end{aligned}
$$

Borrowing countries understand that their choices of debt and default directly affect d_L and that d_L depends on prices $\{q_1, q_2\}$. Borrowing countries also understand that d_L' is directly affected by their choice $\hat{b}_i'$ and $\hat{h}_i$ (which is mapped from $\hat{d}_i$). Moreover, borrowing countries take future functions for borrowing $B_i''(\hat{s}')$, renegotiation $Z_i(\hat{s}')$, default $D_i(\hat{s}')$, prices $Q_i(\hat{s}')$, and recovery $\Phi_i(\hat{s}')$ as given but they understand the their choices today affect the state tomorrow and hence the associated values for these variables.

We now define the bond price functions $q_1(o_1', o_2', s)$ and $q_2(o_2', o_1', s)$.

Definition 1 *When $h_1 = h_2 = 0$, the bond price functions $q_1(o_1', o_2', s)$ and $q_2(o_2', o_1', s)$ solve (23).*

Consider now the case when country i is in good credit standing and country $-i$ is in bad credit standing. Here country i is choosing to default and new loans and country $-i$ is choosing whether to renegotiate or not. Similarly in this case, for every choice $o = \{d_i, b_i', z_{-i}\}$ the bond price and recovery $\{q_i, \phi_{-i}\}$solve

$$
\begin{aligned}
q_i &= \sum_{s'} m(s', s; q_i, \phi_{-i}, o)(1 - D_i(s'))(1 - \zeta_i(s')) \qquad (24) \\
\frac{\theta u'(y_{-i} - \phi_{-i})}{\left[w_{-i}^r(s;\phi_{-i}) - v^{aut}(y)\right]} &= \frac{(1-\theta)g'(d_L(s, q_i, \phi_{-i}, o))}{\left[V^L(s; q_i, \phi_{-i}, o) - V^{L,fail}(s_i)\right]}
\end{aligned}
$$

where the lender's consumption and values are evaluated for every choice o and corresponding price and recovery, q_i, ϕ_{-i}.

Definition 2 *When $h_i = 0$ and $h_{-i} = 1$, the bond price and recovery functions $q_i(o_i', o_{-i}', s)$ and $\phi_{-i}(o_{-i}', o_i', s)$ solve (24)*

Finally, when both countries are in bad credit standing, $h_1 = h_2 = 1$ they are choosing to renegotiate or not and recoveries depend on every choice $o = \{z_i, z_{-i}\}$. If $z_i = 1$ and $z_{-i} = 0$ then ϕ_i solves

$$\frac{\theta u'(y_i - \phi_i)}{[w_i^r(s;\phi_i) - v_i^{aut}(y)]} = \frac{(1-\theta)g'(d_L(\phi_i, o))}{\left[V^L(s;\phi_i,\phi_{-i},o) - V^{L,fail}(s_{-i})\right]} \tag{25}$$

If $z_i = z_{-i} = 1$ then recoveries $\{\phi_i, \phi_{-i}\}$ solve

$$\begin{aligned}\frac{\theta u'(y_i - \phi_i)}{[w_i^r(s;\phi_i) - v_i^{aut}(y)]} &= \frac{(1-\theta)g'(d_L(\phi_i,\phi_{-i}, o))}{\left[V^L(s;\phi_i,\phi_{-i},o) - V^{L,aut}\right]} \\ \frac{\theta u'(y_{-i} - \phi_{-i})}{\left[w_{-i}^r(s;\phi_{-i}) - v_{-i}^{aut}(y)\right]} &= \frac{(1-\theta)g'(d_L(\phi_i,\phi_{-i}, o))}{\left[V^L(s;\phi_i,\phi_{-i},o) - V^{L,aut}\right]}\end{aligned} \tag{26}$$

Note that in our environment the recovery value ϕ_{-i} does not affect the marginal cost of repaying ϕ_i for country i, i.e. the left hand side of equation (26).

Definition 3 *When $h_1 = h_2 = 1$, the recovery functions $\phi_1(o_1', o_2', s)$ and $\phi_2(o_2', o_1', s)$ solve (25) and (26).*

To complete the definitions of prices and recoveries in our model, we need to describe the conditions for the case when renegotiation fails with only one country $-i$ and the lenders trade alone with one country i. When the country is in good credit and it chooses to repay its debt, the price function $q_i^{fail}(o_i, s_i)$ solves

$$q_i^{fail} = \sum_{s'} m^{fail}(s_i', s_i; q_i^{fail}, o_i)\left[1 - D_i^{fail}(s_i')(1 - \zeta_i^{fail}(s_i'))\right]$$

where the decision rules of the country and the lender's kernel are those corresponding to the problem when the country is trading alone with the lender.

When the country is in bad credit and it chooses to renegotiate, the recovery function $\phi_i^{fail}(o_i, s_i)$ solves

$$\frac{\theta u'(y_i - \phi_i^{fail})}{\left[w^r(s;\phi_i^{fail}) - v^{aut}(y)\right]} = \frac{(1-\theta)g'(d_L^{fail}(\phi_i^{fail}, o_i))}{\left[V^{L,fail}(s;\phi_i^{fail}, o_i) - V^{L,aut}\right]}$$

We assume that if renegotiation is not successful in this case both the country and the lenders will remain in permanent financial autarky.

2.5 Equilibrium

We focus on recursive Markov equilibria in which all decision rules are functions only of the state variable $s = \{b_1, b_2, h_1, h_2, y_1, y_2\}$. A recursive equilibrium for this economy consists on (i) the policy functions $O_i(s)$ for every borrowing country i which include policies for debt choices $B_i'(s)$, default and renegotiation decisions $D_i(s)$ and $Z_i(s)$, and for consumption $C_i(s)$, the borrowing countries' value functions $v_i(s)$, $v_i^{nd}(s)$, $v_i^d(s)$, $w_i(s)$, $w_i^{nr}(s)$, and $w_i^r(s)$, (ii) the lender policy functions for dividends $d_L(s)$, debt choices $L_1'(s)$, and $L_2'(s)$, and value function $v^L(\ell_1, \ell_2, s)$ (iii) the recovery functions $\phi_i(o, s)$ for each country, (iv) the bond price functions for every country $q_i(o', s)$, (v) the equilibrium price of debt $Q_i(s)$ and recovery rate $\Phi_i(s)$, (vi) the evolution of the aggregate state $S(s)$, and (vii) the lenders value in the case of renegotiation failure $v^{L,fail}(\ell_i, s)$, such that:

1. Taking as given the bond price function $q_i(o, s)$, the recovery function $\phi_i(o, s)$, and the policy functions for the other country $O_{-i}(s)$, the policy and value functions functions for every country i, $O_i(s) = [B_i'(s), D_i(s),\ Z_i(s)]$, $C_i(s)$, $v_i(s)$, $v_i^{nd}(s)$, $v_i^d(s)$, $w_i(s)$, $w_i^{nr}(s)$, and $w_i^r(s)$ satisfy its optimization problems.

2. Taking as given the bond price $Q_i(s)$, the recovery $\Phi_i(s)$, and the evolution of the aggregate states $S'(s)$, the policy functions and value functions for the lenders $L_1'(s)$, $L_2'(s)$, $c_L(s)$, $v^L(s)$ and $v^{L,1}(s)$ satisfy their optimization problem.

3. The bond price functions $q_1(o, s)$ and $q_2(o, s)$ and recovery functions $\phi_1(o, s)$ and $\phi_2(o, s)$ satisfy equations (23), (24), and (26).

4. The price of debt $Q_i(s)$ clears the bond market for every i

$$\ell_i = b_i \tag{27}$$

5. The recovery $\Phi_i(s)$ exhausts all the recovered funds

$$\phi_i(o,s) = \frac{\Phi_i(s)\ell_i}{b_i}$$

6. The goods market clears

$$c_1 + c_2 + c_L = y_1 + y_2 + y_L \tag{28}$$

7. The law of motion for the evolution countries' choices (7) and the aggregate endogenous states (15) are consistent with the individual decision rules and shocks

8. The lenders value in the case of renegotiation failure $v^{L,fail}(\ell_i, s)$ solves problem (22).

2.6 Strategic decisions

We analyze the strategic interactions resulting from the borrowing countries' problems. For illustration, we assume that the bond prices functions q_1 and q_2 as well as the value function v are differentiable. Consider the case both countries are in good credit standing. Countries know that their choices for borrowing and default affect their current price and their neighbors'. We can write the optimal first order condition for optimal borrowing for country 1 as

$$u'(c_1)\left[q_1(o,s) + \frac{\partial q_1(o,s)}{\partial b_1'}b_1'\right] = -\beta\sum_{s'}\pi(s',s)v_{1,b_1}(s') \tag{29}$$

The left hand side of this expression is the marginal benefit for borrowing an additional unit of debt. As in standard default models, the marginal cost of debt is not only the debt price but also the derivative of the price with respect to their borrowing choices. In our model however, as in standard oligopoly type models, the prices of two countries and are connected through the lender's demand system (23). To understand how the price changes with the choice of the debt, it is useful to write the bond price as a product between the risk-neutral expected repayment times expected kernel

$$q_i = r_i^Q(s_o', y)E[m(s_o', s, q_1, q_2, o)|y] \text{ for } i = 1, 2$$

where expected repayment under the risk neutral measure is
$r_i^Q(s_o', y) = \sum_{s'} \frac{m(s_o', s; q_1, q_2, o)}{E[m(s_o', s; q_1, \phi_2, o)[}[1 - d_1(s_o')(1 - \zeta_1(s_o'))]$.

The demand system implies that the ratio of the prices equal the ratio of the risk neutral repayment, which implies that country 1, for example sees its price determination as

$$q_1 = r_1^Q(s_o', y)E\left[m\left(s_o', s, q_1, q_1\frac{r_2^Q(s_o', y)}{r_1^Q(s_o', y)}, o\right)|y\right]$$

The derivative of the price with respect to debt can be found by differentiating the above expression

$$\frac{\partial q_1(o, s)}{\partial b_1'} = \frac{Em\frac{dr_1^Q(s_o')}{b_1'} + \left[Em_{b_1'} + q_1 Em_{q2}\left(\frac{r_2^Q(s_o', y)}{r_1^Q(s_o', y)}\right)_{b_1'}\right]r_1^Q(s_o', y)}{1 - \left[r_1^Q(s_o', y)Em_{q_1} + r_2^Q(s_o', y)Em_{q2}\right]}$$

where we have suppressed the arguments inside the kernel for short-hand. As the above expression shows, the derivative of the price with respect to debt is a fairly complicated object. The price of debt changes with debt because debt reduces risk neutral repayment $dr_1^Q(s_o')/b_1'$, affects the kernel $Em_{b_1'}$ directly and changes the q_2 through the demand system $\left(\frac{r_2^Q(s_o', y)}{r_1^Q(s_o', y)}\right)_{b_1'}$ which affects the kernel Em_{q2}. The price q_1 also affects the kernel directly and indirectly through its impact on q_2. Borrowing countries weight all these forces when choosing their optimal borrowing.

