Career Concerns with Coarse Information*

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June 2012

Abstract

This paper develops a model of career concerns. The worker's skill is revealed through output, wage is based on expected output, and so on assessed ability. Specifically, effort increases the probability that a skilled worker achieves a one-time breakthrough. Effort levels at different times are strategic substitutes. Equilibrium effort (and, if marginal cost is convex, wage) is single-peaked with seniority. The agent works too little, too late. Both delay and underprovision of effort worsen if effort is observable. If the firm commits to wages but faces competition, the optimal contract features piecewise constant wages as well as severance pay.

Keywords: career concerns, experimentation, career paths, up-or-out, reputation.

JEL Codes: D82, D83, M52.

*A. Bonatti acknowledges the support of MIT's Program on Innovation in Markets and Organizations (PIMO). J. Hörner gratefully acknowledges financial support from NSF Grant SES 092098. We would like to thank Daron Acemoğlu, Glenn Ellison, Bob Gibbons, Michael Grubb, Tracy Lewis, Nicola Persico, Steve Tadelis, Juuso Toikka, Jean Tirole, Alex Wolitsky and especially Joel Sobel, as well as participants in the 2010 European Summer Symposium in Economic Theory, Gerzensee, the 2011 Columbia-Duke-Northwestern IO conference, and seminar audiences at the Barcelona JOCS, Berkeley, Bocconi, CIDE, Collegio Carlo Alberto, the European University Institute, LBS, LSE, MIT, Nottingham, Penn State, Stanford, Toronto, UBC, USC, IHS Vienna and Yale for helpful discussions, and Yingni Guo for excellent research assistance.

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1 Introduction

Perceived talent (reputation) is a key component to a worker's wage. When employers lack commitment power (e.g., because output is not verifiable by a third party), this wage must be paid upfront, unconditionally on current performance. In such an environment, establishing a reputation through success is a powerful motive for hard work: in order to attain higher levels of future pay, an agent is willing to exert higher levels of effort in the present. In a competitive labor market, higher effort levels will in turn yield higher wages: therefore, the equilibrium relationship between wages and the time path of the agent's incentives is crucial for our understanding of reputation in a market context.

Nearly all existing models of career concerns, starting with the seminal contribution of Holmström (1982/99), have investigated the effects of reputation in an environment characterized by: (i) either two periods or time-separable incentives, (ii) frequently observed output, and (iii) short-term contracts. This framework is appropriate for those jobs (e.g., production line workers, fund managers, traders) for which success is related to average performance, and output can be measured almost continuously, although with noise. At the same time, this approach abstracts from several salient features of professions that require strong creativity or problem-solving skills.

The sectors we have in mind include creative industries, professional services such as law and consulting firms, pharmaceutical companies, biotechnology research labs, and academia. Success in these jobs is related to peak performance, and output measures are based on very rare events only.[1] Hence, information is coarse: either a solution is found and the project is successful, or it is not. Because they are so rare, such successes are defining moments in a young professional's career (e.g., receiving a Research Project Grant by the NIH, signing a record deal, attaining partnership in professional services firms, or tenure in academia). As a first approximation, one breakthrough provides all the necessary evidence about the worker's ability.

Furthermore, in all these markets, it is easier to contract on a publicly observable and objectively measurable variable such as employment duration than it is on explicit output-contingent wages.[2] However, depending on the circumstances, the prevalent contractual arrangement might

[1] For example, one first-authored publication in the life sciences is often sufficient to apply for a grant as principal investigator; in addition, over 60% of inventors awarded a patent by the USPTO were awarded only one over the period 1963–1999. See Trajtenberg et al. (2006).

[2] While theoretically attractive, explicit output-contingent contracts are rare in practice. Few R&D firms offer

involve spot contracts or one-sided commitment to wages over a longer horizon.

This paper investigates the incentives and compensation of employees, how they evolve over time, how they depend on the work-performance measurement as well as on other provisions of the labor contract. The environment we consider departs from the literature's in three key respects: (i) output is lumpy, (ii) it is very informative, and (iii) contracts are not necessarily short-term.

Our objective is twofold. First, we want to obtain qualitative insights into the dynamics of a worker's incentives, effort and wages: is aggregate effort provision efficient? Given its aggregate amount, is effort spread optimally over the agent's career? Second, we want to build a framework that is tractable enough to answer specific questions on particular contractual arrangements: what are the properties of optimal long-term wage contracts in a competitive market? What are the implications of improvements in the effort-monitoring technology? Finally, what is the role of deadlines? Indeed, a hallmark of many occupations (law, accounting and consulting firms, etc.) is the use of a probationary period during which employees are expected to obtain promotion to partner; if not, they are meant to quit, if not dismissed forthright.[3] Does this practice enhance workers' incentives and welfare?

In a nutshell, the main theoretical finding of our model is that current and future effort levels are strategic substitutes. The main testable prediction is that, with short-term contracts, wages are a single-peaked function of seniority. However, if firms can commit to wage contracts unilaterally, wage rigidity and severance payments are optimal. Furthermore, if firms can observe the worker's effort, wages increase over time, inasmuch as effort does.

More formally, in our model there are no explicit output-contingent contracts. The firm, or market, must pay the worker, or agent, a competitive wage, given his expected output, which in turn is based on his assessed ability. Information about ability is symmetric at the start.[4]

innovation bonuses. Junior associates in law and consulting firms face fixed wage rates, and even in the motion pictures industry, most contracts involve fixed-payment rather than profit sharing (see Chisholm, 1997).

[3]Though alternative theories have been put forth (e.g., tournament models), agency theory provides an appealing framework to analyze such systems (see Fama, 1980, or Fama and Jensen, 1983). Gilson and Mnookin (1989) offer a vivid account of associate career patterns in law firms, and the relevance of the career concerns model as a possible explanation.

[4]We shall also briefly discuss the case of asymmetric information. Alternatively, one could examine the consequences of overoptimism by the agent. In many cases, however, symmetrical ignorance appears like the

Skill and output are binary and complements: only a skilled agent can achieve a high output –a *breakthrough*. The time at which this output arrives follows an exponential distribution, whose instantaneous intensity increases with the worker's effort. If the agent ever succeeds, and so proves himself, he is promoted and receives a constant exogenously specified compensation thereafter. This specification implies that effort increases not only expected output, but also the speed of learning, unlike in the additively separable Gaussian set-up. The focus is on the effort until this breakthrough occurs, if ever.

Our analysis proceeds in several steps. First, we examine the worker's incentives taking the wage path as given. As we show, career concerns provide insufficient incentives for effort: *for any* wage path, the total amount of effort exerted is inefficiently low. Furthermore, whatever effort is provided, it is done so too late: a social planner constrained to the same total amount of effort would apply it earlier. This backloading of effort contrasts with the inefficient frontloading that arises in Holmström's model.

We then proceed by imposing the following equilibrium restriction: at any point in time, the wage must equal the expected value of output. Fundamental to the dynamics of incentives and wages in our model is the *strategic substitutability* between current and future effort. Substitutability is not a property of the production function –the arrival process of breakthroughs is memoryless– but it arises via the worker's compensation. If career concerns successfully provide incentives for high effort at some point in the worker's career, wages at that time will reflect this increased productivity; in turn, this depresses incentives to exert high effort earlier in the worker's career, as higher future wages makes staying on the current job relatively more attractive. Therefore, substitutability introduces a trade-off in the ability of reputation to generate incentives at different stages in a worker's career.[5]

Substitutability shapes the pattern of effort and compensation: because career concerns are ineffective at the end of the worker's tenure, effort is single-peaked.[6] When marginal cost is

more plausible assumption. See Caves (2003).

[5]Substitutability does not arise in Holmström's additively separable model. It does not arise in Dewatripont, Jewitt and Tirole (1999a,b)'s analysis either, which is not surprising, as it cannot be picked up by a two-period model (career concerns do not arise in the last period). Their analysis focuses instead on the strategic complementarity between expected effort and current effort, which generates, among others, equilibrium multiplicity. We prove that, under mild conditions, the equilibrium is unique.

[6]Of course this does not rule out a monotone effort path as a special case.

convex, wages are single-peaked as well. This stands in contrast with several earlier models, in which effort and wages stochastically decrease over time. Because compensation does not only reflect effort, but also ability, and prolonged failure necessarily increases pessimism regarding this ability, wage dynamics can be more complicated when the marginal cost is not convex. For example, an initial phase of decreasing wages may precede a single-peaked pattern.

Dynamics of this complexity are seldom observed in practice. We therefore leverage the tractability of our model to investigate which specific labor market arrangements are likely to explain other features of labor contracts, such as increasing, rigid wages and severance pay. In particular, we examine three variations of the baseline model.

First, we endow firms with stronger commitment power: employers can commit to contracts that specify an entire wage path, though workers cannot commit not to leave their firm if a competitor offers them a better contract at any point. In that case, the optimal contract is strikingly simple: it is characterized by either a one-step or (if the horizon is long enough) a two-step wage, followed by a lump-sum "severance" payment at the end of tenure, if the worker never succeeded. Effort is highest at the beginning, and constant over each step.

Second, we examine how the quality of monitoring affects our conclusions. In particular, we assume effort is publicly observable but not contractible. In any of the Markov equilibria, effort provision is even lower than under non-observability, and it is further delayed; as a result, effort increases over time (which pushes wages up over time). While this means that, in line with earlier findings in this literature, imperfect observability helps generate incentives, it also points to the fact that empirical patterns might be better explained by models with better monitoring.

Finally, we consider an endogenous deadline (the "tenure clock"): workers may leave the profession whenever they consider it best, though employers rationally anticipate this. In equilibrium, not only is effort single-peaked, it is decreasing at the deadline, and so is the wage. The worker quits too late, relative to what would be optimal, but if he could commit to a deadline, he might choose a longer or a shorter one than without commitment, depending on the circumstances.

The most closely related papers are Holmström, as mentioned, as well as Dewatripont, Jewitt and Tirole (1999a,b). In Holmström, skill and effort enter linearly and additively into the mean of the output that is drawn in every period according to a normal distribution. Wages are as

in our model: the worker is paid upfront the expected value of the output. Our model shares with the two-period model of Dewatripont, Jewitt and Tirole some features that are absent from Holmström's. In particular, effort and talent are complements. We shall discuss the relationship between the three models at length in the paper.

As Gibbons and Murphy (1992), our paper examines the interplay of implicit incentives (career concerns) and explicit incentives (termination penalty). It shares with Prendergast and Stole (1996) the existence of a finite horizon, and thus, of complex dynamics related to seniority. See also Bar-Isaac (2003) for reputational incentives in a model in which survival depends on reputation. Jovanovic (1979) and Murphy (1986) provide models of career concerns that are less closely related: the former abstracts from moral hazard and focuses on turnover when agents' types are match-specific; the latter studies executives' experience-earnings profiles in a model in which firms control the level of capital assigned to them over time.

The binary set-up is reminiscent of Mailath and Samuelson (2005), Bergemann and Hege (2005), and Board and Meyer-ter-Vehn (2010a). The latter two papers use an exponential technology for output. However, in Bergemann and Hege (2005), the effort choice is binary and wages are not based on the agent's reputation, while Board and Meyer-ter-Vehn (2010a) let the agent, who is privately informed, control the evolution of his type through his effort. Here instead, information is symmetric and types are fixed (Subsection 4.2.3 relaxes this assumption).[7] A theory of up-or-out contracts, based on asymmetric learning and promotion incentives, is investigated in Ghosh and Waldman (2010), while Ferrer (2011) studies how lawyers' career concerns impacts litigation. Finally, the empirical work of Chevalier and Ellison (1999) provides evidence of the sensitivity of termination to performance, while Johnson (2011) and Kolstad (2010) quantify the effect of individual and market learning on physicians' incentives and career paths.

[7]Board and Meyer-ter-Vehn (2010b) study the Markov-perfect equilibria of a game in which effort affects the evolution of the player's type both under symmetric and asymmetric information.

2 The model

2.1 Set-up

We consider the incentives of an agent (or *worker*) to exert effort (or *work*). Time is continuous, and the horizon finite: $t \in [0, T]$, $T > 0$. Most results carry over to the case $T = \infty$, as shall be discussed, and the case of endogenous deadlines T will be studied in detail in Section 5.3.

The game (or *project*) can end before $t = T$, if the agent's effort is successful. Specifically, there is a binary state of the world. If the state is $\omega = 0$, the agent is bound to fail, no matter how much effort he exerts. If the state is $\omega = 1$, a success (or *breakthrough*) arrives at a time that is exponentially distributed, with an intensity that increases in the instantaneous level of effort exerted by the agent. The state can be interpreted as the agent's ability, or skill. We will refer to the agent as a high- (resp., low-) ability agent if the state is 1 (resp. 0). The prior probability of state 1 is $p^0 \in (0, 1)$, which measures occupational harshness. Until a success occurs, the agent is "locked in" the game. We shall discuss alternative termination rules in Section 5.