The right hand side of expression in (29) is the marginal cost from having debt the following period. After using the envelope theorem, the marginal value is

$$-v_{1,b_1}(s) = (1 - d_1(s))u'(c)\left[1 - \frac{\partial q_1}{\partial b_1}b_1' - \right]$$

$$-(1-d_1(s))(1-d_2(s))\left[u'(c)\frac{\partial q_1}{\partial B_2'}\frac{\partial B_2'}{\partial b_1}+\beta E v_{1,b_2'}(s)\frac{\partial B_2'}{\partial b_1}\right]$$

In our model, having an extra unit of debt is costly not only because of the cost of paying it in the states in which the borrower does not default, but also because debt affects directly the price of new debt today. Having an extra unit of debt also affects the other country's choices of debt and default which in turn affect the price of debt today through the derivative $\frac{\partial q_1}{\partial B_2'}\frac{\partial B_2'}{\partial b_1}$. Finally, having more debt today also affects all future choices of debt and default of the other country which is encoded in the derivative $v_{1,b_2'}(s)\frac{\partial B_2'}{\partial b_1}$. These future choices of the other country matter because they affect the future debt prices of this country.

3 Joint defaults and renegotiations

In this section we develop a simple two period example to illustrate the linkages in debt markets for the two countries. The two main points we aim to deliver with these examples are that countries default together, and that renegotiations and debt repayments occur together.

Consider a two period version of our model with no uncertainty where countries have identical endowment path y and y'. We first analyze the case when in period 1 both countries have good credit standing to analyze the interdependence of their default decisions. We then analyze the case when the home country has good credit standing and the foreign country has bad credit standing. Here we analyze the interdependence of renegotiations with debt repayment. The formal derivations for these examples are in Appendix A.

3.1 Default dependency

Consider the case when in period 1 the two countries have good credit standing and are deciding whether to repay their current debt or default on it. We denote the foreign country variables with (*). If countries repay their debt, then they choose to borrow. In period 2, countries either repay their debts if they borrowed in period

1, or they pay the recovery ϕ' if they defaulted in period 1. In this example without uncertainty, default does not happen in equilibrium in period 2 because it would be perfectly foreseen and the price of such a loan would be zero. However, default incentives in period 2 limit the borrowing possibilities for period 1. In particular, in period 1 countries effectively face a borrowing limit which is the maximum repayment that countries would be willing to make and equals the default penalty in period 2 $\bar{b} = y' - y^d$, where $y^d < y'$ is the income in case of default. In this example, we consider parameters for which it is optimal for countries to borrow to the limit in period 1. Hence, we abstract from the interdependence across countries in the borrowing decisions to focus on the interdependence in their default decisions.

In period 1, the home and foreign countries repay if the value of repayment is greater than the value of default

$$u(y - b + q(b^*, d^*)\bar{b}) + \beta u(y' - \bar{b}) \geq u(y^d) + \beta u(y' - \phi'(d^*)) \tag{30}$$

$$u(y - b^* + q^*(b, d)\bar{b}) + \beta u(y' - \bar{b}) \geq u(y^d) + \beta u(y' - \phi^{*\prime}(d)) \tag{31}$$

Consider the problem for the home country. It is immediate that default is more likely for the home country when debt b is high, the price q is low and the recovery tomorrow ϕ' is low. The default decisions of the two countries are linked because bond prices today and recoveries tomorrow depend on the decisions of both countries through the lenders' problem.

It is useful to derive the home country's default best response conditional on the foreign country's default decision. The foreign default decision affects home country's future recovery and current debt price. A foreign default today (weakly) decreases the home recovery ϕ' tomorrow because the surplus from renegotiating the debt is lower when both countries are renegotiating relative to when only the home country renegotiates. Intuitively, if the foreign country repays and borrows $\bar{b}$ in period 1, he will repay $\bar{b}$ in period 2. This payment gives the lender a high outside option during renegotiation with the home country which in turn increases the equilibrium ϕ'. This force implies that a foreign default $d^* = 1$ increases the right hand side of equation (30) and thus increases the incentive to default for the home country.

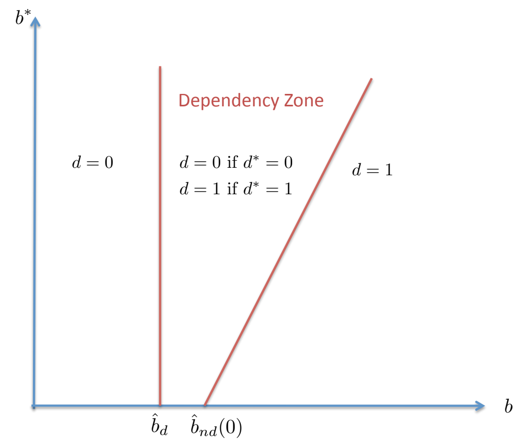
The second force to consider is the effect of the foreign default on the price q. This effect depends on the net capital flows that lenders forego with the foreign default, $b^* - q^*\bar{b}$. The larger the foregone capital flows, the more unfavorable the bond price becomes for the home country with a foreign default. It is easy to show that these capital flows are increasing with b^* and the effect of a foreign default is increasingly detrimental for q the higher b^*.

The home country will default when its current debt b is sufficiently high such that the value of default is greater than the value of repayment. It is useful to consider two cutoffs $\hat{b}_{nd}(b^*)$ and $\hat{b}_d(b^*)$ above which home country will default conditional on the foreign country repaying $\hat{b}_{nd}(b^*; d^* = 0)$ and conditional on the foreign country defaulting $\hat{b}_d(b^*; d^* = 1)$.

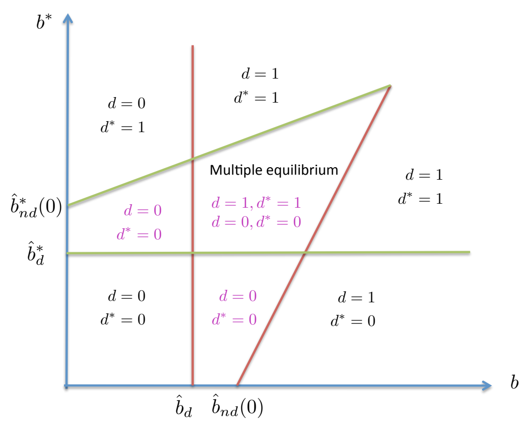
The effects of a foreign default on the price q and the future recovery ϕ' imply that $\hat{b}_{nd}(b^*)$ is increasing in b^* and that $\hat{b}_d(b^*)$ is independent of b^*. The ranking of $\hat{b}_{nd}(b^* = 0)$ and $\hat{b}_d(b^* = 0)$ depend on the details of the utility of lenders but if the effect of default on recovery is big enough $\hat{b}_{nd}(b^* = 0) > \hat{b}_d(b^* = 0)$.

To summarize this analysis, Figure (1a) plots the best response of the home country as a function of its own debt level b and the foreign country's debt level b^* conditional on the foreign decision to default or repay. For sufficiently low (or high) levels b, the home country always repays (or defaults) independently on the foreign decision. For intermediate levels of b, however, the home country repays only if the foreign country repay. We label this region, the *dependency zone.* By symmetry, the best response of the foreign country is identical to that of the home country, such that for intermediate levels of debt, the foreign country only repays if the home country repays.

Figure (1b) plots the equilibrium taking into consideration the best responses for the two countries. The figure shows that in the dependency zones, both countries have joint repayments and joint defaults. Consider the dependency zone for country 1. When the foreign debt is low enough, the foreign repayment guarantees a home repayment. For high foreign debt, a foreign default guarantees a home default. When the foreign debt is in the intermediate region, our model features multiple equilibrium: either both countries default or both countries repay. Nevertheless,



(a) Best Response of Home Country



(b) Equilibrium

Figure 1: Correlated Default

even in this region the equilibrium features either joint defaults or joint repayments.

3.2 Renegotiation dependency

We now analyze the case when in period 1 the home country has bad credit standing and is deciding whether to renegotiate today or wait to renegotiate tomorrow, while the foreign country has good credit and is deciding to repay its debt or default. As in the previous example, we assume that if the foreign country repays, it will borrow to the limit $\bar{b} = y' - y^d$ and focus on the default and renegotiation decisions.

The home country renegotiates today if the value of renegotiating today is greater than the value of delaying renegotiation until tomorrow

$$u(y - \phi(b^*, d^*) + \beta u(y') \geq u(y^d) + \beta u(y' - \phi'(d^*)) \tag{32}$$

As above, the foreign country repays if the value of repayment is greater than the value of defaulting

$$u(y^* - b^* + q^*(z)\bar{b}) + \beta u(y^{*\prime} - \bar{b}) \geq u(y^d) + \beta u(y^{*\prime} - \phi^{*\prime}(z))$$

Home renegotiation is more likely if the recovery ϕ today is low and the recovery ϕ' is high tomorrow. Likewise, foreign default is more likely when debt b^* is high, the price q^* is low and the recovery tomorrow $\phi^{*\prime}$ is low. The renegotiation and default decisions are linked because prices ϕ and q^* are connected through lenders' problem. Recall that ϕ and q^* satisfy these conditions

$$\begin{aligned} & \frac{\theta u'(y-\phi)}{[u(y-\phi)+\beta u(y')-u(y^d)(1+\beta)]} \\ = \; & \frac{(1-\theta)g'(y_L+\phi+(1-d^*)(b^*-q^*\bar{b}))}{\left[g(y_L+\phi+(1-d^*)(b^*-q^*\bar{b}))+\beta g(y_L+\bar{b})\right]-\left[g(y_L+(1-\tilde{d}^*)(b^*-\tilde{q}^*\bar{b}))+\beta g(y_L+(1-\tilde{d}^*)\bar{b}+\tilde{\phi}^{*'})\right]} \end{aligned} \tag{33}$$

$$q^* = \frac{\delta g'(y_L + \bar{b}(d^* = 0))}{g'(y_L + z\phi + b^* - q^*\bar{b})}$$

where $\tilde{d}^*$ is the default decision for the foreign country in case renegotiation fails and it deals alone with the lender. The equilibrium prices in this off-equilibrium state are $\tilde{q}^*$ and $\tilde{\phi}^*$.