Effort is a (measurable) function from time to the interval $[0, \bar{u}]$, where $\bar{u} \in \bar{\mathbb{R}}$ represents an upper bound (possibly infinite). If a high-ability agent exerts effort u_t over the time interval $[t, t + \mathrm{d}t)$, the probability of a success over that time interval is $(\lambda + u_t)\mathrm{d}t$. The parameter $\lambda \geq 0$ can be interpreted as the luck of a talented agent. Alternatively, it measures the minimum effort level that the principal can force upon the agent by direct oversight, i.e. the degree of contractibility of the worker's effort. Formally, the instantaneous arrival rate of a breakthrough at time t is given by $\omega \cdot (\lambda + u_t)$. Note that, unlike in Holmström's model, but as in the model of Dewatripont, Jewitt and Tirole, work and talent are complements.

As long as the game has not ended the agent receives a flow wage w_t. For now, let us think of this wage as an exogenous (integrable, non-negative) function of time only that accrues to the agent as long as the game has not ended. Eventually, equilibrium constraints will be imposed on this function, and this wage will reflect the market's expectations of the agent's effort and ability, given that the market values a success. This value is normalized to one.

In addition to receiving this wage, the agent incurs a cost of effort: exerting effort level u_t over the time interval $[t, t + \mathrm{d}t)$ entails a flow cost $c(u_t)\mathrm{d}t$. We shall consider two cases: in the convex case, we assume that $\bar{u} = \infty$, c is increasing, thrice differentiable and convex, with

$c(0) = 0$, $\lim_{u\to 0} c'(u) = 0$, $\lim_{u\to\infty} c'(u) = \infty$, $c'' > 0$ and $c''' \geq 0$.[8] In the linear case, $\bar{u} < \infty$ and $c(u) = \alpha \cdot u$, where $\alpha > 0$. The linear case is not a special case of what is called the convex one, but it yields similar results, while allowing for illustrations and sharper characterizations.

Achieving a success is desirable on two accounts: first, a known high-ability agent can expect a flow outside wage of $v \geq 0$, so that this outside option v is a (flow) opportunity cost for him that is incurred as long as no success has been achieved.[9] The outside option of the low-ability agent is normalized to 0. Second, we allow for a fixed penalty of $k \geq 0$ for reaching the deadline (i.e., for not achieving a success by time T). This might represent diminished career opportunities to workers with such poor records. Alternatively, this penalty might be an adjustment cost, or the difference between the wage he could have hoped for had he succeeded, and the wage he will receive until retirement. In the linear cost case, we assume $k > \alpha$, for the penalty plays essentially no role otherwise. There is no discounting.[10]

The worker's problem can then be stated as follows: to choose $u : [0, T] \to [0, \bar{u}]$, measurable, to maximize his expected sum of rewards, net of the outside wage v:

$$\mathbb{E}_u \left[\int_0^{T\wedge\tau} \left[w_t - v\chi_{\omega=1} - c(u_t) \right] \mathrm{d}t - \chi_{\tau\geq T} k \right],^{11}$$

where $\mathbb{E}_u$ is the expectation conditional on the worker's strategy u and τ is the time at which a success occurs (a random time that is exponentially distributed, with instantaneous intensity at time t equal to 0 if the state is 0, and to $\lambda + u_t$ otherwise) and χ_A is the indicator of event A.

Of course, at time t effort is only exerted, and the wage collected, conditional on the event that no success has been achieved. We shall omit to say so explicitly, as those histories are the only nontrivial ones. Given his past effort choices, the agent can compute his belief p_t that he is of high ability by using Bayes' rule. It is standard to show that, in this continuous-time

[8] The assumption that c' is convex is only required for three results: Lemma 2.1, equilibrium uniqueness and single-peakedness of equilibrium wage in Section 4 (Theorem 4.2).

[9] A natural case is the one in which v equals the flow value of success given that the agent has established that $\omega = 1$. This would be his payoff in the Markov equilibrium of the complete information game. Since successes arrive at rate λ and are worth 1, $v = \lambda$ in that case.

[10] At the beginning of the appendix, we explain how to derive the objective function from its discounted version as discounting vanishes. Values and optimal policies converge (pointwise) as the discount rate vanishes.

[11] Stating the objective as a net payoff ensures that the program is well-defined even when $T = \infty$.

environment, Bayes' rule reduces to the ordinary differential equation

$$\dot{p}_t = -p_t\left(1-p_t\right)\left(\lambda+u_t\right),\ p_0 = p^0. \tag{1}$$

By the law of iterated expectations, we can then rewrite our objective as

$$\int_0^T e^{-\int_0^t p_s(\lambda+u_s)\mathrm{d}s}\left[w_t - p_t v - c\left(u_t\right)\right]\mathrm{d}t - ke^{-\int_0^T p_t(\lambda+u_t)\mathrm{d}t}.$$

The exponential term captures the possibility that time t is never reached. Using eqn. (1), we can write the exponential term as

$$e^{-\int_0^t p_s(\lambda+u_s)\mathrm{d}s} = \frac{1-p_0}{1-p_t}. \tag{2}$$

Alternatively, observe that

$$\mathbb{P}\left[\tau \geq t\right] = \frac{\mathbb{P}\left[\omega = 0 \cap \tau \geq t\right]}{\mathbb{P}\left[\omega = 0 | \tau \geq t\right]} = \frac{\mathbb{P}\left[\omega = 0\right]}{\mathbb{P}\left[\omega = 0 | \tau \geq t\right]} = \frac{1-p_0}{1-p_t}.$$

Hence, the problem simplifies to the maximization of

$$\int_0^T \frac{1-p_0}{1-p_t}\left[w_t - c\left(u_t\right) - v\right]\mathrm{d}t - \frac{1-p_0}{1-p_T}k,^{12} \tag{3}$$

given w, over all measurable $u : [0,T] \to [0,\bar{u}]$, subject to (1).

Considering this last maximization, there appears to be two drivers to the worker's effort. First, if the wage falls short of the outside option (i.e. if $w_t - v$ is negative), he has an incentive to exert high effort to stop incurring this flow deficit. Achieving this is more realistic when the belief p is high, so that this incentive should be strongest early on, when he is still sanguine about the project. Second, if the penalty is strictly positive, there is an incentive to succeed so as to avoid paying it. This incentive should be most acute when the deadline looms close, as success becomes unlikely to arrive without effort. Taken together, this suggests an effort pattern that is a convex function of time. However, this ignores that, in equilibrium, the wage reflects the agent's expected effort. As a result, we shall show that the worker's effort pattern is the exact opposite of what this first intuition suggests.

[12]Note that we have replaced $p_t v$ by the simpler v in the bracketed term inside the integrand. This is because

$$\int_0^T \frac{p_t}{1-p_t} v\mathrm{d}t = \int_0^T \frac{v}{1-p_t}\mathrm{d}t - vT,$$

and we can ignore the constant vT, at least until Section 5.3, where the deadline is endogenized.

2.2 The social planner

Before solving the agent's problem, we start by analyzing the simpler problem faced by a social planner. What is the expected value of a breakthrough? Recall that the value of a realized breakthrough is normalized to one. But a breakthrough only arrives with instantaneous probability $p_t(\lambda + u_t)$, as it occurs at rate $\lambda + u_t$ only if $\omega = 1$. Therefore, the planner maximizes

$$\int_0^T \frac{1-p_0}{1-p_t}\left[p_t(\lambda+u_t) - v - c(u_t)\right]\mathrm{d}t - k\frac{1-p_0}{1-p_T}, \tag{4}$$

over all measurable $u : [0,T] \to [0,\bar{u}]$, given (1). As for most of the optimization programs considered in this paper, we apply Pontryagin's maximum principle to get a characterization. The proof of the next lemma and of all formal results can be found in the appendix. A strategy u is *extremal* if it only takes extreme values: $u_t \in \{0,\bar{u}\}$, for all t.

Lemma 2.1 *The optimum exists. At any optimum:*

1. *effort u is monotone (in t); it is non-increasing if and only if the deadline exceeds some finite length;*

2. *in addition, in the case of linear cost, the optimal strategy is extremal and maximum effort precedes zero effort if and only if $v > \alpha\lambda$;*

3. *if effort is non-increasing, so is the marginal product $p(\lambda+u)$; if it is non-decreasing, then the marginal product is single-peaked in the convex cost case, and piecewise decreasing with at most one upward jump in the linear cost case.*

Monotonicity of effort can be roughly understood as follows, in the linear cost case. There are two reasons why effort can be valuable: because it helps reduce the time over which the waiting cost v is incurred, and because it helps avoid paying the penalty k. The latter encourages late effort, the former early effort, provided the belief is high. But, in the absence of discounting, it makes little sense to work early if one plans on stopping before working eventually again: it is then better to postpone exerting this effort to this later stage where no effort is planned. Hence, if effort is exerted eventually, it is exerted only at the end. Conversely, if the penalty does not motivate late effort, effort is only exerted at the beginning.

Because the belief p is decreasing over time, note that the marginal product is decreasing whenever effort is decreasing, but the converse need not hold (as the product $p(\lambda + u)$ might vary in either direction). The interval over which the marginal product is non-decreasing can be empty, or the entire horizon. Conversely, it is straightforward to construct examples in which effort is increasing, and the marginal product is first increasing, then decreasing. Note that, for the critical deadline mentioned in the first part of the lemma, effort is constant.

With linear cost, whether effort is non-increasing or non-decreasing depends only on the sign of $v - \alpha\lambda$. This does not contradict the first part of the lemma: for long enough deadlines, effort is constant (and 0) if $v \leq \alpha\lambda$, and first maximal then zero if $v > \alpha\lambda$. Note that neither the initial belief (p^0), nor the terminal cost (k) affect whether maximum effort is exerted first or last. Of course, they affect the total amount of effort, but given this amount, they do not affect its timing. The role of the sign $\alpha\lambda - v$ in the ordering of these intervals can be seen as follows: consider exerting some bit of effort now or at the next instant (thus, keeping the total amount of planned effort fixed); by waiting, a loss $v\mathrm{d}t$ is incurred; on the other hand, with probability $\lambda\mathrm{d}t$, the marginal cost of this effort, α, will be saved. Therefore, if $v > \alpha\lambda$, it is better to work early than late, if at all. We shall maintain this assumption from now on.

Assumption 2.2 *In the linear cost case, the parameters α, v and λ are such that*

$$v > \alpha\lambda.$$

Under this assumption, effort can be efficient even far from the deadline. An example of such a path is given by the left panel in Figure 1. The right panel gives the corresponding path for the value of output (i.e., $p_t(\lambda + u_t)$).

Whether effort is exerted at the deadline depends on how pessimistic the planner is at that point. By standard arguments (see Appendix A), full effort is exerted then if and only if

$$p_T(1 + k) \geq \alpha.^{13} \tag{5}$$

This states that the expected marginal social gains from effort (success and penalty avoidance) should exceed the marginal cost. If the social planner becomes too pessimistic, he "gives up"

[13] In the convex case, the social planner exerts an effort level that solves $p_T(1 + k) = c'(u_T)$.

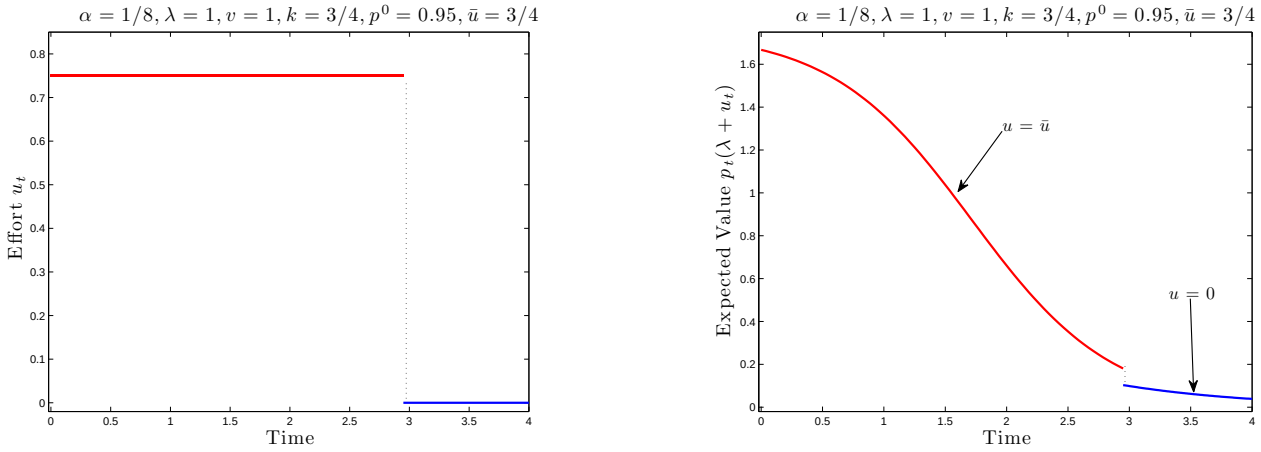

Figure 1: Effort and expected value at the social optimum

before the end. Note that the flow loss v no longer plays a role at that time, as the terminal (lump-sum) penalty overshadows any such flow cost.

It is straightforward to solve for the switching belief in the linear case. This belief decreases in α and increases in v and k: the higher the cost of failing, or the lower the cost of effort, the longer effort is exerted. More generally, we have the following result.

Lemma 2.3

1. *Both in the convex and linear cost case, the final belief decreases with the deadline.*
2. *Total effort exerted increases with the deadline*
   (a) *in the linear case, if and only if* $\lambda(1+k) < v$;
   (b) *in the convex case, if*
   $$\max_u [(\lambda + u)(1+k) - c(u)] < v.$$

Hence, total effort need not increase with the deadline; the sufficient condition given in the convex case (implying $\lambda(1+k) < v$) is not necessary; weaker, but less concise conditions can be given for the convex case, as well as examples in which total effort decreases with the deadline.

3 The agent's problem: The role of wages

Before solving for an equilibrium in which wages are market-based, consider an arbitrary *exogenous* (integrable) wage path $w : [0, T] \to \mathbb{R}_+$. The agent's problem given by (3) differs from the social planner's in two respects: the agent disregards the expected value of a success (in particular, at the deadline), which increases with effort; he takes into account future wages, which are less likely to be pocketed if more effort is exerted. We start with a "technical" result.

Lemma 3.1 *A solution to the maximization problem* (3) *exists. With convex cost, the optimal trajectory p is unique; with linear cost, if p_1 and p_2 are optimal trajectories, and $p_{1,t} \neq p_{2,t}$ over some interval $[a, b] \in [0, T]$, then $w_t = v - \alpha\lambda$ (a.e.) on $[a, b]$.*

That is, there is a unique solution (in terms of trajectories and hence control) in the convex case, and multiplicity in case of linear cost is confined to time intervals over which the wage is equal to a particular constant. While this last case might appear non-generic, it plays an important role in the equilibrium analysis nonetheless.

3.1 Level of effort

What determines the instantaneous *level* of effort? Transversality implies that, at the deadline, the agent exerts an effort level that solves

$$p_T k = c'(u_T).^{14}$$

This is similar to the social planner's trade-off, except that the worker does not take into account the lump-sum value of success (compare with eqn. (5)). Hence, given p_T, his effort level is smaller.

It follows from Pontryagin's theorem that the amount of effort put in at time t solves

$$c'(u_t) = -\int_t^T (1 - p_t) \frac{p_s}{1 - p_s} [w_s - c(u_s) - v] \, \mathrm{d}s + (1 - p_t) \frac{p_T}{1 - p_T} k. \tag{6}$$

The left-hand side is the instantaneous marginal cost of effort. The marginal benefit (right-hand side) can be understood as follows. Conditioning throughout on reaching time t, the expected

[14] In the linear case, this must be understood as: the agent chooses $u = \bar{u}$ if and only if $p_T k \geq \alpha$, and chooses $u = 0$ otherwise.

flow utility over some interval $\mathrm{d}s$ at time $s \in (t, T)$ is

$$\mathbb{P}[\tau \geq s](w_s - c(u_s) - v)\,\mathrm{d}s.$$

From (2), recall that

$$\mathbb{P}[\tau \geq s] = \frac{1-p_t}{1-p_s} = (1-p_t)\left(1 + \frac{p_s}{1-p_s}\right);$$

that is, effort at time t affects the probability that time s is reached only through the likelihood ratio $p_s/(1-p_s)$. From (1),

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{p_t}{1-p_t} = -\frac{p_t}{1-p_t}(\lambda + u_t),$$

and so a slight increase in u_t decreases the likelihood ratio at time s precisely by $-p_s/(1-p_s)$. Combining, such an increase changes expected revenue from time s by an amount

$$-(1-p_t)\frac{p_s}{1-p_s}[w_s - c(u_s) - v]\,\mathrm{d}s,$$

and integrating over s (including $s = T$) yields the result.

The trade-off captured by eqn. (6) illustrates a key feature of career concerns in this model. Information is coarse: either a success is observed or not. This structure only allows the agent to affect the probability that the relationship terminates.[15] As is intuitive, increasing the wedge between the future rewards from success and failure $(v - w_s)$ encourages high effort, *ceteris paribus*. Higher wages in the future depress incentives to exert effort today, as they reduce this wedge. Given eqn. (6), it is straightforward to prove the following lemma, whose proof is omitted.

Lemma 3.2 *Consider the case of convex cost, and fix $T > 0$. Let w, w' be two wage paths, and denote by p, p' the corresponding beliefs. Then $w_t < w'_t$ for all $t \in [0, T]$ implies that $p_T < p'_T$.*

That is, if wages are higher throughout, total effort, as measured by terminal belief, is lower. However, because of the transversality condition, the effort paths must cross at some point.

[15] This is a key difference with Holmström's model in which signals and posterior beliefs are one-to-one. Although the log-likelihood ratio is linear in effort, as is the principal's posterior belief in Holmström's model, here there is no scope for future wage to adjust linearly in output, so as to provide incentives that would be independent of the wage level itself. As we will see in Section 4, the level of future compensation does affect incentives to put in effort in equilibrium.

Higher wages far in the future have a smaller effect on current-period incentives for two reasons, as is clear from eqn. (6). The game is less likely to last until then, and conditional on reaching these times, the agent's effort is less likely to be productive (as the probability of a high type then is very low).[16] Hence, as we shall see in Section 5.1, it is not true that pushing wages back, holding the total wage bill constant, necessarily depresses total effort.

Similarly, a higher penalty for termination or a lower cost of effort provide stronger incentives.

3.2 Timing of effort

To understand how effort is allocated over time, let us differentiate eqn. (6). (See the proof of Proposition 3.3.) We obtain the following equation:

$$p_t \cdot \underbrace{c\left(u_{t+\mathrm{d}t}\right)}_{\text{cost saved}} + \underbrace{p_t\left(v - w_t\right)}_{\text{wage premium}} + \underbrace{c''\left(u_t\right)\dot{u}_t}_{\text{cost smoothing}} = \underbrace{p_t\left(\lambda + u_t\right)}_{\text{Pr. of success at } t} \cdot c'\left(u_t\right). \tag{7}$$

The right-hand side of eqn. (7) represents the gains from shifting an effort increment $\mathrm{d}u$ from the time interval $[t, t+\mathrm{d}t)$ to $[t+\mathrm{d}t, t+2\mathrm{d}t)$ (*backloading*): the agent saves the marginal cost of this effort increment $c'(u_t)\mathrm{d}u$ with instantaneous probability $p_t(\lambda + u_t)\mathrm{d}t$ –the probability with which this additional effort will not have to be carried out. The left-hand side of eqn. (7) represents the gains from exerting this additional effort early instead (*frontloading*): the agent increases by $p_t\mathrm{d}u$ the probability that the entire cost of tomorrow's effort $c(u_{t+\mathrm{d}t})\mathrm{d}t$ will be saved. He also increases at the same rate the probability that he gets the "premium" $(v - w_t)\mathrm{d}t$ an instant earlier. Finally, if effort is increasing at time t, frontloading improves the workload balance, which is worth $c''(u)\mathrm{d}u\mathrm{d}t$. This yields the arbitrage eqn. (7).[17]

With linear cost, cost-smoothing is irrelevant, and since this is the only term that is not proportional to the belief p_t, the condition simplifies: frontloading effort is preferred if the wage

[16] Note also that, although learning is valuable, the value of information cannot be read off this first-order condition directly: the maximum principle is an "envelope theorem," and as such does not explicitly reflect how future behavior adjusts to current information.

[17] Note that all these terms are "second order" terms. Indeed, to the first order, it does not matter whether effort is slightly higher over $[t, t+\mathrm{d}t)$ or $[t+\mathrm{d}t, t+2\mathrm{d}t)$. Similarly, while doing such a comparison, we can ignore the impact of the change on later revenues, as it is the same under both scenarios.

premium exceeds the value of luck in cost units:

$$v - w_t \geq \alpha\lambda. \tag{8}$$

That the belief is irrelevant to the timing of effort (absent the cost-smoothing motive) is intuitive: if the state is 0, the cost of the effort increment is incurred either way, so that the comparison can be conditioned on the event that the state is 1.

What do we learn from eqn. (8) regarding the dynamics of effort in the linear case? First, note that, unless $w = v - \alpha\lambda$ holds identically over some interval, effort is extremal. Second, suppose that w is increasing. Then the left-hand side decreases over time, and the agent prefers frontloading up to some critical time, after which backloading becomes optimal (the critical time might be 0 or T). This does not imply that his effort is non-increasing; rather, if he puts in low effort, he must do so over some intermediate time interval.

Similarly, if wages decrease over time, the agent first backloads, then frontloads effort. That is, if he ever puts in high effort, he will do so in some intermediate phase.

The same observations can be made by considering eqn. (7) for the convex case, though effort will not be extremal. We summarize this discussion with the following proposition.

Proposition 3.3

1. *If w is decreasing, u is a quasi-concave function of time; if w is increasing, u is quasi-convex; if w is constant, u is monotone.*

2. *With linear cost and strictly monotone wages, the optimal strategy is extremal.*

To conclude, even when wages are monotone, the worker's incentives need not be so. Not surprisingly then, equilibrium wages, as determined in Section 4, will not be either.

3.3 Comparison with the social planner

Note that eqn. (8) reduces to the social planner's trade-off when $w_t = 0$ (see Lemma 2.1.2). Hence, the social planner's arbitrage condition coincides with the agent's if there were no wages. The same holds with convex cost, although the social planner internalizes the value of possible

success at future times. This is because the corresponding term in eqn. (4) can be "integrated out,"

$$\int_0^T \frac{1-p_0}{1-p_t} p_t \left(\lambda + u_t\right) \mathrm{d}t = -(1-p_0) \int_0^T \frac{\dot{p}_t}{(1-p_t)^2} \mathrm{d}t = (1-p_0) \ln \frac{1-p_T}{1-p_0},$$

so that it only affects the final belief, and hence the transversality condition. But the agent's and the social planner's transversality conditions do not coincide, even when $w_s = 0$. As mentioned, the agent fails to take into account the value of a success at the last instant. Hence, his incentives at T, and hence his strategy for the entire horizon, differ from the social planner's. The agent works too little, too late.

The next proposition formalizes this discussion. Given w, denote by p^* the belief trajectory solving the agent's problem, and p^{FB} the corresponding trajectory for the social planner.

Proposition 3.4 *Consider the convex cost case, and fix $T > 0$ and $w > 0$.*

1. *The agent's aggregate effort is lower than the planner's, i.e. $p_T^* > p_T^{FB}$. Furthermore, instantaneous effort at any t is lower than the planner's, given the current belief p_t^*.*

2. *Suppose the planner's aggregate effort is constrained so that $p_T = p_T^*$. Then the planner's optimal trajectory p lies below the agent's trajectory, i.e. for all $t \in (0, T)$, $p_t^* > p_t$.*

The first part states that aggregate effort is too low, but also instantaneous effort, *given* the agent's belief. Nevertheless, as a function of calendar time, effort might be higher for the agent at some dates, because the agent is more optimistic than the social planner at that point. The next example (Figure 2) illustrates this phenomenon in the case of equilibrium wages.

The second part of the proposition asserts that, fixing the aggregate effort chosen by the agent, this effort is allocated too late relative to what would be optimal: the prospect of collecting future wages encourages "procrastination."

The same result holds in the linear case, up to the strictness of the inequalities: if the agent's optimum effort is maximum throughout, he is working just as much as the social planner.

4 Equilibrium

Suppose now that the wage is set by a principal (or *market*) without commitment power. The principal does not observe the agent's past effort, but only the lack of success. Non-commitment

motivates the assumption that wage equals expected marginal product, i.e.

$$w_t = \mathbb{E}_t[p_t(\lambda + u_t)],$$

where p_t and u_t are the agent's belief and effort, respectively, at time t, given his private history of past effort (of course, it is assumed that he has had no successes so far), and the expectation reflects the principal's beliefs regarding the agent's history (in case the agent mixes).[18] However, given Lemma 3.1, the agent will not use a chattering control (i.e., a distribution over measurable functions u), but rather a single function (unless the cost is linear and $w = v - \alpha\lambda$ over some interval, but even then the multiplicity is limited to the distribution of effort over this interval).[19] Therefore, we may write

$$w_t = \hat{p}_t(\lambda_t + \hat{u}_t), \tag{9}$$

where $\hat{p}_t$ and $\hat{u}_t$ denote the belief and anticipated effort at time t, as viewed from the principal.