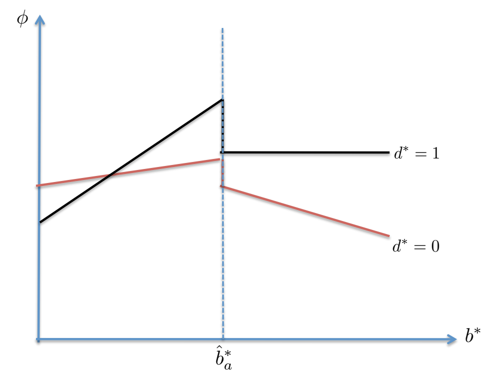
A foreign default affects the renegotiation decision at home because it influences the recoveries ϕ and ϕ'. Let's first consider the effect of a foreign default on ϕ which varies with the level of foreign debt b^*. Note that b^* affects ϕ by its impact on three pieces that make up the marginal surplus for lenders in (33): the lenders' marginal utility today, the lender's value in the equilibrium, and the lender's threat value in case of renegotiation failure. An important cutoff of b^* is the level of debt above which the foreign country defaults when it deals alone with the lender in case of renegotiation failure. We label such cutoff as $\hat{b}^*_a$: $\tilde{d}^* = 0$ if $b^* \leq \hat{b}^*_a$ and $\tilde{d}^* = 1$ otherwise.

In order to derive the home country's renegotiation best response z we need to establish how ϕ varies with b^* conditional on the foreign country's default decision. If $d^* = 1$, $\phi(b^*; d^* = 1)$ is increasing in b^* for $b^* \leq \hat{b}^*_a$, it then jumps down and remains flat for $b^* > \hat{b}^*_a$. Recovery ϕ increases with b^* because the threat value is increasing for $b^* \leq \hat{b}^*_a$. After the cutoff $\hat{b}^*_a$, ϕ is independent of b^*. The jump down at $\hat{b}^*_a$ happens because of discrete decline in the lenders' threat value at the default point.

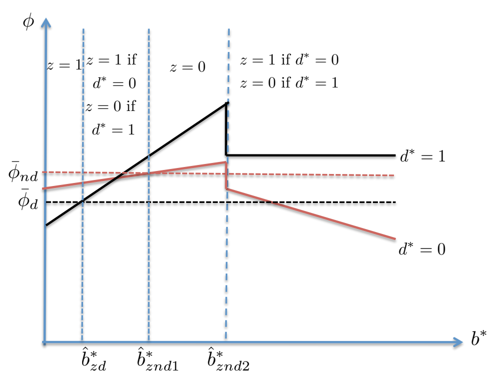
If $d^* = 0$, one can show that $\phi(b^*; d^* = 0)$ is also increasing in b^* for $b^* \leq \hat{b}^*_a$. Here, the positive relation between b^* and ϕ for $b^* \leq \hat{b}^*_a$ is weaker because when $d^* = 0$, b^* also affects the lenders' marginal utility and equilibrium value. Such effects creates a force towards a negative relation beween b^* and ϕ. For levels of debt $b^* > \hat{b}^*_a$, however, the effect on the threat value vanishes, and recovery ϕ decreases with b^*. Figure (2a) plots the recovery $\phi(b^*)$ conditional on d^*.

An interesting implication of the recovery function is that, when $d^* = 0$, high enough foreign debt ($b^* > \hat{b}^*_a$), leads to a lower recovery at home than a low level of foreign debt ($b^* \leq \hat{b}^*_a$). The intuition for this result is that when foreign debt is high enough, renegotiation at home has positive spillovers for the foreign country because it prevents a foreign default. In such states, the surplus for the lenders from renegotiation are larger and hence recovery in equilibrium is lower.

The home decision to renegotiate or not conditional on the foreign default decision



(a) Optimal Recovery ϕ conditional on d^*



(b) Best Response of z on d^*

Figure 2: Optimal Recovery and Renegotiation

depend no only on the value for ϕ but also on the value of ϕ'. A low ϕ and a high ϕ' lead to renegotiation $z = 1$. As before, a foreign default decreases the home recovery tomorrow $\phi'(d^* = 1) \leq \phi'(d^* = 0)$. This ranking implies, that the cutoff value of recovery today ϕ below which the home country renegotiates is lower when the foreign country defaults than when the foreign country repays. Let $\bar{\phi}_d$ and $\bar{\phi}_{nd}$ be the cutoff values for ϕ below which the home country renegotiates if the foreign country is defaulting or not defaulting, where $\bar{\phi}_d \leq \bar{\phi}_{nd}$. The non-monotonic shape of recovery ϕ imply that in general there are two cutoffs for b^* conditional on the foreign default decision where the recovery function $\phi(b^*; d^*)$ intersects with $\bar{\phi}_d$ or $\bar{\phi}_{nd}$. We label these cutoffs with $\{\hat{b}_{zd1}, \hat{b}_{zd2}\}$ and $\{\hat{b}_{znd1}, \hat{b}_{znd2}\}$, with $\hat{b}_{j1} < \hat{b}_{j2}$. In what follows, we assume that $\hat{b}_{zd} \equiv \hat{b}_{zd1} = \hat{b}_{zd2}$ and that $\hat{b}_{zd1} < \hat{b}_{znd1}$.[1]

Using this analysis, the right panel of Figure (2b) plots the home's renegotiation best response conditional on the foreign default decision. As shown in the figure when b^* is sufficiently low ($b^* < \hat{b}_{zd}$), the home country always renegotiates and $z = 1$. When $\hat{b}_{znd1} < b^* < \hat{b}_{znd2}$ the home country never renegotiates and $z = 0$. However, for intermediate levels of debt $\hat{b}_{zd} < b^* < \hat{b}_{znd1}$ and high levels of debt $\hat{b}_{znd1} < b^*$, the home country renegotiates only if the foreign country repays. Hence, the *dependency zone* for the home country is $\hat{b}_{zd} < b^* < \hat{b}_{znd1} \cup \hat{b}_{znd1} < b^*$.

Let's now consider the foreign country's default best response as a function of its own debt b^* conditional on the home country renegotiation decision. There will be two cutoffs depending on the home country's renegotiation choices $\{\hat{b}^*_{z0}, \hat{b}^*_{z1}\}$ below which the foreign country defaults. It is easy to show that $\hat{b}^*_{z0} < \hat{b}^*_{z1}$. The default region is larger when the home country renegotiation because the bond price is tighter for the foreign country if the home country does not renegotiate. This best response function implies that when $b^* < \hat{b}^*_{z0}$, the foreign country for sure repays its debt and when $\hat{b}^*_{z1} < b^*$ it for sure defaults. However, in the foreign *dependency zone* the foreign country repays, only if the foreign country renegotiates.

With the characterization of the best responses for the two countries, we are now

[1] We choose parameters that highlight the interaction across countries. For other parameter configurations, however, the ranking and existence of these cutoffs could be different. For example, if $\bar{\phi}_{nd}$ and $\bar{\phi}_d$ are sufficiently low, there are not intersections across these functions and the home country always renegotiate.

ready to discuss the equilibrium. We assume the following ranking across the cutoffs: $\hat{b}_{zd} < \hat{b}_{z0} < \hat{b}_{znd1} < \hat{b}_{znd2} < \hat{b}_{z1}$. The equilibrium default and renegotiation choices are shown in Figure (3). We plot b^* on both x-axis and y-axis. The vertical lines correspond to the renegotiation cutoffs for the home country. The horizontal lines correspond to the default cutoffs for the foreign country. The equilibrium can be read in the 45 degree line.

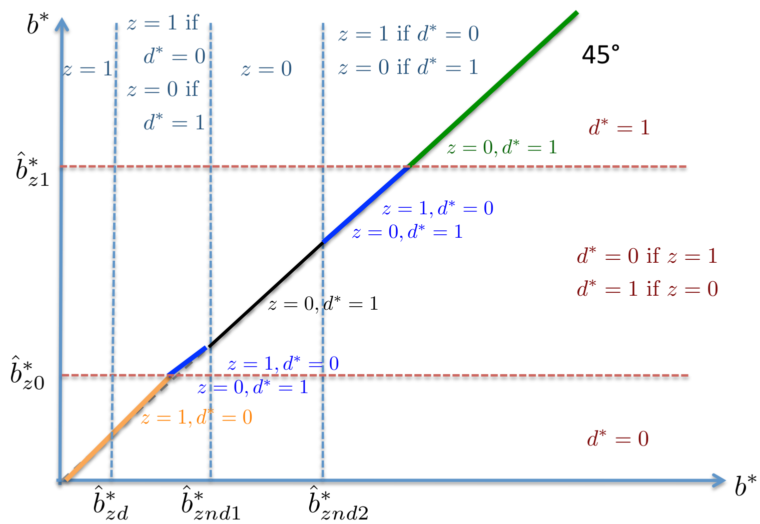


Figure 3: Equilibrium: Default and Renegotiation Choice

In equilibrium, countries always repay and renegotiate together. Specifically, when debt is small, $b^* < \hat{b}_{z0}$ the home country renegotiates and the foreign country repays ($z = 1, d^* = 0$). In a subset of this region $\hat{b}_{zd} < b^* < \hat{b}_{z0}$, which is part of the dependency zone for the home country, the home country renegotiates only because the foreign country repays. When $\hat{b}_{z0} < b^* < \hat{b}_{znd1}$ the model feature multiple equilibrium. Either the home country renegotiates and the foreign defaults, or vice-versa. For the intermediate range of debt $\hat{b}_{znd1} < b^* < \hat{b}_{znd2}$, the home country does not renegotiate, and such decision leads to a foreign default. When debt is larger $\hat{b}_{znd2} < b^* < \hat{b}_{z1}$ the model again features multiple equilibrium: either the home country renegotiates and the foreign defaults, or vice versa. For large debt levels $\hat{b}_{z1} < b^*$ the foreign default leads to a non-renegotiation at home.

In this section we have analyzed the forces behind the linkages across defaults and renegotiations across countries. Joint defaults and non-renegotiations in response to foreign defaults are natural implications when countries share a common lender. In this example, however we have abstracted from any debt dynamics. In practice the level of debt is endogenous to countries decisions and their choices will interact with defaults and renegotiations. In the following section, we analyze our general dynamic model.

4 Quantitative Analysis

We solve the model numerically and analyze the linkages across the two borrowing countries in country interest rates, defaults and renegotiations. The general model predicts that borrowing rates and default probabilities are higher for one of the borrowing countries when the other country has a high risk of default. The model also predicts recoveries are lower and renegotiations faster for one of the borrowing countries when the other country has a high risk of default.