In equilibrium, expected effort must coincide with actual effort.

Definition 4.1 *An equilibrium is a measurable function u and a wage path w such that:*

1. *u is a best-reply to w given the agent's private belief p, which he updates according to (1);*
2. *the wage equals the marginal product, i.e. (9) holds for all t;*
3. *beliefs are correct on the equilibrium path, that, is, for every t,*

$$\hat{u}_t = u_t,$$

and therefore, also, $\hat{p}_t = p_t$ at all $t \in [0, T]$.

[18] A lot is buried in this assumption. In discrete time, if $T < \infty$, and under assumptions that guarantee uniqueness of the equilibrium (see below), non-commitment *implies* that wage is equal to marginal product in equilibrium, by a backward induction argument, assuming that the agent and the principal share the same prior. Alternatively, this is the outcome if a sequence of short-run principals (at least two at every instant), whose information is symmetric and no worse than the agent's, compete through prices for the agent's services. We shall follow the literature by directly *assuming* that wage is equal to marginal product.

[19] If there are such time intervals (as equilibrium existence will require for many parameter values), the multiplicity of best-replies over this interval is of no importance: the expected effort at any time during this interval, as well as the aggregate effort over this interval will be uniquely determined, and the agent is indifferent over all effort levels over this time interval; the multiplicity does not affect wages, effort or beliefs before or after such an interval.

Note that, if the agent deviates, the market will typically hold incorrect beliefs.

To understand the structure of equilibria, consider the following example, illustrated in Figure 2. Suppose that the principal expects the agent to put in the efficient amount of effort, which decreases over time in this example. Accordingly, the wage paid by the firm decreases as well. The agent's best-reply, then, is quasi-concave: effort first increases, and then decreases (see left panel). The agent puts in little effort at the start, as he has no incentive "to kill the golden goose." Once wages come down, effort becomes more attractive, so that the agent increases his effort, before fading out as pessimism sets in. The market's expectation does not bear out: marginal product is single-peaked. In fact, it would decrease at the beginning if effort was sufficiently flat.

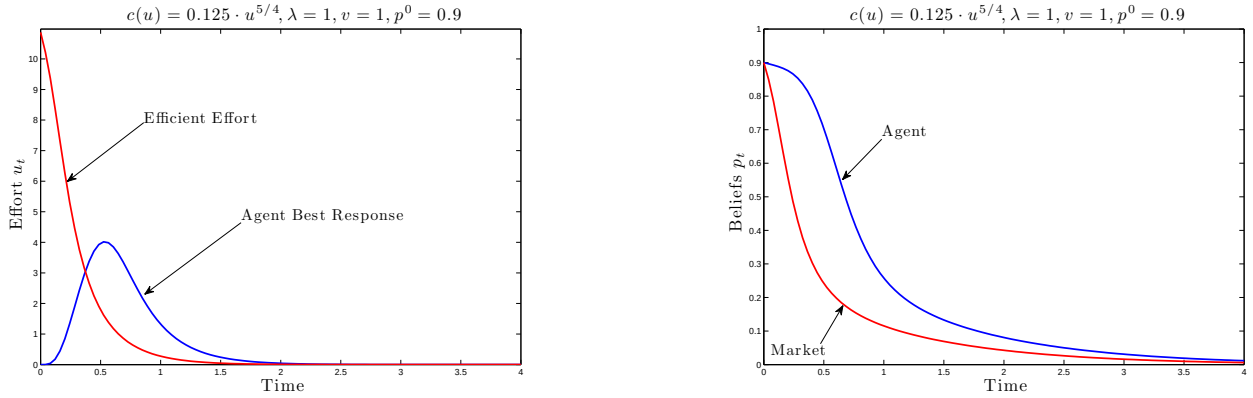

Figure 2: Agent's best-reply and beliefs to the efficient wage scheme

Eventually the agent exerts more effort than the social planner would. This is because the agent is more optimistic at those times, having worked less in the past (see right panel). Effort is always too low given the actual belief of the agent, but not necessarily given calendar time.

As this example makes clear, effort, let alone wages, are not monotone in general. However, it turns out that the equilibrium structure remains quite simple.

Theorem 4.2

1. *An equilibrium exists. It is unique in the linear case if $\alpha < k$, and in the convex case if*

$$c''(0) \geq \frac{1}{\lambda}\left(\frac{v}{\lambda} - k\right)\frac{p^0}{1-p^0}. \tag{10}$$

2. *In every equilibrium, (on path) effort is single-peaked, and the wage is non-decreasing in at most one interval. In the convex case, the wage is single-peaked.*

The complete proof is in Appendix C. A sketch of the argument for uniqueness is as follows: we know from Lemma 3.1 that the agent's best response to any wage yields a optimal unique trajectory p; *given* the value of the terminal belief p_T, we argue there is a unique path of effort and beliefs that is consistent with the equilibrium restriction on wages; we then look at the time required, along the equilibrium path, to drive beliefs from p^0 to p_T and prove that it is strictly increasing. Therefore, given the deadline T, there exists a unique value of p_T that can be reached in equilibrium. Condition (10) is sufficient to establish the last step in the convex case.

Equilibrium wages are not single-peaked in general for the linear case, and single-peakedness in the convex case relies on our assumption that the marginal cost is convex (as does the uniqueness proof). Figure 3 illustrates that this is not quite true otherwise (note that the cost is convex, but not the marginal cost). The mode of the wage lies to the left of the mode of effort: if the wage is increasing over time, it must be that effort is increasing, but not conversely.

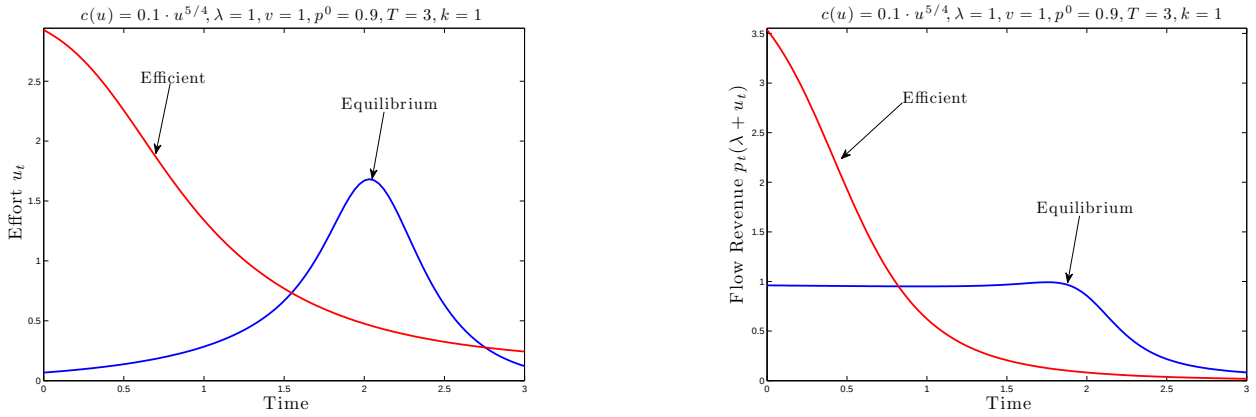

Figure 3: Effort and wages with convex costs

As stated in the theorem, there are simple sufficient conditions that guarantee equilibrium uniqueness (in addition to convexity of c'), for example assuming that the penalty k is large enough. It does not imply that there are multiple equilibria otherwise: we have been unable to construct any example of multiple equilibria.

How does this structure depend on parameters? Fixing all other parameters, if initial effort is increasing for some prior p^0, then it is increasing for higher priors –in particular, effort would be increasing throughout if the prior were 1. This highlights the fact that, although effort can be decreasing for other motives (a long enough deadline, for instance), growing pessimism plays a key role in turning increasing into decreasing effort.

Numerical simulations suggest that the payoff is single-peaked (with possibly interior mode) in p^0. This is a recurrent theme in the literature on reputation: uncertainty is the lever for reputational incentives. (Recall however that the payoff is net of the outside option, which is not independent of p^0; otherwise, it is increasing in p^0.)

A more precise description of the overall structure of the equilibrium can be given in the case of linear cost.

Proposition 4.3 *With linear cost, any equilibrium path consists of at most four phases, for some* $0 \leq t_1 \leq t_2 \leq t_3 \leq T$*:*

1. *during* $[0, t_1]$*, no effort is exerted;*
2. *during* $(t_1, t_2]$*, effort is interior, i.e.* $u_t \in (0, \bar{u})$*;*
3. *during* $(t_2, t_3]$*, effort is maximal;*
4. *during* $(t_3, T]$*, no effort is exerted.*

Any of these intervals might be empty.[20]

As stated in Proposition 4.3, any of the intervals might be empty, and it is easy to compute instances of each of the different possibilities.[21] If the deadline is very short, effort is first zero, then maximum. For deadlines of intermediate lengths, an intermediate phase kicks in, in which effort is interior. Finally, for long deadlines, a final phase appears, in which no effort is exerted.

[20] Here and elsewhere, the choices at the extremities of the intervals are irrelevant, and our specification is arbitrary in this respect.

[21] In particular, if $k < \alpha$, the agent never works at the deadline; if $1 + \alpha < v/\lambda$, and no effort is exerted at some point, it is then exerted until the end; if, contrary to our maintained assumption, $v/\lambda < \alpha$, the characterization simplifies to at most two intervals, with zero effort being followed by maximum effort.

When a phase with interior effort exists, effort grows, so as to keep the wage constant at $v - \alpha\lambda$, which guarantees that the agent is indifferent between all effort levels. It is continuous at t_1 (i.e., $\lim_{t \downarrow t_1} u(t) = 0$), but jumps up at time t_2 (assuming the third interval is non-empty). See Figure 4 for an example of effort (left panel) and corresponding wage dynamics (right panel). (The parameters are the same as those used in Figure 1 above.)

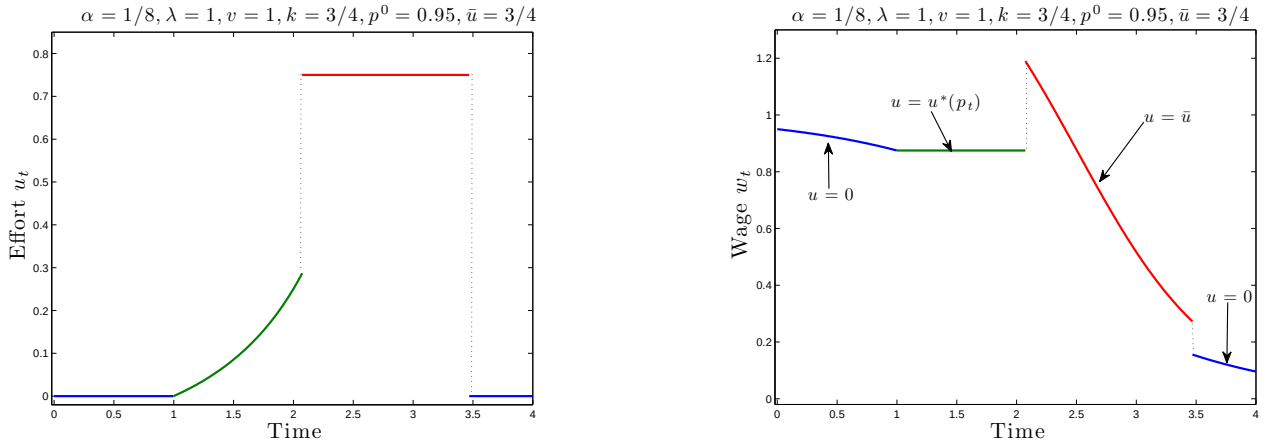

Figure 4: Effort and wages with linear cost

Note that we have not specified the worker's equilibrium strategy entirely, as we have not described his behavior following his own (unobservable) deviations. Yet it is not difficult to describe the worker's optimal behavior off-path, as it is the solution of the optimization problem studied before, for the belief that results from the agent's history, given the wage path.

The linear cost case provides a simple way to understand what drives incentives. On-path equilibrium effort is a function of belief and time. Define the boundaries $p_k : [0, T] \to [0, 1]$ that divide the state space into regions according to equilibrium effort: p_3 is the boundary below which all effort stops; $p_2 > p_3$ is the boundary at which maximum effort starts (that is, maximum effort is exerted between those two curves); and $p_1 > p_2$ is the boundary below which interior effort starts. By Proposition 4.3, the boundaries are each crossed at most once on path. It turns out that p_1 is independent of t: if interior effort is on path, it begins at a belief that is independent of the specific path. As for p_2 and p_3, their structure hinges on parameters. As the following figures illustrate, there are two circumstances in which effort is exerted: either effort is exerted because the belief is "right," given the remaining time, or because there is very little

time left. See right panel of 5.