4.1 Parameterization

The utility function for the borrowing countries is $u(c) = \dfrac{c^{1-\sigma}}{1-\sigma}$. We set the risk aversion coefficients σ to 2, which is a common value used in real business cycle studies. The utility of lenders is exponential given by $g(d_L) = -e^{-d_L}$. The length of a period is one year. The stochastic process for output for the borrowing countries is independent from one another and follow a log-normal AR(1) process, $\log(y_{t+1}) = \rho \log(y_t) + \varepsilon_{t+1}$ with $E[\varepsilon^2] = \eta^2$. We discretize the shocks into a nine-state Markov chain using a quadrature-based procedure (Tauchen and Hussey, 1991). We use annual series of linearly detrended GDP for Greece for 1960–2011 taken from the World Development Indicators to calibrate the volatility and persistence of output.

Note that the parameter controlling the lenders' constant resources y_L does not affect any result because $g(d_L)$ is CARA.

We calibrate four parameters: the lender and borrowers' discount rates δ and β , the default cost λ, and the borrower's bargaining parameter θ, to match three moments: the average German yield 4%, the average spread of 2.2% found in Greek euro bonds, and average recovery of 60%, which is the mean recovery rate reported in Cruces and Trebesch (2012) across 182 sovereign restructures for 1970-2010. Table 1 summarizes the parameter values.

Table 1: Parameters Values

	Value	Target
Countries' risk aversion	$\sigma = 2$	Standard value
Stochastic structure for shocks	$\rho_y = 0.93$, $\eta_y = 0.02$	Greece output
Calibrated parameters		
Output cost after default	$\lambda = 0.012$	Spread 2.2%
Borrowers' discount factor	$\beta = 0.82$	Risk free rate 4%
Lender's discount factor	$\delta = 0.96$	Recovery rate 60%
Bargaining power	$\theta = 0.70$	

4.2 Results

We simulate the model and report statistics summarizing debt markets for one of the borrowing countries. Due to symmetry, statistics for the second country are equal. The equilibrium we consider is the one that maximizes the joint values for the two borrowing countries.[2]

Borrowing countries do not interact directly with one another or have common shocks. Linkages across borrowing countries are encoded in the bond price and recovery schedules $q(o, s)$ and $\phi(o, s)$ which themselves depend on other states and choices of the borrowing countries.

[2]As in the simple example, for a certain region of the parameter space our model features multiple equilibrium. Our equilibrium selection consists on choosing the one that maximize $v_1 + v_2$. Appendix B discusses in detail the numerical algorithm.

Table (2) reports statistics default probabilities, spreads, recovery rates, length of renegotiation and risk free rates for country 1, home, conditional on the debt market conditions for country 2, foreign.

The risk free rate is defined as the inverse of the lender's kernel $r^f = 1/Em - 1$. Spreads are defined as the difference between the country interest rate and the risk free rate $spr_i = 1/q_i - r_f - 1$. Recovery rates are defined as the recovery relative to the debt in default ϕ/b_i.

Consider first the overall average statistics of the model. The calibration of the model is successful. The model predicts an average spread 2.2%, risk free rate of 4.1% and recovery rate of about 60%. The model also predicts that the average default probability is 2.0% and that renegotiations happen with 99% probability.

Table 2: Debt Linkages

	Overall	Foreign Good Credit			Foreign Bad Credit	
Home	Average	spr>p50	spr ≤p50	Default	Renego	Not renego
Spread	2.2	3.5	0.8	0.9	1.7	–
Default prob.	2.0	0.2	0.5	85.1	0.0	100
Recovery	60	61	68	97	59	–
Renegotiation prob.	99	100	100	0.0	100	0
Risk free rate	4.1	4.3	4.2	3.8	-1.2	4.2

To illustrate debt market linkages across borrowing countries, Table (2) reports debt market statistics for the home country conditional on the debt market conditions for the foreign country. Consider first the case when the foreign country is in good credit standing, where we partition the states into those where the foreign rate is above and below its median.

We find that spreads are correlated: the average spread for home equals 3.5% when the spread for the foreign country is above its median while it is 0.8% when the foreign spread is below the median. The main reason for this positive correlation of spreads is that both countries default together. The average default probability for home is less than 1% when the foreign country is in good credit and repaying its debt, i.e. in states when spreads are defined. When the foreign defaults, however, the probability of default at home jumps to 85%. This strong incidence of joint defaults implies that high foreign spreads forecast a foreign default and a home default, which in turn is priced in home spreads. Recoveries for home also vary with the conditions in the foreign country. When foreign spreads are low, home recovery equals 61% and when foreign spreads are high recoveries are higher and equal to 68%. When the foreign country defaults, however, recoveries are highest and equal to 97%. Nevertheless, these high recoveries during defaults are almost never observed in equilibrium because the home country renegotiates during these states with less than 0.01 probability.

The joint incidence of defaults happen in our model because a foreign default makes the home debt price schedule tighter as illustrated in the simple example of section 2. In the dependent zone, the default best response for the home country depends on the foreign country default decision. In the general model this force continues to operate, the home country prefers to default when the foreign defaults because the bond price schedule is very unfavorable. To illustrate such effect, consider the off-equilibrium home spread when the foreign defaults if the home would repay. This off-equilibrium spread is 10% even though the home is already reducing debt in response to the tight schedule. In face of such unfavorable debt terms, the home country prefers to default.

The lower home recoveries when the foreign spread is high occur because renegotiations prevent a foreign default. The simple example of section 2 shows that home recoveries are decreasing in the level of foreign debt when foreign debt are above a threshold because as debt increases the surplus from renegotiation is larger. Such a relation is also present in the general model: when foreign debt is above a cutoff, the recovery at home decreases with foreign debt for the states in which the foreign repays its debt. Such higher debt levels are associated with higher spreads and hence our model predicts low recoveries in times of high spreads. When foreign debt becomes, however, the foreign starts to default. Here recoveries jump up, because renegotiation lose the benefits of prevent the foreign default. Quantitatively though, the differential recovery is small relative to the recent Greek experience. To provide some context, note that during the recent European debt crises many countries have high spreads and the recovery after the Greek default was 35%. Such a recovery is much lower than the historical average of 60%. Our model produces the right co-movement, but the magnitudes are smaller.

Now consider the case when the foreign country is in bad credit standing because of a previous default. The implications for the home country depend dramatically on whether the foreign country renegotiates its debt or not. When the foreign country renegotiate, default probabilities, spreads, and recoveries are low. Hence the home country defaults rarely (less than 1% of the time) and renegotiates 100% of the time. The risk free rate during times of renegotiations are quite low because the lender receives the recovery which lowers its marginal value.

If the foreign does not renegotiate, then the home country always defaults if it's in good credit and never renegotiates if it's in bad credit. These results imply that defaults and lack of renegotiations happen together as in the example of section 2. If the foreign country fails to renegotiates, the bond price function for the home country are so unfavorable that default is preferred.

In Table 2 we showed debt market linkages across countries conditional on different credit standing states for the foreign and home countries. In our model, however, the probability of the different credit standing states is endogenous to the decisions of countries. In Table 3 we report the probability in the limiting distribution of both countries being in a good and bad credit standing as well as the frequency of events with joint or single default and renegotiations. The table shows that about 95.5% of the time countries are borrowing and repaying their debt. Most of the defaults are joint with both countries defaulting together 1.65% of the time. Countries do default alone at times. About 0.29% of the time single defaults occur when the foreign country has good credit. Most of the renegotiations in our model also happen together. Countries are renegotiating together about 1.67% of the time. Single renegotiations also occur, but infrequently. About 0.29% of the time the home country is renegotiating when the foreign country is repaying its debt.

Table 3: Probabilities in the limiting distribution

		Foreign Good Credit		Foreign Bad Credit	
		Repay	Default	Reneg.	Not reneg.
Home Good Credit	Repay	95.51	0.29	0.29	0.0
	Default	0.29	1.65	0.0	0.004
Home Bad Credit	Reneg.	0.29	0.0	1.67	0.0
	Not reneg.	0.0	0.004	0.0	0.0

4.3 Crises events

We now illustrate how joint debt crises manifest themselves over time. To this end, we plot time series of various debt market measures for the home country when the foreign country is in a debt crisis. We define a debt crisis in the foreign country as the periods where the foreign country has a spread above its 75 percentile. To construct the times series in Figures 4 we start with the limiting distribution over states $s = \{b, h, y\}$ and follow each point in the distribution through its law of motion $S'(s)$ over 3 periods. The figures are the corresponding average statistics across these 3 periods for the subset of points in the limiting distribution that at time 0 correspond to a foreign crisis.

The top four panels in Figure 4 plots a time series for both spreads, default incidence, output and the risk free rate in three periods. These statistics correspond to the states when both countries, i.e. have access to borrowing and can default. Period 0 is the *crisis period* where the foreign country has high spreads. Consider first the time series of spreads. The time series of spreads shows that the foreign country experiences a rapid increase in spreads from 4.4 to 6.3% in crisis period. The home spreads follow the foreign spreads increasing from about 3.9 to 5.2% in the crisis period. After the crisis period, if the foreign country manages to continue to repay its debts, its spreads fall to about 4.6%. The home spreads also declines to a level similar to the one pre crisis.

Now consider the incidence of default. By construction, the foreign country does not default in periods -1 and 0 because we are precisely considering periods prior to a foreign default. After the crisis period however, the foreign country has a high

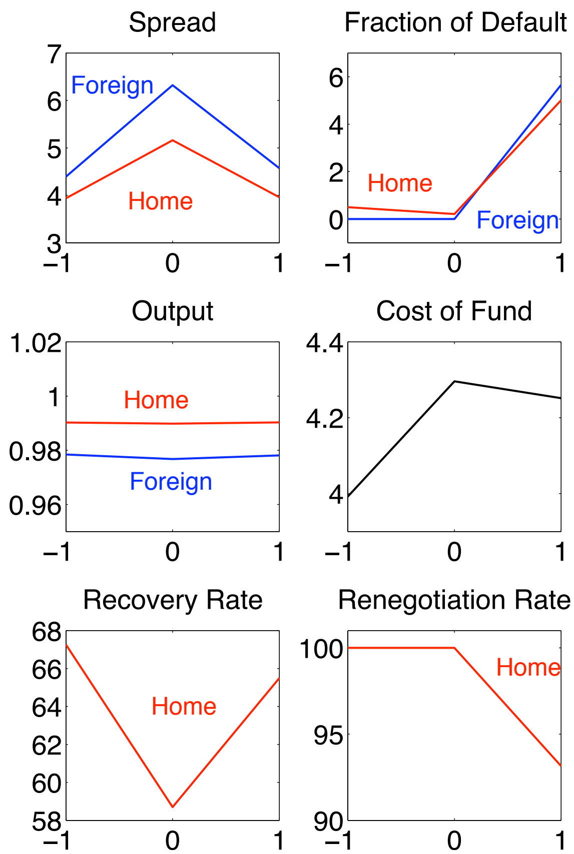


Figure 4: Crisis Events

probability of default of about 5.6%. The home probability of default decreases about 0.2% into the crisis period and increases about 5% the following period. Hence, the joint increase in spreads leading to the crisis is driven by the higher default incidence in period 1 of foreign and home as the current spread incorporates the future default probability.