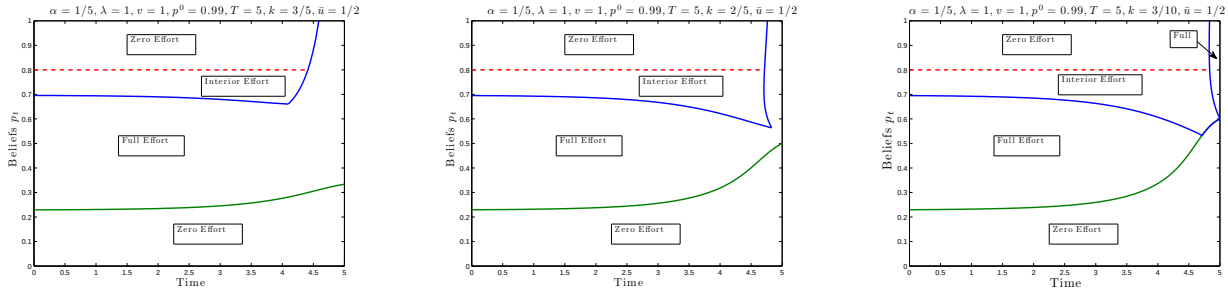

Figure 5: High ($k = .6$), medium ($k = .4$) and low k ($k = .3$)

Proposition 4.4 *For all $t \leq T$,*

1. *the no effort frontier $p_3(t)$ is decreasing in k and v. It is increasing in α and λ;*

2. *the full effort frontier $p_2(t)$ is decreasing in α, λ and $\bar{u}$. It is increasing in k and v.*

This result holds regardless of whether the full effort region is connected. It confirms the intuition that (in terms of beliefs) the agent works longer when the prize and the penalty are higher, and works less when the marginal cost of effort and the luck component are more significant.

One might wonder whether the penalty k is really hurting the worker. After all, it endows him with some commitment to work. In the linear cost case, simple algebra shows that increasing k increases the amount of work performed; furthermore, if parameters are such that working at some point is optimal, then the optimal (i.e. payoff-maximizing) termination penalty is strictly positive.

Finally, holding $\lambda + \bar{u}$ constant, we can interpret λ as a degree of contractibility of effort. Appending a linear cost for this amount of effort, we can ask whether aggregate effort increases with the level of contractible effort. Numerically, it appears that the optimal choice is always extremal, but choosing a high λ can be counter-productive: forcing the worker to maintain a high effort level prevents the agent to scale it back as it should be when the project appears to be unlikely to succeed. This lack of flexibility can be more costly than the benefits from direct oversight.

4.1 Discussion

The key driver behind the structure of equilibrium, as described in Theorem 4.2, is the strategic substitutability between effort at different dates. If more effort is expected "tomorrow," wages tomorrow will be higher in equilibrium, which depresses incentives, and hence effort "today." To be clear, there is substitutability between effort at different dates for the social planner as well, because higher planned effort tomorrow makes effort today less useful, but wages provide an additional channel.

This substitutability appears to be new to the literature on career concerns. As we have mentioned, in the model of Holmström, the optimal choices of effort today and tomorrow are entirely independent, and because the variance of posterior beliefs is deterministic with Gaussian signals, the optimal choice of effort is deterministic as well. Dewatripont, Jewitt and Tirole emphasize the complementarity between expected effort and incentives for effort (at the same date): if the agent is expected to work hard, failure to achieve a high signal will be particularly detrimental to tomorrow's reputation, which provides a boost to incentives today. Substitutability between effort today and tomorrow does not appear in their model, because it is primarily focused on two periods, and at least three are required for this effect to appear. With two periods only, there are no reputation-based incentives to exert effort in the second (and final) period anyhow.[22]

Conversely, complementarity between expected and actual effort at a given time is not discernible in our model, because time is continuous. But this complementarity appears in discrete time versions of it, and three-period examples can be constructed that illustrate this point.

As a result of this novel effect, effort and wage dynamics display original features. Both in Holmström's and in Dewatripont, Jewitt and Tirole's models, the wage is a supermartingale. Here instead, effort can be first increasing, then decreasing, and wages can be decreasing first, increasing then, and decreasing again. These dynamics are not driven by the deadline.[23] They are not driven either by the fact that, with two types, the variance of the public belief need not

[22] It is worth noting that this substitutability does not require the multiplicative structure that we have assumed. If instead, we had posited that instantaneous success probability is given by $\lambda\chi_{\omega=1} + u_t$, effort would be similarly single-peaked, as is readily verified.

[23] This is unlike for the social planner, for which we have seen that effort is non-increasing with an infinite horizon, while it is monotone (and possibly increasing) with a finite horizon.

be monotone.[24] The same pattern emerges in examples with an infinite horizon, and a prior $p^0 < 1/2$ that guarantees that this variance only decreases over time, see Figure 6. As equation (6) makes clear, the provision of effort is tied to the capital gain that the agent obtains if he breaks through. Viewed as an integral, this capital gain is too low early on, it increases over time, and then declines again, for a completely different reason. Indeed, this wedge depends on two components: the wage gap, and the impact of effort on the (expected) arrival rate of a success. Therefore, high initial wages would depress the first component, and hence kill incentives to exert effort early on. The latter component declines over time, so that eventually effort fades out.

Similarly, one might wonder whether the possibility of non-increasing wages in this model is driven by the fact that effort and wage paths are truly conditional paths, inasmuch as they assume that the agent has not succeeded. Yet it is not hard to provide numerical examples which illustrate that the same phenomenon arises for the unconditional flow payoff (v in case of a past success), though the increasing cumulative probability that a success has occurred by a given time, leading to higher payoffs (at least if $w_t < v$) dampens the downward tendency.

We have assumed –as is usually done in the literature– that the agent does not know his own skill. The analysis of the game in which the agent is informed is simple, as there is no scope for signaling. An agent who knows that his ability is low has no reason to exert any effort, so we focus on the high-skilled agent. Because of the market's declining belief, the same dynamics arise, and this agent's effort is single-peaked (in general, it is not monotone). The high-skilled agent's effort need not vanish, though expected effort from the market's point of view does so.

Dynamics of the complexity just described are not often observed in practice: while it is difficult to ascertain effort patterns, wages do typically go up over time. See Abowd, Kramarz and Margolis (1999), Murphy (1986) and Topel (1991) among others, and Hart and Holmström (1987) and Lazear and Gibbs (2007) for surveys. Lazear (1981) obtains a positive impact of wages on seniority by (among others) assuming that the worker's outside option is increasing over time, and also derives the optimal deadline, or retirement age (Lazear, 1979). Nevertheless, it is not hard to find examples in which compensation goes down with repeated failure. For instance, the income of junior analysts is proportional to the number of job assignments they are being given, so that struggling analysts end up earning lower wages. Similar patterns arise

[24]Recall that, in Holmström's model, this variance decreases (deterministically) over time, which plays an important role in his results.

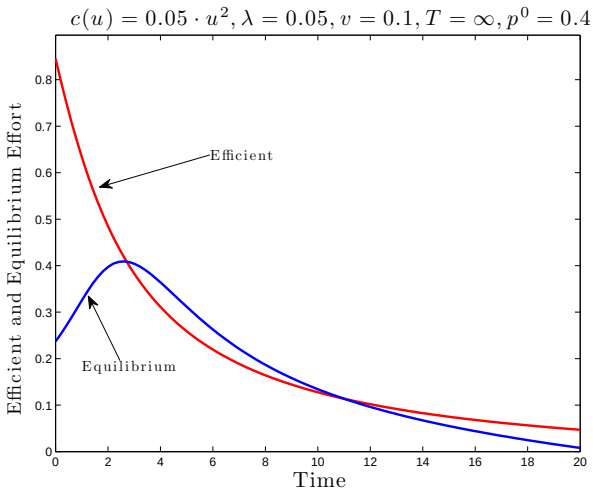

Figure 6: The same pattern in the case of $T = \infty$, $p^0 < 1/2$

in the art and entertainment industries.

Furthermore, our model provides insights into what labor market arrangements are likely to explain other prominent features of actual labor contracts. These include signing bonuses, rigid wages, and severance pay. In the next section, we shall consider three such possibilities: one-sided commitment by the principal (which is typically ruled out in career concerns models), better effort-monitoring technology, and endogenously determined deadlines. Before considering alternative labor market arrangements, we conclude this section by arguing that our findings are robust to some of our specific modeling assumptions.

4.2 Robustness

Undoubtedly, our model has stylized features: all uncertainty is resolved after one breakthrough, there is no learning-by-doing, and the quality of the project cannot change over time. We briefly consider three extensions, each of which introduces one of these features, and embeds our baseline model as a special case. We argue that none of these features is critical to our main findings.

4.2.1 Multiple breakthroughs

Suppose that one breakthrough does not resolve all uncertainty. Specifically, assume that there are three states of the world, $\omega = 0, 1, 2$, and two consecutive projects. The first one can be

completed if and only if the state is not 0; assume (instantaneous) arrival rates of $\lambda_1 + u_t$ and $\lambda_2 + u_t$, respectively, conditional on $\omega = 1$ or $\omega = 2$; if the first project is completed, an observable event, the agent tackles the second one, which in turn can only be completed if $\omega = 2$; assume again an arrival rate of $\lambda_2 + u_t$ if $\omega = 2$. Suppose that the horizon is infinite for both projects.

Such an extension can be solved by "backward induction." Once the first project is completed, the continuation game reduces to the game of Section 4. The value function of this problem then serves as continuation payoff to the first stage. While this value function cannot be solved in closed-form, it is easy to derive the solution numerically. The following example illustrates the structure of the solution. The parameters are $v = 1, \alpha = 1/2, \mathbb{P}[\omega > 0] = 0.85, \mathbb{P}[\omega = 2 \mid \omega > 0] = 0.75, \lambda_2 = 1, \lambda_1 = 0.6, c(u) = u^2/8$. See Figure 7. The left panel shows effort and wages during the first stage. As is clear, the same pattern as in our model emerges: effort is single-peaked, and as a result, wages can be first decreasing, then single-peaked.

The right panel shows how efforts and beliefs evolve before and after the first success. The green curves represent the equilibrium belief that $\omega = 2$, before and after the success (the light green curve is the belief as long as no success has occurred, and the dark green one the belief right after a success has occurred); the blue curves are equilibrium effort (the light blue curve is effort as long as no success has occurred, the dark blue one is the effort right after a success). Note that effort at the start of the second project is also single-peaked as a function of the time at which this project is started (the later it is started, the more pessimistic the agent at that stage, though his belief has obviously jumped up given the success).

4.2.2 Learning-by-doing

Memorylessness is a very stark, if convenient property of the exponential distribution. This means that past effort plays no role in the probability of instantaneous breakthrough, conditional on the state. Surely, in many applications, agents learn from the past not only about their skill levels, but about the best way to achieve a breakthrough. While considering such learning-by-doing formally is beyond the scope of this paper, it is simple easy enough to gain some intuition from numerical simulations. We model human capital accumulation as in Doraszelski (2003). The evolution of human capital is given by

$$\dot{z}_t = u_t - \delta z_t, \quad z_0 = 0,$$

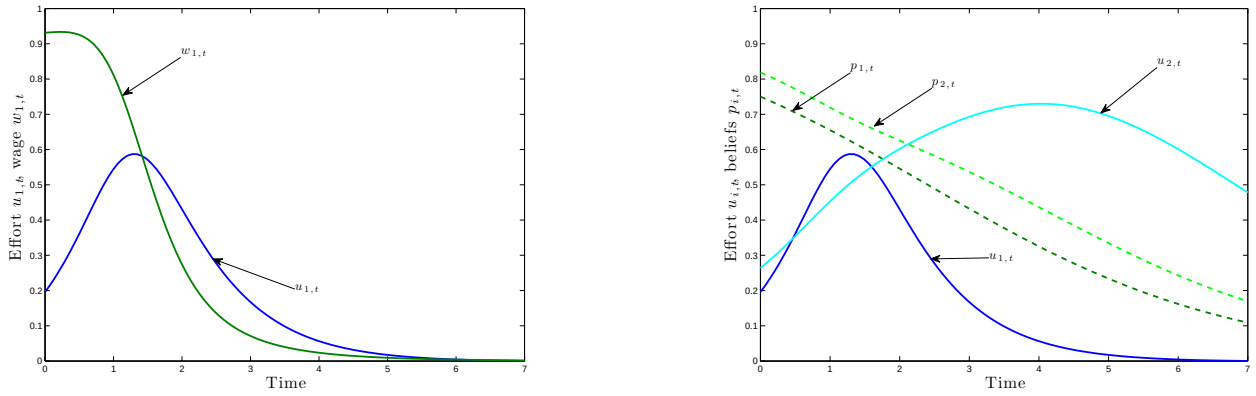

Figure 7: Efforts and beliefs with two breakthroughs

while its productivity is

$$h_t = u_t + \rho z_t^{\phi}.$$

That is, the probability of success over the interval $[t, t + \mathrm{d}t)$ is $(\lambda + h_t)\mathrm{d}t$, given human capital h_t and effort u_t. Here δ, ρ and ϕ are positive constants that measure how fast human capital depreciates, its importance relative to instantaneous effort, and the returns to scale from human capital. Not surprisingly, the main new feature is a spike of effort at the beginning, whose