Figure 4 also shows time series for output and the risk free interest rate. Output is pretty flat across these periods, although the output of the foreign country is below its mean because crises are associated with downturns. The risk free interest rate increases too, from 4% to 4.3%. The increase in risk free rate reflects the increase in cost of fund for home country when foreign country is more likely to default.

Let's now consider what happens to the home country if it is in bad credit standing during a crisis in the foreign country. The bottom two panels of Figure 4 plots time series of the recovery rate and the renegotiation rate at home. During the crisis period, recovery rates fall for the home country because the surplus to the lender from the renegotiating with the home country is larger if the foreign country has high spreads. The surplus is larger because a renegotiation prevents an immediate foreign default, which is costly to the lender. After the crisis period, recovery rates increase a bit, this is largely due to the fact that a large incidence of foreign defaults events raise the recovery for home. Renegotiation rates are pretty high, but after the crisis they fall due to a higher recovery.

4.4 Policy intervention during crises

In this section we analyze the effects of reducing the concavity of the payoff of lenders during crises. This exercise is motivated by the recent policy reponse during the European debt crisis, where the European Central Bank implemented measures aimed at increasing bank funding, such as broadening collateral eligibility requirements. Broadly speaking, these measures allow banks to better insure risk which effectively makes them behave in a more risk neutral fashion with respect to their debt holding with any one borrowing country. Specifically, we compute our model with a continuum of risk neutral lenders and study the default and renegotiation behavior of home and foreign countries during the crisis states that we identified in the previous section.

The dashed lines in Figure 5 are the time series of the variables of interest during the crisis states if countries suddenly face risk neutral lenders. Both home and foreign countries reduce their default rate in period 1 greatly, from more than 4% to about 0.8%. The interest spreads of both countries also decline from more than 4% to about 1%. As a result, risk free rate is also low. Risk neutral lending also increase home recoveries and renegotiation rates relative to the benchmark.

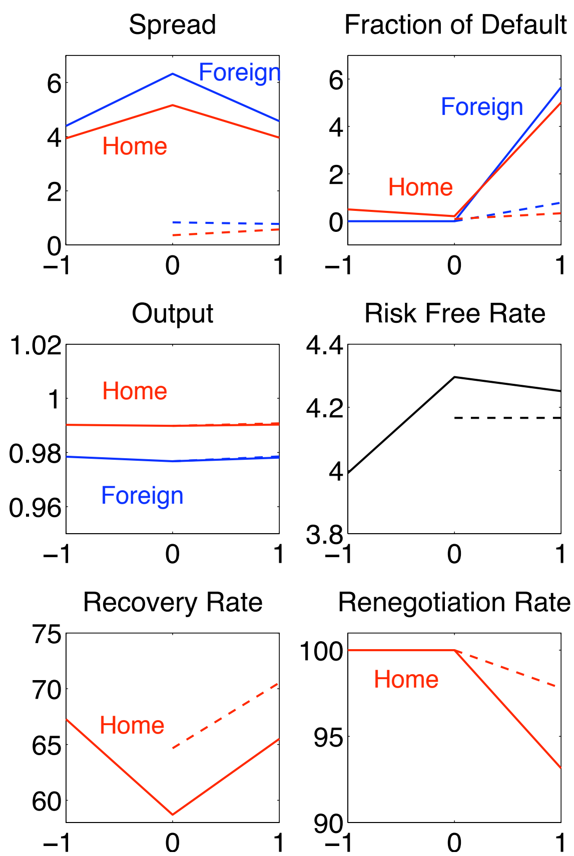


Figure 5: Crisis: Policy Intervention

5 Conclusion

We developed a multi-country model of sovereign default and renegotiation. Debt market conditions for borrowing countries are linked to one another because they borrow from a common lender with concave payoffs. In our model country interest rates are correlated and renegotiations with one country have spillover effects to other countries. The model provides a rationale for larger haircuts with one country to prevent default in other countries. Our model provides a framework to study some of the recent events in Europe.

References

[1] Aguiar, M., and G. Gopinath (2006). Defaultable Debt, Interest Rates and the Current Account. *Journal of International Economics,* 69(1): 64–83.

[2] Arellano, C. (2008). Default Risk and Income Fluctuations in Emerging Economies. *American Economic Review,* 98(3): 690–712.

[3] Benjamin, D. and M. Wright (2009). Recovery Before Redemption: A Theory of Delays in Sovereign Debt Renegotiations. Working Paper, State University of New York at Buffalo.

[4] Borri, N. and A. Verdelhan (2009). Sovereign Risk Premia. Working Paper, LUISS.

[5] Cruces and Trebesch (2012)

[6] D'Erasmo, P. (2011). Government Reputation and Debt Repayment. Working Paper, University of Maryland.

[7] Eaton, J., and M. Gersovitz (1981). Debt with Potential Repudiation: Theoretical and Empirical Analysis. *Review of Economic Studies*, 48(2):289–309.

[8] Lizarazo, S. (2010a). Default Risk and Risk Averse International Investors. Working Paper, ITAM.

[9] Lizarazo, S. (2010b). Contagion of Financial Crises in Sovereign Debt Markets. Working Paper, ITAM.

[10] Presno, I. and D. Puozo (2011). Sovereign Default Risk and Uncertainty Premia. Working Paper, University of California at Berkeley.

[11] Yue, V. (2010). Sovereign Default and Debt Renegotiation. *Journal of International Economics*.

APPENDIX

A Characterization of Simple Example

In this section we characterize the simple example in Section 3 by solving the model backwardly.

A.1 Period 2

We consider three cases here, both in good credit standings, one in good and one in bad, and both in bad credit standings.

Both in good credit standings In this case, there is no interaction between the two countries. Each chooses to default if and only if the repayment value is smaller than the default value. Thus there exists a $\bar{b}$ such that countries default if $b > \bar{b}$ where the cutoff debt $\bar{b}$ is given by

$$\bar{b} = y'^d.$$

One in good and one in bad In this case, the country with the good credit standing still chooses to default if its debt level is larger than $\bar{b}$, independent of the renegotiation decision of the country in bad credit. The country in bad credit standing renegotiates with lender. The recovery value, however, depends on the repayment value of the country in good credit standing. Without loss of generality, we assume home country is in the good credit and foreign country is in the bad credit. The recovery value of foreign country, $\phi^{*\prime}$ solves the following Nash bargaining problem,

$$\frac{\theta u_c(y^{*\prime} - \phi^{*\prime})}{u(y^{*\prime} - \phi^{*\prime}) - u(y^d)} = \frac{(1-\theta)g'(y_L + b'^{*\prime})}{g(y_L + b'^{*\prime}) - g(y_L + b')} \tag{34}$$

where b' is home country's debt holding at the beginning of period 1. In equilibrium, $b' = \bar{b}$.In addition, given that there is no uncertainty in the second period, the lending contract at period 1 makes sure that Home country chooses not to default for sure in the second period.

One question is how $\phi^{*\prime}$ varies with home country's debt b'. Let's call the right-hand side of equation (34) as the marginal surplus of lender, $MS(\phi^{*\prime}; b')$. It's straightforward to show that $\partial MS/\partial \phi^{*\prime} < 0$ since lender has concave utility function g.

Proposition 4 *In period 1, the recovery is always weakly smaller than $\bar{b}$.*

Proposition 5 *If $g(c) = c^{1-\sigma}/(1-\sigma)$, $\frac{\partial MS}{\partial b'} \geq 0$ and $\phi^{*\prime}$ increases with b'. If $g(c) = -\exp(-\alpha c)$, $\frac{\partial MS}{\partial b^{*\prime}} = 0$ and $\phi^{*\prime}$ is independent of b'.*

Proof.

$$\frac{\partial MS}{\partial b'} = \frac{g''(y_L + b'^{*\prime})\left[g(y_L + b'^{*\prime}) - g(y_L + b')\right] - g'(y_L + b'^{*\prime})\left[g'(y_L + b'^{*\prime}) - g'(y_L + b')\right]}{\left(g(y_L + b'^{*\prime}) - g(y_L + b')\right)^2 / (1-\theta)}$$

The necessary and sufficient condition for $\frac{\partial MS}{\partial b'} \geq 0$ to hold is

$$g''(y_L + b'^{*\prime})\left[g(y_L + b'^{*\prime}) - g(y_L + b')\right] \geq g'(y_L + b'^{*\prime})\left[g'(y_L + b'^{*\prime}) - g'(y_L + b')\right]$$

or

$$\frac{g''(y_L + b'^{*\prime})}{g'(y_L + b'^{*\prime})} \geq \frac{g'(y_L + b'^{*\prime}) - g'(y_L + b')}{g(y_L + b'^{*\prime}) - g(y_L + b')}.$$

It is easy to show that if $g(c) = -\exp(-\alpha c)$, the equality holds, $\frac{\partial MS}{\partial b'} = 0$.

Under the CRRA utility function, we have

$$\frac{-\sigma}{y_L + b'^{*\prime}} \geq (1-\sigma)\frac{(y_L + b'^{*\prime})^{-\sigma} - (y_L + b'^{-\sigma}}{(y_L + b'^{*\prime})^{1-\sigma} - (y_L + b'^{1-\sigma}}.$$

Reorganize it,

$$\begin{aligned}
\frac{\sigma}{\sigma-1} &\leq \frac{(y_L+b'^{*\prime})^{1-\sigma}-(y_L+b'^{*\prime})(y_L+b'^{-\sigma}}{(y_L+b'^{*\prime})^{1-\sigma}-(y_L+b'^{1-\sigma}} \\
&= \frac{1-\left(\frac{y_L+b'^{*\prime}}{y_L+b'}\right)^{\sigma}}{1-\left(\frac{y_L+b'^{*\prime}}{y_L+b'}\right)^{\sigma-1}} \\
&= \frac{1-x^{\sigma}}{1-x^{\sigma-1}} \\
&= \frac{x^{\sigma}-1}{x^{\sigma-1}-1}
\end{aligned}$$

where $x=\left(\frac{y_L+b'^{*\prime}}{y_L+b'}\right)\geq 1$*. Thus, after minus 1 on both hand side of the inequality, we have*

$$\frac{1}{\sigma-1} \leq \frac{x^{\sigma}-x^{\sigma-1}}{x^{\sigma-1}-1}.$$

Given $x\geq 1$*, the above inequality holds if and only if*

$$\sigma x^{\sigma-1}-(\sigma-1)x^{\sigma}\leq 1,$$

which holds for sure when $x\geq 1$*. Here is why, the left-hand side of the inequality decreases with* x*. When* $x=1$*, the equality holds. Thus when* $x>1$*, the inequality holds for sure. Q.E.D.*

The intuition is as follows. If lender's outside value is independent of b', an increase in b' increases lender's wealth and lower lender's marginal utility of consumption. Lender would like to accept lower recovery $\phi^{*\prime}$. However, when lender's outside value increases with b', lender has higher threat value.

Both in bad credit The recovery value ($\phi'^{*\prime}$) solve

$$\frac{\theta u_c(y'-\phi')}{u(y'-\phi'^d)} = \frac{(1-\theta)g'(y_L+\phi'^{*\prime})}{g(y_L+\phi'^{*\prime})-g(y_L)}$$
$$\frac{\theta u_c(y^{*\prime}-\phi^{*\prime})}{u(y^{*\prime}-\phi^{*\prime})-u(y^d)} = \frac{(1-\theta)g'(y_L+\phi'^{*\prime})}{g(y_L+\phi'^{*\prime})-g(y_L)}$$

Proposition 6 *The recovery of home country becomes smaller when foreign country is also in bad credit compared to the case when foreign country is in good credit, i.e.* $\phi'^{*}=1) < \phi'^{*}=0)$.

Proof.

$$\begin{aligned}\frac{\theta u_c(y'-\phi')}{u(y'-\phi'^d)} &= \frac{(1-\theta)g'(y_L+\phi'^{*\prime})}{g(y_L+\phi'^{*\prime})-g(y_L)} \\ &< \frac{(1-\theta)g'(y_L+\phi'^{*\prime})}{g(y_L+\phi'^{*\prime})-g(y_L+\phi^{*\prime})} \\ &\leq \frac{(1-\theta)g'(y_L+\phi'+\bar{b})}{g(y_L+\phi'+\bar{b})-g(y_L+\bar{b})}\end{aligned}$$

The last inequality holds because $\bar{b} \geq \phi^{*'}$ *and lender's marginal surplus weakly increases with its wealth. Q.E.D.*

A.2 Period 1

Similarly as in period 2, we consider three cases here, both in good credit standings, one in good and one in bad, and both in bad credit standings.

To emphasize the interaction within period 1 and abstract the future impact, we assume that in the second period both two countries have the same income y. Furthermore, we assume that countries have low discount factor and always borrow to the limit of defaulting tomorrow $\bar{b}$.

Both in good credit standing We are interested in understanding the default decisions of both countries in period 1. In this environment, a country repays and sets $d = 0$ if the value of repayment is greater than the value of default,

$$u(y - b + q(d^*)\bar{b}) + \beta u(y' - \bar{b}) \geq u(y^d) + \beta u(y' - \phi'(d^*)) \tag{35}$$

$$u(y^* - b^* + q(d)\bar{b}) + \beta u(y^{*\prime} - \bar{b}) \geq u(y^d) + \beta u(y^{*\prime} - \phi^{*\prime}(d)) \tag{36}$$

It is immediate that default is more likely when debt is high, the price is low and the recovery tomorrow is low. The default decisions of the two countries are linked because prices and recoveries depend on the decisions of both countries through the lenders' problem.

Let's first derive home country's best response of default conditional on foreign country's default decision. Foreign default matters for home country's default decision from two perspectives. First, foreign country's default decision affects home country's future recovery of debt . As we show in Proposition 6, home country's future recovery weakly decreases with foreign country's default in period 1. Thus, default in foreign country increases home country's default incentive. Second, foreign country's default decision affects current bond price schedule of home country.

Lemma 7 *Conditional on not default, a country's trade balance* $TB(b) = b - q(b)\bar{b}$ *increases with initial debt holding* b.

Proof. *Suppose lender's wealth in period 1 and period 2 are given by* ω_L *and* ω_L' *which include foreign country's state and decisions. The trade balance of a country is then given by*

$$TB(b) \equiv b - q(b)\bar{b} = b - \frac{\delta g'(\omega_L' + \bar{b})}{g'(\omega_L + b - q\bar{b})}\bar{b}$$

where $q(b)$ *solves*

$$q(b) = \frac{\delta g'(\omega_L' + \bar{b})}{g'(\omega_L + b - q\bar{b})}.$$

Taking derivative of the trade balance over b, we have

$$
\begin{aligned}
\frac{\partial TB}{\partial b} &= 1 - q'(b)\bar{b} \\
&= 1 + \frac{qg''(\omega_L + b - q\bar{b})\bar{b}}{g'(\omega_L + b - q\bar{b}) - qg''(\omega_L + b - q\bar{b})\bar{b}} \\
&= \frac{g'(\omega_L + b - q\bar{b})}{g'(\omega_L + b - q\bar{b}) - qg''(\omega_L + b - q\bar{b})\bar{b}} \geq 0
\end{aligned}
$$

Q.E.D.

Lemma 8 *If a country defaults with low debt, it also defaults with high debt for any given fixed states.*

Proof. *Based on condition (35), a country defaults if and only if*

$$b - q(b)\bar{b} \geq y - u^{-1}\left[u(y^d) + \beta u(y' - \phi') - \beta u(y' - \bar{b})\right]$$

According to Lemma (7), the left-hand side of the inequality increases with b. The right-hand side of the inequality is independent of b. We thus prove the lemma. Q.E.D.

Thus there exists a cutoff of $\bar{b}(s)$ as a function of current state s, such that a country defaults if and only if $b \geq \bar{b}(s)$. This is the same as in the case with risk-neutral lender. Note that the bond price q increases with a country's debt b, this tends to decrease a country's default incentive for large debt. However, we still have here that default decision is monotonic is b.

Let's call $\bar{b}_d$ as the default cutoff of home country when Foreign country defaults, $\hat{b}_{nd}(b^*)$ as the default cutoff of home country when foreign country repays its debt b^*.

Proposition 9 *Home country's cutoff* $\hat{b}_{nd}(b^*)$ *increases with foreign debt* b^*.

Proof. . *When foreign country with* b^* *chooses not to default, home country's default* *cutoff* $\hat{b}_{nd}$ *satisfies,*

$$\begin{aligned}
\hat{b}_{nd} - q_{nd}\bar{b} &= y - u^{-1}\left[u(y^d) + \beta u(y' - \phi'_{nd}) - \beta u(y' - \bar{b})\right] \equiv R(\phi'_{nd}) \\
q_{nd} &= \frac{\delta g'(y_L + \bar{b} + \bar{b})}{g'(y_L + \hat{b}_{nd} + b^* - 2q_{nd}\bar{b})}.
\end{aligned}$$

where ϕ'_{nd} *is the recovery value of home country who deals alone with lender in period 2 given foreign country repays in period 1. Here we also used the condition that none of the two countries will default in period 2 and thus face the same bond price* $q^* = q$. *Let's check how* $\hat{b}_{nd}$ *responds to* b^*,

$$\frac{\partial \hat{b}_{nd}(b^*)}{\partial b^*} = -\frac{\bar{b}\frac{q_{nd}g''(y_L+\hat{b}_{nd}+b^*-2q_{nd}\bar{b})}{g'(y_L+\hat{b}_{nd}+b^*-2q_{nd}\bar{b})-2q_{nd}\bar{b}g''(y_L+\hat{b}_{nd}+b^*-2q_{nd}\bar{b})}}{\frac{\partial TB}{\partial \hat{b}_{nd}}} \geq 0.$$

Q.E.D.

The relationship between $\hat{b}_{nd}(b^*)$ and $\hat{b}_d$ depends on parameters. When foreign country defaults, home country's cutoff $\hat{b}_d$ satisfies,

$$\begin{aligned}
\hat{b}_d - q_d\bar{b} &= y - u^{-1}\left[u(y^d) + \beta u(y' - \phi'_d) - \beta u(y' - \bar{b})\right] \equiv R(\phi'_d) \\
q_d &= \frac{\delta g'(y_L + \bar{b} + \phi^{*\prime}_{nd})}{g'(y_L + \hat{b}_d - q_d\bar{b})}.
\end{aligned}$$

where ϕ'_d is the recovery value of home country when Foreign country defaults and both of the two countries work together to renegotiate with lender, $\phi^{*\prime}_{nd}$ is the recovery of foreign country given home country repays. In particular, $\phi'_d \leq \phi'_{nd}$ from Proposition 6. Thus the $R(\phi'_d) \leq R(\phi'_{nd})$. When $b^* = 0$, q_{nd} becomes

$$q_{nd}(b^* = 0) = \frac{\delta g'(y_L + \bar{b} + \bar{b})}{g'(y_L + b - 2q_{nd}\bar{b})} \leq q_d$$

for the same level of b. The last inequality holds because $\bar{b} \geq \phi^{*\prime}_{nd}$ and $2\bar{b} > \bar{b}$. Therefore, there are two opposing effects. On the one hand, when foreign country repays, home country prefers to repay since future repayment is relatively high ($R(\phi'_{nd})$ is high). On the other hand, when foreign repays with low debt (here zero), foreign country's borrowing lowers lender's wealth and makes borrowing expensive for home country (q is low). Under CARA utility of lender, $\phi'_{nd} = \phi'_d$ and so the first effect disappears. It is for sure $\hat{b}_{nd}(b^* = 0) < \hat{b}_d$. If the utility if CRRA, the difference between ϕ'_{nd} and ϕ'_d is big, it is likely that $\hat{b}_{nd}(b^* = 0) > \hat{b}_d$. In this case, given $\hat{b}_{nd}$ increases with b^*, it is for sure that $\hat{b}_{nd}(b^*) \geq \hat{b}_d$ for any b^*.

The left panel of Figure (1) plots the best response of home country's default over foreign country's default. The right panel of Figure (1) plots the equilibrium.

One Good and One Bad Without loss of generality, we assume that home country is bad credit and foreign country is in good credit. Home country considers whether to renegotiate with the lender, while foreign country considers whether to default over its debt.