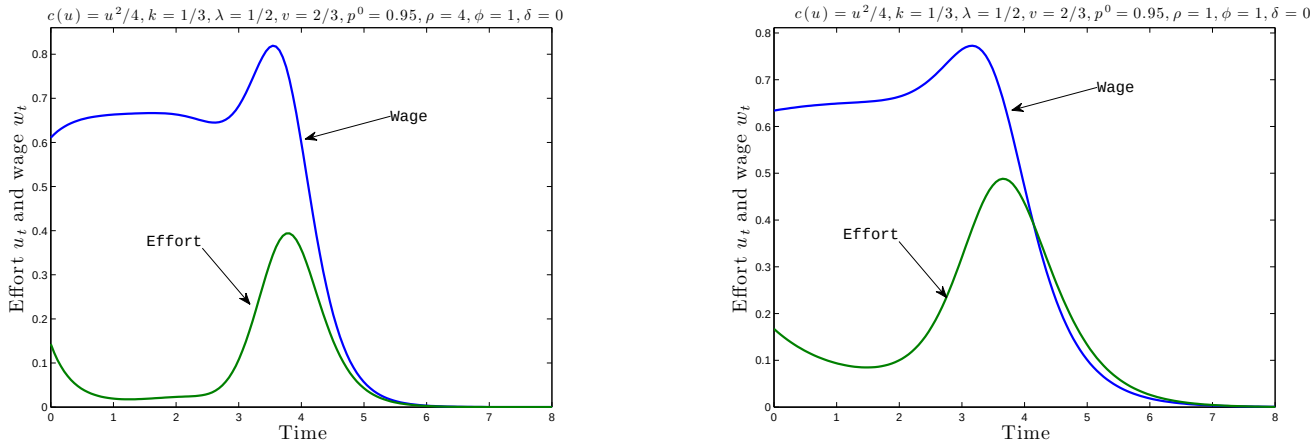

Figure 8: Two possible configurations with learning-by-doing

purpose is to build human capital. This spike might lead to decreasing initial effort, before it

becomes single-peaked (Figure 8 illustrates), though this need not be the case. Beyond this new twist, features from the baseline model appear quite robust.

4.2.3 Changing state

Suppose finally that, unbeknownst to the agent and the principal, the state of the world is reset at random times, exponentially distributed at rate $\rho > 0$; whenever it is reset, the state is reset to 1 with probability $p^* \in (0,1)$.[25] In our environment, this is "equivalent" to the stationary version developed by Holmström in the Gaussian case, though we do not restrict attention to steady states. Specifically, suppose that with instantaneous probability $\rho > 0$ the ability is reset, in which case it is high with probability p^*. Such an event remains unobserved by all parties. As before, a breakthrough ends the game, and the environment remains the same as before, with linear cost (and $v > \alpha\lambda$, as before) and an infinite horizon. (Thus, the baseline model with linear cost and $T = \infty$ is a special case in which $\rho = 0$.)

By Bayes' rule, the (agent's) belief p obeys

$$\dot{p}_t = \rho\left(p^* - p_t\right) - p_t\left(1 - p_t\right)\left(\lambda + u_t\right), \quad p_0 = p^0,$$

and this is the same as the principal's belief in equilibrium. The equilibrium is unique, and effort and belief converge to some limiting value, which is independent of the prior, and decreasing in the eventual effort level u, as follows. (The proof of the following is available from the authors.)

Proposition 4.5 *There exists $\alpha\lambda < \underline{v} < \bar{v}$ and $0 < \underline{p} < \bar{p} < p^*$ such that, if:*

1. *$v > \bar{v}$, effort is eventually maximum, and p tends to a limit below $\underline{p}$;*
2. *$v \in (\underline{v}, \bar{v})$, effort is eventually interior, with p tending to a limit in $(\underline{p}, \bar{p})$;*
3. *$v < \underline{v}$, effort is eventually zero, and p tends to a limit above $\bar{p}$.*

The higher the value, the more effort is exerted, the lower is the asymptotic belief. This eventual belief is non-decreasing in p^* and ρ, and non-increasing in $\bar{u}$. It is decreasing in λ when effort is extremal, but increasing otherwise. As is easy to check, asymptotic effort (or stationary effort if

[25] This specification bears a close similarity to Board and Meyer-ter-Vehn (2010a,b), though it also differs in some key respects; among others, the state-resetting process is exogenous and information is symmetric (on path).

$p^0 = p^*$) is decreasing in α, the marginal cost of effort, and in λ, the "luck" component of the arrival of breakthroughs. Comparative statics of effort with respect to p^* and ρ are ambiguous.

Note that, if the prior belief is below the limiting value, effort and hence wage can be increasing over time. (It is easy to construct examples in which wage increases throughout, see right panel of Figure 9, but it need not be so, see left panel.) It would be interesting to consider the game that does not end with a success, but rather continues with a value reset at the prior of 1 (which immediately starts declining towards p^*), but we have not pursued this here.

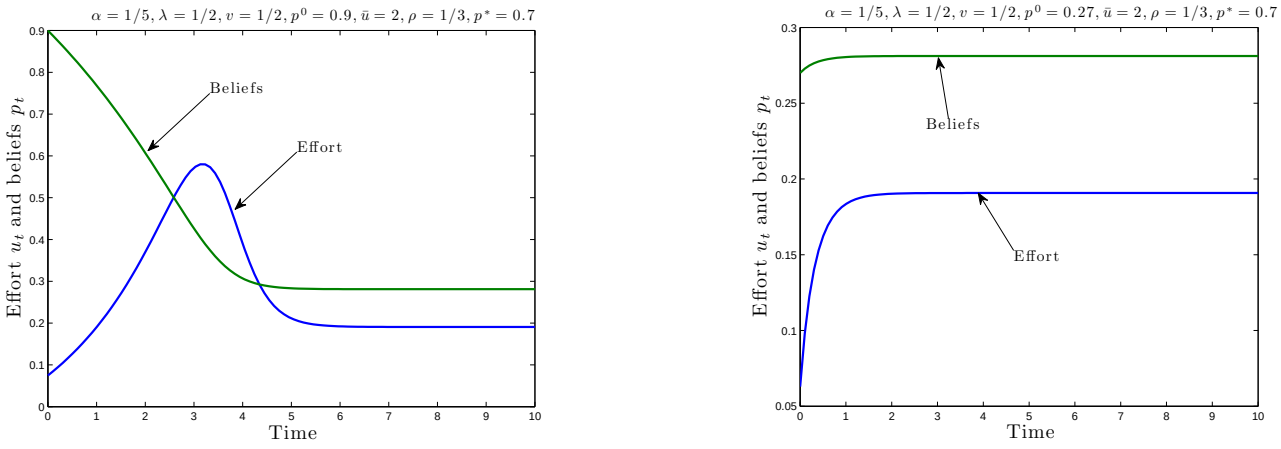

Figure 9: Changing states: two possible configurations

5 Alternative labor market arrangements

Throughout the new two subsections, attention is restricted to the case of linear cost.

5.1 Commitment by the principal

So far, as in most of the literature, the wage has been set equal to the worker's marginal product. This is a reasonable premise in a number of industries, in which lack of transparency or volatility in the firm's revenue stream might inhibit commitment by the firm to a particular wage scheme. Alternatively, it is sometimes argued that competition for the agent's services leads to a similar

outcome. Our model does not substantiate such a claim: if the principal can commit to a wage path, matters change drastically, even under competition.

In particular, if the principal could commit to a breakthrough-contingent wage scheme, the moral hazard problem would be solved entirely: under competition, the principal would offer the agent the value of a breakthrough, 1, in case of a success, and nothing otherwise.

If at least the principal could commit to a time-contingent wage scheme that involved payments after a breakthrough (even if such payments were not contingent on its realization), the moral hazard would also be mitigated. Promised payments at time t in case of no breakthrough should also be made in case of breakthrough, so as to eliminate all disincentives due to wages.

Here, we examine a weaker form of commitment. Suppose that the agent cannot be forced to stay with a principal (he can leave at any time), but the principal can commit to a wage path that is conditioned to no breakthrough. That is, wages can only be paid in the continued absence of a breakthrough. Think of the agent moving on once a breakthrough occurs, with the firm being unable to retain him in this event. At all times, other firms, who are symmetrically informed (they observe the wages paid by all past employers), compete by offering wage paths. The same deadline applies to all of them, i.e. the tenure clock is not reset. The deadline could be the agent's retirement age, for instance, so that switching firms does not affect the work horizon.

We write the principal's problem as of maximizing the agent's welfare subject to constraints. Formally, we solve the following optimization problem $\mathcal{P}$.[26] The principal chooses $u : [0,T] \to [0,\bar{u}]$ and $w : [0,T] \to \mathbb{R}_+$, integrable, to maximize $W(0,p^0)$, where, for any $t \in [0,T]$,

$$W(t,p_t) := \max_{w,u} \int_t^T \frac{1-p_t}{1-p_s}(w_s - v - \alpha u_s)\,\mathrm{d}s - k\frac{1-p_t}{1-p_T},$$

such that, given w, the agent's effort is optimal,

$$u = \arg\max_u \int_t^T \frac{1-p_t}{1-p_s}(w_s - v - \alpha u_s)\,\mathrm{d}s - k\frac{1-p_t}{1-p_T},$$

[26] We are not claiming that this optimization problem yields the equilibrium of a formal game, in which the agent could deviate in his effort scheme, leave the firm, and competing firms would have to form beliefs about the agent's past effort choices, etc. Given the well-known modeling difficulties that continuous time raises, we view this merely as a convenient shortcut. Among the assumptions that it encapsulates, note that there is no updating based on an off-path action (e.g., switching principals) by the agent.

and the principal offers as much to the agent at later times than the competition could offer at best, given the equilibrium belief,

$$\forall \tau \geq t : \int_{\tau}^{T} \frac{1-p_\tau}{1-p_s} \left(w_s - v - \alpha u_s\right) \mathrm{d}s - k \frac{1-p_\tau}{1-p_T} \geq W\left(\tau, p_\tau\right); \tag{11}$$

finally, the firm's profit must be non-negative,

$$0 \leq \int_{t}^{T} \frac{1-p_t}{1-p_s} (p_s(\lambda + u_s) - w_s) \mathrm{d}s.$$

Note that competing principals are subject to the same constraints as the principal under consideration: because the agent might ultimately leave them as well, they can offer no better than $W\left(\tau, p_\tau\right)$ at time τ, given belief p_τ. This leads to an "infinite regress" of constraints, with the value function appearing in the constraints themselves. To be clear, $W\left(\tau, p_\tau\right)$ is not the continuation payoff that results from the optimization problem, but the value of the optimization problem if it started at time τ.[27] Because of the constraints, the solution is not time-consistent, and dynamic programming is of little help. Fortunately, this problem can be solved, as shown in Appendix D.1–at least as long as $\bar{u}$ and v are large enough. Formally, we assume that

$$\bar{u} \geq \left(\frac{v}{\alpha\lambda} - 1\right) v - \lambda, \text{ and } v \geq \lambda(1+k).^{28} \tag{12}$$

Before describing its solution, let us provide some intuition. Recall the first-order condition (6) that determines the agent's effort. Clearly, the lower the future total wage bill, the stronger the agent's incentives to exert effort, which is inefficiently low in general. Now consider two times $t < t'$: to provide strong incentives at time t', it is best to frontload any promised payment to times before t', as such payments will no longer matter at that time. Ideally, the principal would pay what he owes upfront, as a "signing bonus." However, this violates the constraint (11), as an agent left with no future payments would leave right after cashing in the signing bonus.

But from the perspective of incentives at time t, backloading promised payments is better. To see this formally, note that the coefficient of the wage w_s, $s > t$, in eqn. (6) is (up to the

[27] Harris and Holmström (1982) impose a similar condition in a model of wage dynamics under incomplete information. However, because their model abstracts from moral hazard, constraint (11) reduces to a non-positive continuation profit condition.

[28] We do not know whether these assumptions are necessary for the result.

factor $(1-p_t)$) the likelihood ratio $p_s/(1-p_s)$, as explained before eqn. (6). Alternatively,

$$(1-p_t)\frac{p_s}{1-p_s} = \mathbb{P}[\omega=1|\tau\geq s]\,\mathbb{P}[\omega=1] = \mathbb{P}[\omega=1\cap\tau\geq s];$$

that is, effort at time t is affected by wage at time $s>t$ inasmuch as time s is reached and the state is 1: otherwise effort plays no role anyhow.