Home country renegotiates ($z = 1$) if

$$u(y - \phi) + \beta u(y') \geq u(y^d) + \beta u(y' - \phi'(d^*)) \tag{37}$$

Foreign country repays if

$$u(y^* - b^* + q^*\bar{b}) + \beta u(y^{*\prime} - \bar{b}) \geq u(y^d) + \beta u(y^{*\prime} - \phi^{*\prime}(z))$$

where prices ϕ and q^* are connected through lenders' problem:

$$\frac{\theta u'(y - \phi)}{[u(y - \phi) + \beta u(y') - u(y^d)(1 + \beta)]}$$
$$= \frac{(1 - \theta) g'(y_L + \phi + (1 - d^*)(b^* - q^*\bar{b}))}{\left[g(y_L + \phi + (1 - d^*)(b^* - q^*\bar{b})) + \beta g(y_L + \bar{b})\right] - \left[g(y_L + (1 - \tilde{d}^*)(b^* - \tilde{q}^*\bar{b})) + \beta g(y_L + (1 - \tilde{d}^*)\bar{b} + d^*\phi^{*\prime})\right]}$$

$$q^* = \frac{\delta g'(y_L + \bar{b} + (1 - z)\phi'(d^* = 0))}{g'(y_L + z\phi + b^* - q^*\bar{b})}.$$

where $\tilde{d}^*$ and $\tilde{q}^*$ are the default and bond price of foreign country when it deals alone with the lender.

Lemma 10 *Conditional on foreign country defaults $d^* = 1$, optimal recovery $\phi_d(b^*)$*

first increases with b^* then dumps down to a flat level after $b^* \geq \hat{b}_a^*$ where $\hat{b}_a^*$ denotes the default cutoff when foreign country deals alone with lender.

Proof. *when foreign country defaults, $\phi_d(b^*)$ solves*

$$\frac{\theta u'(y-\phi)}{[u(y-\phi)+\beta u(y')-u(y^d)(1+\beta)]}$$
$$= \frac{(1-\theta)g'(y_L+\phi)}{\left[g(y_L+\phi))+\beta g(y_L+\bar{b})\right]-\left[g(y_L+(1-\tilde{d}^*)(b^*-\tilde{q}^*\bar{b}))+\beta g(y_L+(1-\tilde{d}^*)\bar{b}+d^*\phi^{*\prime})\right]}$$

According to Lemma 7, the trade balance $b^*-\tilde{q}^*(b^*)\bar{b}$ increases with b^* when $b^* \leq \hat{b}_a^*$. This implies that lender's outside option increases with b^*. Thus recovery value $\phi_d(b^*)$ conditional on $b^* \leq \hat{b}_a^*$ increases with b^*. When $b^* > \hat{b}_a^*$, foreign country defaults when it deals with lender alone. Lender's outside value lowers since $0 \leq \hat{b}_a^* - \tilde{q}(\hat{b}_a^*)\bar{b}$ and $\phi^{*\prime} \leq \bar{b}$.

Q.E.D.

The black line in Figure (2) plots $\phi(b^*)$ conditional on foreign defaults $d^* = 1$.

Lemma 11 *Conditional on foreign country not default, $d^* = 0$, the optimal recovery $\phi_{nd}(b^*)$ first increases with b^* then decreases with b^*. The turning point is at $b^* = \hat{b}_a^*$ where $\hat{b}_a^*$ denotes the default cutoff when foreign country deals alone with lender.*

Proof. *when foreign country not default and $b^* \leq \hat{b}_a^*$, $\phi_{nd}(b^*)$ solves*

$$\begin{aligned}
\frac{\theta u'(y-\phi)}{[u(y-\phi)+\beta u(y')-u(y^d)(1+\beta)]} &= \frac{(1-\theta)g'(y_L+\phi+(b^*-q^*\bar{b}))}{\left[g(y_L+\phi+(b^*-q^*\bar{b}))\right]-\left[g(y_L+(b^*-\tilde{q}^*\bar{b}))\right]} \\
q^* &= \frac{\delta g'(y_L+\bar{b})}{g'(y_L+\phi+b^*-q^*\bar{b})} \\
\tilde{q}^* &= \frac{\delta g'(y_L+\bar{b})}{g'(y_L+b^*-\tilde{q}^*\bar{b})}
\end{aligned}$$

It's easy to see that $q^* > \tilde{q}^*$ as $\phi_{nd} \geq 0$. Let $TB(b^*) = b^* - q^*\bar{b}$ and $\tilde{TB}(b^*) = b^* - \tilde{q}^*\bar{b}$.

Also we have

$$\frac{\partial TB}{\partial b^*} = \frac{g'(y_L + \phi_{nd} + b^* - q^*\bar{b})}{g'(y_L + \phi_{nd} + b^* - q^*\bar{b}) - qg''(y_L + \phi_{nd} + b^* - q^*\bar{b})\bar{b}}$$
$$\frac{\partial \tilde{TB}}{\partial b^*} = \frac{g'(y_L + b^* - \tilde{q}^*\bar{b})}{g'(y_L + b^* - \tilde{q}^*\bar{b}) - qg''(y_L + b^* - \tilde{q}^*\bar{b})\bar{b}}$$

Let's take derivative of lender's marginal surplus over b^*,

$$\begin{aligned}\frac{\partial MS}{\partial b^*} = & \frac{g''(y_L + \phi + (b^* - q^*\bar{b}))\frac{dTB}{db^*}\left[g(y_L + \phi + (b^* - q^*\bar{b})) - g(y_L + (b^* - \tilde{q}^*\bar{b}))\right]}{\left(g(y_L + \phi + (b^* - q^*\bar{b})) - g(y_L + (b^* - \tilde{q}^*\bar{b}))\right)^2/(1-\theta)} \\ & - \frac{g'(y_L + \phi + (b^* - q^*\bar{b}))\left(\frac{dTB}{db^*}g'(y_L + \phi + (b^* - q^*\bar{b})) - \frac{d\tilde{TB}}{db^*}g'(y_L + (b^* - \tilde{q}^*\bar{b}))\right)}{\left[g(y_L + \phi + (b^* - q^*\bar{b})) - g(y_L + (b^* - \tilde{q}^*\bar{b}))\right]^2/(1-\theta)}\end{aligned}$$

The necessary and sufficient condition for $\frac{\partial MS}{\partial b^*} \geq 0$ *is*

$$\frac{g''(y_L + \phi + (b^* - q^*\bar{b}))}{g'(y_L + \phi + (b^* - q^*\bar{b}))} \geq \frac{g'(y_L + \phi + (b^* - q^*\bar{b})) - \frac{\frac{d\tilde{TB}}{db^*}}{\frac{dTB}{db^*}}g'(y_L + (b^* - \tilde{q}^*\bar{b}))}{\left[g(y_L + \phi + (b^* - q^*\bar{b})) - g(y_L + (b^* - \tilde{q}^*\bar{b}))\right]} \quad (38)$$

Under a CARA utility function $g(c) = -\exp(-\alpha c)$,

$$\frac{\frac{d\tilde{TB}}{db^*}}{\frac{dTB}{db^*}} = \frac{1 + \alpha q^*\bar{b}}{1 + \alpha\tilde{q}^*\bar{b}} \geq 1$$

The necessary and sufficient condition (38) becomes

$$-\alpha \geq \frac{g'(y_L + \phi + (b^* - q^*\bar{b})) - \frac{\frac{d\tilde{TB}}{db^*}}{\frac{dTB}{db^*}}g'(y_L + (b^* - \tilde{q}^*\bar{b}))}{\left[g(y_L + \phi + (b^* - q^*\bar{b})) - g(y_L + (b^* - \tilde{q}^*\bar{b}))\right]}.$$

Note that

$$
\begin{aligned}
-\alpha &= \frac{g'(y_L+\phi+(b^*-q^*\bar{b}))-g'(y_L+(b^*-\tilde{q}^*\bar{b}))}{\left[g(y_L+\phi+(b^*-q^*\bar{b}))-g(y_L+(b^*-\tilde{q}^*\bar{b}))\right]} \\
&\geq \frac{g'(y_L+\phi+(b^*-q^*\bar{b}))-\frac{\frac{d\tilde{T}B}{db^*}}{\frac{dTB}{db^*}}g'(y_L+(b^*-\tilde{q}^*\bar{b}))}{\left[g(y_L+\phi+(b^*-q^*\bar{b}))-g(y_L+(b^*-\tilde{q}^*\bar{b}))\right]}
\end{aligned}
$$

Thus under CARA utility function, the marginal surplus of lender increases with b^* *when* $b^* \leq \hat{b}_a^*$*. Thus* ϕ_{nd} *increases with* b^**.*

Under a CRRA utility function, the necessary and sufficient condition (38) becomes,

$$
-\sigma\frac{1}{y_L+\phi+(b^*-q^*\bar{b})} \geq (1-\sigma)\frac{(y_L+\phi+(b^*-q^*\bar{b}))^{-\sigma}-\frac{\frac{d\tilde{T}B}{db^*}}{\frac{dTB}{db^*}}(y_L+(b^*-\tilde{q}^*\bar{b}))^{-\sigma}}{(y_L+\phi+(b^*-\tilde{q}^*\bar{b}))^{1-\sigma}-(y_L+(b^*-\tilde{q}^*\bar{b}))^{\sigma-1}}
$$