In terms of the firm's profit (or the agent's payoff), the coefficient placed on the wage at time s (see (3)) is

$$\mathbb{P}[\tau\geq s],$$

i.e., the probability that this wage is paid (or collected). Because players grow more pessimistic over time, the former coefficient decreases faster than the latter: backloading payments is good for incentives at time t. Of course, to provide incentives with later payments, those must be increased, as a breakthrough might occur until then, which would void them; but it also decreases the probability that these payments must be made in the same proportion. Thus, what matters is not the probability that time s is reached, but the fact that reaching those later times is indicative of state 0, which is less relevant for incentives. Hence, later payments depress incentives less.

To sum up: from the perspective of time t, backloading payments is useful; from the point of view of $t'>t$, it is detrimental, but frontloading is constrained by (11). Note that, as $T\to\infty$, the planner's solution tends to the agent's best response to a wage of $w=0$. Hence, the firm can approach first best by promising a one-time payment arbitrarily far in the future (and wages equal to marginal product thereafter). This would be almost as if $w=0$ for the agent's incentives, and induce efficient effort. The lump sum payment would then be essentially equal to $p^0/(1-p^0)$.

Note finally that, given the focus on linear cost, there is no benefit in giving the agent any "slack" in his incentive constraint at time t; otherwise, by frontloading slightly future payments, incentives at time t would not be affected, while incentives at later times would be enhanced.

The following result should come as no surprise.

Theorem 5.1 *The following is a solution to the optimization problem $\mathcal{P}$, for some $t\in[0,T]$. Maximum effort is exerted up to time t, and zero effort is exerted afterwards. The wage is equal to $v-\alpha\lambda$ up to time t, so that the agent is indifferent between all levels of effort up to then, and it is 0 for all times $s\in(t,T)$; a lump-sum is paid at time T.*[29]

[29]The wage path that solves the problem is not unique in general.

The proof is in Appendix D.1, and it involves several steps: we first conjecture a solution in which effort is first full (and the agent is indifferent), then nil; we relax the objective in program $\mathcal{P}$ to maximization of aggregate effort, and constraint (11) to a non-positive continuation profit constraint; we verify that our conjecture solves the relaxed program, and finally that it also solves the original program. In the last step we show that (a) given the shape of our solution, maximizing total effort implies maximizing the agent's payoff, and (b) the competition constraint (11) is slack at all times $t > 0$.

Hence, under one-sided commitment, high effort might be exerted throughout. This happens if T is short: a phase in which no effort is exerted exists if and only if the deadline is long enough. As for the worker's continuation utility, it is increasing throughout when v is low enough (precisely, $v < \alpha(\lambda + \bar{u})$).

The lump-sum at time T can be interpreted as severance pay. As time proceeds, the agent produces revenue that exceeds the flow wage collected: the liability recorded by the principal grows over time, shielding it from the threat of competition, as this liability will eventually be settled via this severance pay.

While it would be desirable to know to which extent these results extend to more general settings, note that it is all the more striking that the result obtains in this "linear" environment. Indeed, rigid wages and severance pay are usually attributed to risk aversion by the agent (see Azariadis, 1975, among many others). Commitment provides an alternative rationale.

5.2 Observable effort

5.2.1 Set-up

As is often the case for output, the inability to contract on effort might be attributable to the subjectivity in its measurement rather than to the impossibility of monitoring it. Often, co-workers have a fairly good idea of their colleagues' work ethic. Typically, the evaluation of a worker's immediate superior affects his future compensation. To understand the worker's incentives in such an environment, we assume now that effort is observed, so that the perceived ability and hence the wage evolve with the agent's actual effort level. As before, the principal pays upfront the value of the agent's expected output.

Because effort is monitored, the belief of the principal coincides with the agent's at all times,

on and off the equilibrium path, and the payment flow is given by

$$w_t = p_t(\lambda + \hat{u}_t),$$

where p_t is the common belief and $\hat{u}_t$ is the expected effort level given p_t. The agent maximizes

$$\int_0^T \frac{1-p_0}{1-p_t}\left[p_t(\lambda+\hat{u}_t) - \alpha u_t - v\right] \mathrm{d}t - k\frac{1-p_0}{1-p_T}.$$

In contrast to (3), the revenue is no longer a function of time only, as chosen effort affects future beliefs, hence future wages. At the very least, then, wages and behavior are a function of time t and current belief p. In fact, we shall restrict attention to equilibria in Markov strategies

$$u : [0,1] \times [0,T] \to [0, \bar{u}],$$

such that u is upper semicontinuous in (p,t), and such that the value function

$$V(p,t) = \sup_u \left\{ \int_t^T \frac{1-p_t}{1-p_s}\left[p_s\left(\lambda + u\left(p_s,s\right)\right) - \alpha u(p_s,s) - v\right] \mathrm{d}s - k\frac{1-p_t}{1-p_T} \right\},$$

with $p_t = p$, is piecewise differentiable.[30] We shall prove that such *Markov* equilibria exist.

5.2.2 Equilibrium structure

We first argue that if the agent ever exerts low effort, he has always done so before.

Lemma 5.2 *Fix a Markov equilibrium. If $u = 0$ on some open set $\Omega \subset [0,1] \times [0,T]$, then also $u(p',t') = 0$ if the equilibrium trajectory that starts at (p',t') intersects Ω.*

This lemma implies that every equilibrium has a relatively simple structure: if the agent is ever willing to exert high effort, he keeps being willing to do so at any later time, at least on the equilibrium path. In any equilibrium involving extremal effort levels only, there are at most two phases: first, the worker exerts no effort, and then full effort. This is precisely the opposite of the optimal policy for the social planner (under our assumption $v > \alpha\lambda$), in which high effort comes

[30] That is, there exists a finite partition of $[0,1] \times [0,T]$ into closed subsets S_i with non-empty interior, such that V is differentiable on the interior of S_i, and the intersection of any pair S_i, S_j is either empty or a smooth 1-dimensional manifold.

first (see Lemma 2.1). The agent can only be trusted by the market to put in high effort if he is "back to the wall," so that maximum effort will remain optimal at any later time, no matter what he does until then; if the market paid the worker for high effort, yet he was supposed to let up his effort later on, then the worker would gain by deviating to low effort, pocketing the high wage in the process; because the observable deviation to no effort would make everyone more optimistic, it would only increase his incentives to exert high effort later and thus increase his wage at later times. This, of course, relies heavily on the Markov assumption.

As the next theorem states, there are multiple Markov equilibria.

Theorem 5.3 *Given $T > 0$, there exists continuous, distinct, non-increasing $\underline{p}, \bar{p} : [0,T] \to [0,1]$, with $\underline{p}_t \leq \bar{p}_t$ and $\underline{p}_T = \bar{p}_T$, such that:*

1. *all Markov equilibria involve maximum effort above $\bar{p}$:*

$$p_t > \bar{p}_t \Rightarrow u(p,t) = \bar{u};$$

2. *all Markov equilibria involve no effort below $\underline{p}$:*

$$p_t \leq \underline{p}_t \Rightarrow u(p,t) = 0;$$

3. *these bounds are tight: there exists a Markov equilibrium $\underline{\sigma}$ (resp. $\bar{\sigma}$) in which effort is either 0 or $\bar{u}$ if and only if p is below or above $\underline{p}$ (resp. $\bar{p}$).*

The proof of Theorem 5.3, in Appendix D.2, provides an explicit description of these boundaries. Along any equilibrium trajectory, they are crossed at most once, from below. Note that these boundaries for the belief might be as high as one, in which case effort is never exerted at the corresponding time: indeed, there exists t^* (independent of T) such that effort is zero at all times $t < T - t^*$ (if $T > t^*$). The threshold $\underline{p}$ is decreasing in the cost of effort α, and increasing in the outside option v and penalty k. Considering the equilibrium with maximum effort, the agent works more, the more desirable success is. These results are illustrated in Figure 10 for the same parameters as in Figure 4 in the unobservable case.

It is worth noting that, while $\underline{\sigma}$ and $\bar{\sigma}$ provide upper and lower bounds on the equilibrium effort exerted in an equilibrium (in the sense of (1.)–(2.)), these equilibria are not the only

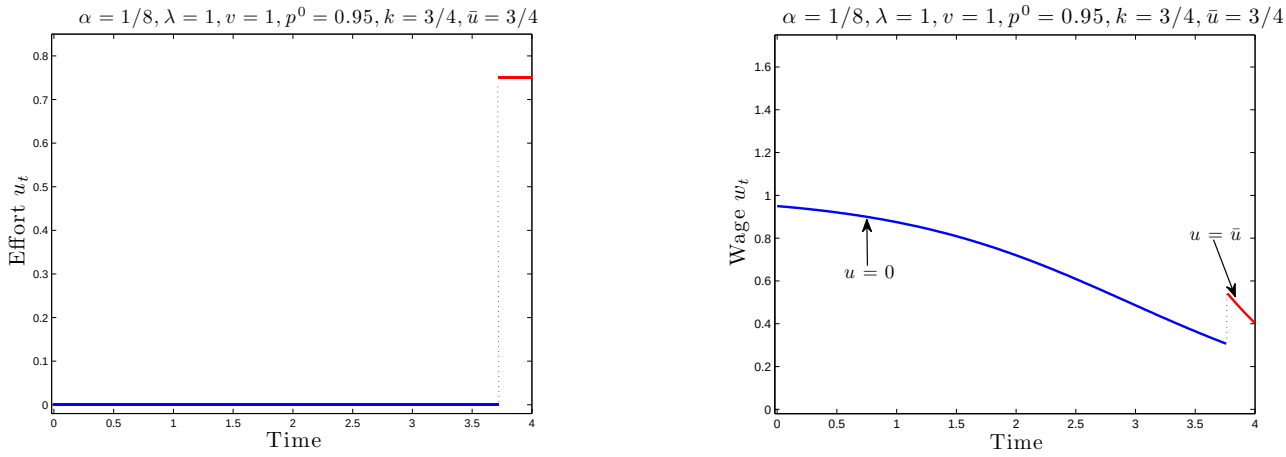

Figure 10: Effort and wages in the observable case

ones. There exist other Markov equilibria involving only extremal effort levels, whose switching boundary lies between $\underline{p}$ and $\bar{p}$, as well as equilibria in which interior effort levels are exerted at some states. In particular, the proof constructs an equilibrium in which the agent exerts effort in $(0, \bar{u})$ at all times t for all values of p in $[\underline{p}_t, \bar{p}_t]$. This effort is equal to $\bar{u}$ once the curve $\underline{p}$ is reached, decreases continuously along the equilibrium trajectory from that point on, until the upper boundary is reached (which, unless a breakthrough occurs, necessarily happens before time T, as $\underline{p}_T = \bar{p}_T$), at which point the effort level jumps up to $\bar{u}$.[31]

In the extremal equilibria, wages are decreasing over time, except for an upward jump at the point at which effort jumps up to $\bar{u}$. In the interior-effort equilibrium described in the proof (in which effort is interior everywhere between $\underline{p}$ and $\bar{p}$), wages decrease continuously over time.

Equilibrium multiplicity can be understood as follows. Because the principal only expects high effort if the belief is high and the deadline is close, such states (belief and times) are desirable for the agent, as the higher wage more than outweighs the effort cost. Yet low effort is the best way to reach those states, as effort depresses beliefs: hence, if the principal expects the agent to shirk until a high boundary is reached (in (p, t)-space), the agent has strong incentives to shirk to reach this boundary; if the principal expected the agent to shirk until an even higher boundary,

[31] It is not possible to strengthen (4) further to the statement that, once maximum effort is exerted, it is exerted throughout: there is considerable leeway in specifying equilibrium strategies between $\bar{p}$ and $\underline{p}$, and nothing precludes maximum effort to be followed by interior effort. (Of course, if $\bar{p}$ is crossed, effort remains maximal.)

this would only reinforce this incentive –up to some point.

This intuition foreshadows already what is stated in the next subsection: observability further depresses incentives and reduces effort, relative to non-observability. But as explained, it is also more in consonance with increasing wages: effort is non-decreasing over time, hence the only force that pushes down wages is growing pessimism, not declining work. For the purpose of comparative statics, we focus on the equilibrium that involves the largest region of effort.

Proposition 5.4 *The boundary of the maximal effort equilibrium $\underline{p}(t)$ is non-increasing in k and v and non-decreasing in α and λ.*

The effect of the maximum effort level $\bar{u}$ is ambiguous. Also, one might wonder whether increasing the penalty k increases welfare, for some parameters, as it helps resolve the commitment problem. Unlike in the non-observable case, this turns out never to occur, at least in the maximum-effort equilibrium: increasing the penalty decreases welfare, though it increases total effort. The proof is in Appendix D.2. Similarly, increasing v, the value of succeeding, increases effort (in the maximum-effort equilibrium), though it decreases the worker's payoff.