After reorganizing, we have

$$
\begin{aligned}
\sigma &\leq (\sigma-1)\frac{1-\frac{\frac{d\tilde{T}B}{db^*}}{\frac{dTB}{db^*}}\left(\frac{y_L+\phi+(b^*-q^*\bar{b})}{y_L+(b^*-\tilde{q}^*\bar{b})}\right)^{\sigma}}{1-\left(\frac{y_L+\phi+(b^*-q^*\bar{b})}{y_L+(b^*-\tilde{q}^*\bar{b})}\right)^{1-\sigma}} \\
&= (\sigma-1)\frac{1-\frac{\frac{y_L+b^*-\tilde{q}^*\bar{b}}{y_L+b^*+(\sigma-1)\tilde{q}^*\bar{b}}}{\frac{y_L+\phi+b^*-q^*\bar{b}}{y_L+\phi+b^*+(\sigma-1)q^*\bar{b}}}\left(\frac{y_L+\phi+(b^*-q^*\bar{b})}{y_L+(b^*-\tilde{q}^*\bar{b})}\right)^{\sigma-1}}{1-\left(\frac{y_L+\phi+(b^*-q^*\bar{b})}{y_L+(b^*-\tilde{q}^*\bar{b})}\right)^{\sigma-1}} \\
&= (\sigma-1)\frac{1-\frac{y_L+\phi+b^*+(\sigma-1)q^*\bar{b}}{y_L+b^*+(\sigma-1)\tilde{q}^*\bar{b}}\left(\frac{y_L+\phi+(b^*-q^*\bar{b})}{y_L+(b^*-\tilde{q}^*\bar{b})}\right)^{\sigma-1}}{1-\left(\frac{y_L+\phi+(b^*-q^*\bar{b})}{y_L+(b^*-\tilde{q}^*\bar{b})}\right)^{\sigma-1}} \\
&= (\sigma-1)\frac{\frac{y_L+\phi+b^*+(\sigma-1)q^*\bar{b}}{y_L+b^*+(\sigma-1)\tilde{q}^*\bar{b}}\left(\frac{y_L+\phi+(b^*-q^*\bar{b})}{y_L+(b^*-\tilde{q}^*\bar{b})}\right)^{\sigma-1}-1}{\left(\frac{y_L+\phi+(b^*-q^*\bar{b})}{y_L+(b^*-\tilde{q}^*\bar{b})}\right)^{\sigma-1}-1}
\end{aligned}
$$

The inequality holds for sure since $q^* \geq \tilde{q}^*$ *and* $\phi+b^*-q^*\bar{b} \geq b^*-\tilde{q}^*\bar{b}$*, which holds because lender receives at least the outside option after renegotiation. Thus we show* *that under CRRA utility, the marginal surplus of lender and so ϕ_{nd} both increase with foreign debt b^* when $b^* \leq \hat{b}^*_a$.*

When $b^* \geq \hat{b}^*_a$, the optimal ϕ_{nd} solves

$$
\begin{aligned}
\frac{\theta u'(y-\phi)}{[u(y-\phi)+\beta u(y')-u(y^d)(1+\beta)]} &= \frac{(1-\theta)g'(y_L+\phi+(b^*-q^*\bar{b}))}{g(y_L+\phi+(b^*-q^*\bar{b}))-g(y_L)} \\
q^* &= \frac{\delta g'(y_L+\bar{b})}{g'(y_L+\phi+b^*-q^*\bar{b})}
\end{aligned}
$$

In this case, the lender's marginal surplus (the right-hand side of the equation) decreases with b^* since g function is concave and the trade balance $b^*-q^*\bar{b}$ increases with b^*.

As long as $\hat{b}^*_a - \tilde{q}_a\bar{b} \geq 0$, there is a dump in ϕ_{nd} see the red line in Figure 2, the reason is because lender has a loss in outside value and would like to accept lower repayment. When b^* keeps increasing, recovery of Home starts to decrease due to the wealth effect of lender.

Q.E.D.

Under which case the recovery of Home is higher, when foreign defaults $d^*=1$ or when foreign repays $d^*=0$? Suppose $b^* \leq \hat{b}^*_a$, foreign country is not defaulting when alone with lender. In this case, ϕ_d and ϕ_{nd} satisfies

$$
\begin{aligned}
\frac{\theta u'(y-\phi_d)}{[u(y-\phi_d)+\beta u(y')-u(y^d)(1+\beta)]} &= \frac{(1-\theta)g'(y_L+\phi_d)}{g(y_L+\phi_d)-g(y_L+b^*-\tilde{q}^*\bar{b})} \\
\frac{\theta u'(y-\phi_{nd})}{[u(y-\phi_{nd})+\beta u(y')-u(y^d)(1+\beta)]} &= \frac{(1-\theta)g'(y_L+\phi_{nd}+b^*-q^*\bar{b})}{g(y_L+\phi_{nd}+b^*-q^*\bar{b})-g(y_L+b^*-\tilde{q}^*\bar{b})}
\end{aligned}
$$

Clearly that when $b^*=0$, $\phi_d \leq \phi_{nd}$ since $q^*\bar{b} \geq 0$. When b^* is large enough such that $b^*-q^*\bar{b} \geq 0$, $\phi_d \geq \phi_{nd}$.

According to the condition (37), there exists two recovery cutoffs $\bar{\phi}_d$ and $\bar{\phi}_{nd}$ that solve

$$
\begin{aligned}
\bar{\phi}_d &= y - u^{-1}\left[u(y^d)+\beta u(y'-\phi'_{zd})-\beta u(y')\right] \\
\bar{\phi}_{nd} &= y - u^{-1}\left[u(y^d)+\beta u(y'-\phi'_{znd})-\beta u(y')\right]
\end{aligned}
$$

where ϕ'_{zd} is home recovery when foreign defaults and ϕ'_{znd} is the home recovery when foreign chooses not to default in period 2. According to Proposition 6, $\phi'_{zd} \leq \phi'_{znd}$, which implies that $\bar{\phi}_d \leq \bar{\phi}_{nd}$. Use these two cutoffs, we can determine the area of b^* where Home repays. See Figure (2). When Foreign defaults with $b^* \leq \hat{b}^*_{zd}$, Home renegotiates. When Foreign is not default, Home renegotiates $z = 1$ when $b^* \leq \hat{b}^*_{znd1}$ or $b^* \geq \hat{b}^*_{znd2}$.

For foreign country in the good credit, there are two default cutoffs depending on home country's renegotiation choices. In particular, $\bar{b}^*_{z1} \geq \bar{b}^*_{z0}$. Two reasons, first when $z = 0$, foreign country has more incentive to default so that it can renegotiate together with home country in period 2. This tends to lower the default cutoff. On the other hand, the bond price under $z = 0$ is also lower since lender suffer wealth loss in period 1 but get wealth increase in period 2. This also increases the default incentive and lower the default cutoff when $z = 0$. Notice that $\bar{b}^*_{z0} \leq \hat{b}^*_a \leq \bar{b}^*_{z1}$

The equilibrium default and renegotiation choices are shown in Figure (3). We plot b_2 on both x-axis and y-axis. We are interested in 45 degree lines the equilibrium.

B Computational Algorithm

We assume there are finite periods T and solve the model backwards. In the last period, countries have zero future values and do not borrow from lenders anymore. They choose whether to default or renegotiate their debts. Using the optimal default $d^T(s)$, repayment $\phi^T(s)$, and renegotiation choices, $z^T(s)$ we can update the value function at period T and bond price schedule at period $T-1$, $v^T(s)$, $w^T(s)$ and $q^{T-1}(o, o^*, s)$ where

$$
\begin{aligned}
v^T(s) &= (1 - d^T(s))u(y - b) + d^T(s)u(y^{def}) \\
w^T(s) &= z^T(s)u(y - \phi^T(s)) + (1 - z^T(s))u(y^{def}) \\
q^T(o, o^*, s) &= \sum_{s'} m(s', s, o, o^*)[1 - d^T(s')]
\end{aligned}
$$

At period $t \leq T-1$, for each state s we solve the equilibrium between the two countries taking as given the future decision rules. There are four possible states in terms of countries credit standing: both good, one good and one bad, both bad.

- *Both in good credit*

  In this case, we look at four possible default patterns: both repay, one defaults and one doesn't, and both default.

  - Assuming both repay. In this case, we look for fixed point of two countries' debt choices $(b', b^{*\prime})$. In particular, we solve home country's optimal $\tilde{b}'$ by taking as given that foreign country is not defaulting and chooses $\tilde{b}^{*\prime}$. With home's choice over $\tilde{b}'$, we then solve foreign country's optimal debt $\tilde{b}^{*\prime}$. We oscillate between the two countries until we find a fixed point of $(b', b^{*\prime})$. We then check each country's incentive to default. If both choose not to default, we find one candidate of equilibrium.
  - Assuming one default and doesn't. Without loss of generosity, let's assume home country defaults and foreign doesn't. We first solve foreign country's $b^{*\prime}$ taking as given home defaults. We then solve home country's optimal b' if it weren't default. Once we get the value of default and non-default of each country, we check whether their default choices are consistent with the initial assumption that home defaults and foreign doesn't. If so this is a candidate of equilibrium.
  - Assuming both default. We solve each country's off-equilibrium b' and value if it weren't default. We compare its default value versus non-default value. If both countries prefer to default, we have a candidate of equilibrium.

  In total, we could have at most four equilibria in terms of default pattern. We compare which one equilibrium default pattern gives the highest total value of the two borrowers. We choose that as the solution for the current state s in period $t \leq T - 1$.

- *One good one bad*

  We also look at four possibilities here: repay and renegotiate, repay and not renegotiate, default and renegotiate, and default and not renegotiate. Let's assume home is in good credit and chooses whether to default, and foreign is in bad credit and chooses whether to renegotiate.

  - Repay and renegotiate. We solve home country's optimal choice of b'. In particular, for each choice b' of home country, we solve (q, ϕ^*) jointly according to equations (24). We find the b' that maximizes home country's non-default value. We then check home country's default incentive and foreign country's renegotiation choice given $\phi^*(b')$. If home chooses to repay and foreign chooses to renegotiate, we have one candidate of equilibrium.
  - Default and renegotiate. Given home defaults, we solve ϕ^* for foreign country and check whether foreign country wants to renegotiate. If so, we check home country's default incentive given ϕ^* and foreign renegotiates. If home wants to default, we have a candidate of equilibrium.
  - Default and not-renegotiate. Given that foreign does not renegotiate, we check home's default incentive. Given home defaults, we check foreign's renegotiating incentive. If home chooses default and foreign chooses not to renegotiate, we have a candidate of equilibrium.

  Similarly as in the case of both in good credit, we compare over the candidates of equilibrium and choose the one giving highest value of the two countries.

- *Both bad*

  We look four possibilities: both renegotiate, only one renegotiate, and both do not renegotiate. When both renegotiate, we solve two countries' recovery (ϕ, ϕ^*) jointly and check wether it's the case that both prefer to renegotiate. If only home country chooses to renegotiate, we solve home country's recovery ϕ assuming foreign chooses not to renegotiate. We then check whether foreign has incentive to default taking as given home's optimal ϕ and the assumption that home renegotiates. In the last case, assuming both choose not to renegotiate, we check each country's renegotiation incentive given the other country does not renegotiate. In the last, we compare over the candidates of equilibrium and choose the one giving highest value of the two countries.

After we finish computation of equilibrium for period t, we update the bond price for period $t-1$. We continue until t reaches period 0 or the distance between period t's value and price $(v^t(s), w^t(s), q^t(o, o^*, s))$ and period $t-1$'s value and price $(v^{t-1}(s), w^{t-1}(s), q^{t-1}(o, o^*, s))$ is small enough.