5.2.3 Comparison with the non-observable case

Along the equilibrium path, the dynamics of effort look very different when one compares the social planner, the agent when effort is unobservable, and the agent when effort is observable. Yet it turns out that effort can easily be ranked across those cases. To do so, the key is to describe effort in terms of the state (p, t), i.e., the public belief and calendar time.

For the observable case, it is enough to focus on the region (i.e., subset of the (p, t)-space) defined by the frontier $\underline{p}$, as this characterizes the maximum effort equilibrium, and it will turn out that even in this equilibrium, the agent works less than under non-observability.

Proposition 5.5 *The maximal effort region for the observable case is contained in the full effort region(s) for the non-observable case.*

Proposition 5.5 confirms the intuition that observability of effort reduces the incentives to work. In particular, the highest effort equilibrium in the observable case involves lower effort levels than the (unique) equilibrium in the unobservable case. Recall also from Proposition

3.4.(1) that the (interior or full) effort region in the non-observable case is in turn contained in the full effort region for the social planner.

How about non-Markov equilibria? Defining such equilibria formally in our continuous-time environment is problematic, but it is clear that threatening the agent with reversion to the Markov equilibrium $\bar{\sigma}$ provides incentives for high effort that extend beyond the high-effort region defined by $\underline{\sigma}$ –in fact, beyond the high-effort region in the unobservable case. The planner's solution, however, remains out of reach, since the punishment is restricted to beliefs below $\underline{p}$.

5.3 Endogenous deadlines

The first two variations of our model have shown how more commitment power or better monitoring drastically affect the effort and wage pattern. In both cases, the deadline for the agent's career was held constant. We now consider an endogenous deadline, and show that the effects on wages and effort are not as pronounced.

By an endogenous deadline, we mean that the worker decides when to quit the profession. We assume (for now) that he has no commitment power. The principal anticipates the quitting decision, and takes this into account while determining the agent's equilibrium effort, and therefore, the wage he should be paid.

More specifically, in each interval $[t, t+\mathrm{d}t)$ such that the agent has not quit yet, the principal pays a wage $w_t \mathrm{d}t$, the agent then decides how much effort to exert over this time interval, and at the end of it, whether to stay or leave, which is an observable choice.

This raises the issue of the principal's beliefs if the agent were to deviate and stay beyond what the equilibrium specifies. For simplicity, we adopt passive beliefs. That is, if the agent is supposed to drop out at some time but fails to do so, the principal does not revise his belief regarding the past effort choices, ascribing the failure to quit to a mistake, and anticipates equilibrium play in the continuation (which means, as it turns out, that he anticipates the agent quitting at the next opportunity).[32]

[32]In the linear cost case, this means that we fix the off-equilibrium beliefs to specify $\hat{u}_t = \bar{u}$ if $p_t > p^*$, where p^* is the lowest belief at which it would be optimal for the agent to exert maximum effort if he anticipated quitting at the end of the interval $[t, t+\mathrm{d}t)$ (see Appendix B for p^* in closed-form), and $\hat{u}_t = 0$ otherwise. In other words, the market does not react to a failure to quit, anticipates the agent quitting immediately afterwards and expects instantaneous effort to be determined as if $p = p_T$ were the terminal belief.

We have argued above that endogenous deadlines do not affect the qualitative features of the equilibrium effort and wage patterns. Indeed, with convex cost, effort is *always* decreasing at the equilibrium deadline (i.e. at the agent's optimal quitting time). This implies, in particular, that the wage is decreasing at that stage. Furthermore, it is simple to construct examples in which effort is not decreasing throughout. Hence, effort is single-peaked and wages are first decreasing, and then single-peaked. The agent's belief at the deadline is always too high relative to that of the social planner at the first-best deadline. Furthermore, both effort and the worker's marginal product are decreasing throughout in the first-best solution.

How about if the worker could commit to the deadline (but still not to effort levels)? The optimal deadline with commitment can be either shorter or longer than without. In either case, however, the deadline is set so as to increase aggregate effort, and so increase wages. This may require increasing the deadline –so as to increase the duration over which higher effort levels are sustained, even if that means quitting at a point where staying is unprofitable– or decreasing the deadline –so as to make high effort levels credible. Figure 11 illustrates the two possibilities.

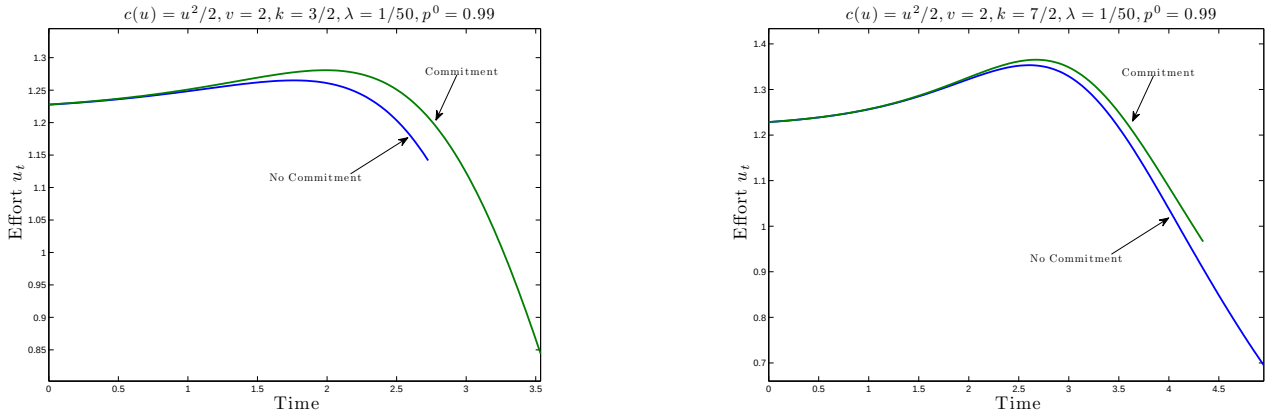

Figure 11: Setting the deadline with commitment can push it higher or lower than without (the curves stop at the respective deadlines).

Finally, having the worker quit when it is best for him to do so (without commitment to the deadline) reinforces our comparison between observable and non-observable effort. As we show, the deadline chosen is shorter, and the total effort exerted is lower, when effort is observed by the principal (in the linear cost case). These results are summarized in the following proposition.

Exact characterizations are provided in the proof (see Appendix D.3).

Proposition 5.6 *With convex cost,*

1. *effort is always decreasing at the optimal deadline without commitment;*
2. *the belief of the planner at the deadline is lower than the agent's at the optimal deadline without commitment;*
3. *the deadline with commitment can be shorter or longer than without.*

Furthermore, with linear cost and without commitment, total effort is lower, and the deadline chosen shorter, when effort is observable than when it is not.

One might also wonder how optimal deadlines affect the structure of the optimal contract with commitment but competition developed in Section 5.1. A complete analysis (for the linear cost case) is provided in Appendix D.3.3. By an optimal deadline, we mean the deadline that society would like to impose to maximize total welfare, and would apply to all competing firms simultaneously. In the absence of such external enforcement, it is not hard to see that, if the deadline were part of the contract, firms might as well offer contracts with an infinite deadline. With external enforcement, however, the deadline can be finite (depending on parameters). For all parameters, it is such that the second phase –in which effort and wages are zero– is non-empty: the value of extending the deadline beyond the point at which the worker would start shirking is always positive. This is unlike what the social planner would impose in terms of effort: it would be optimal to choose a deadline and an effort path that specifies full effort until the deadline.

We conclude this section by comparing the performance of a deadline with a finishing line. A deadline T is a time at which the game stops. A finishing line, instead, is a value of the belief, $\hat{x}$, at which the game stops, and the penalty k is incurred. Given some finishing line, what is the agent's optimal strategy? As a consequence, what is the optimal finishing line, and is setting a finishing line preferable to a deadline? A finishing line makes more sense when effort is observable than not, and so we assume it is. Attention is restricted to Markov strategies, which, given the absence of deadline, reduce to measurable functions u of the belief only. As usual, equilibrium requires that expected effort, which determines wages, coincides with actual effort.

Proposition 5.7

1. *Given the finishing line $\hat{x}$, the optimal strategy involves first full effort, then interior and decreasing effort, then zero effort.*[33]

2. *The optimal finishing line involves the same belief as the optimal deadline without commitment and unobservable effort.*

6 Concluding remarks

There are several extensions that we think could potentially yield interesting insights.

Policy Design While Section 5 has investigated the role of several important assumptions of our model, it has not touched upon the issue of corrective measures that can be taken to remedy the inefficiency of the outcome of the baseline model. Different arrangements could be considered, that differ in terms of flexibility and coerciveness: taxes, pensions, or even, in the extreme case, slavery, etc. For instance, an obvious institution to examine within our framework is apprenticeship length (see Malcolmson, Maw and McCormick, 2003).

Search equilibrium We have discussed when the worker would choose to quit, not when the firm would like to lay off the worker. To examine this issue, it is necessary to introduce some friction in the model: as the firm is paying a fair wage at all times in the current model, it has nothing to lose nor to gain by firing the worker. Yet this is an important question, in light of the rigid tenure policies adopted by many professional service firms. Why not keep the employee past the probationary period, adjusting the wage for the diminished incentives and lower assessed ability?[34] Firms have a cost of hiring (or firing) workers –possibly due to the delay in filling a vacancy– but derive a surplus from the worker in excess of the competitive wage they have to pay. Studying the efficiency properties and the characteristics of the resulting labor market (composition of the working force, duration of unemployment) seems to be an interesting undertaking. Section 5.1 has assumed perfect competition. A natural starting point would be a

[33] Of course, depending on the finishing line, the project might stop before effort drops from maximum effort.

[34] See Gilson and Mnookin (1989) for a discussion of this puzzle for the case of law firms.

market in which marginal productivities are heterogenous, and there are matching frictions, as in Postel-Vinay and Robin (2002).

Partnerships Also, this framework appears suitable for an analysis of partnerships.[35] After all, in law or consulting firms, projects are often assigned to a team of employees, who combine partners with junior associates. The team must achieve possibly conflicting objectives: incentivizing both the partner and the associate, and eliciting information about the associate's ability. How should profits be shared in the team to do so? When should the project be terminated, or the junior associate replaced? Is it indeed optimal to combine workers whose assessed ability differs, as opposed to workers about whom information is symmetric? A related issue is yardstick competition: in our model, there is no distinction between the agent's skill and the feasibility of the project that he tackles. In practice, the market learns both about the worker and the project's feasibility, and this learning occurs also through the progress of other agents' work on similar issue. Clearly then, yardstick competition affects incentives. Finally, and relatedly, workers of different perceived skills might choose different types of projects; more challenging projects, or tougher environments, might foster learning of very high skilled workers, but be redhibitory for workers with lower perceived skills. Examining how the market allocates employees and firms and how this allocation differs from the efficient match is an interesting problem.

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Appendix

Throughout this appendix, we shall use the log-likelihood ratio

$$x_t := \ln \frac{1-p_t}{p_t}$$

of state $\omega = 0$ vs. $\omega = 1$. We set $x^0 := \ln(1-p^0)/p^0$. Note that x increases over time and, given u, follows the O.D.E.

$$\dot{x}_t = \lambda + u_t,$$

with $x_0 = x^0$. We shall also refer to x_t as the "belief," hoping that this will create no confusion.

We start by explaining how the objective function can be derived as the limit of a discounted version of our problem. Suppose that V is the value of a success and $V_L = V - k$ is the value of failure. Given the discount rate r, the agent's payoff is given by

$$\left(1+e^{-x_0}\right) V_0 = \int_0^T e^{-rt}\left(1+e^{-x_t}\right)\left(\frac{\dot{x}_t}{1+e^{x_t}} V + w_t - c\left(u_t\right)\right) \mathrm{d}t + e^{-rT}\left(1+e^{-x_T}\right) V_L,$$

where V_0 is his *ex ante* payoff.
Integrating by parts we obtain

$$\begin{aligned}
\left(1+e^{-x_0}\right) V_0 &= \int_0^T e^{-rt} e^{-x_t} \dot{x}_t V \mathrm{d}t + \int_0^T e^{-rt}\left(1+e^{-x_t}\right)\left(w_t - c\left(u_t\right)\right) \mathrm{d}t + e^{-rT}\left(1+e^{-x_T}\right)(V-k) \\
&= -e^{-rt} e^{-x_t} V\big|_0^T + \int_0^T e^{-rt}\left(\left(1+e^{-x_t}\right)\left(w_t - c\left(u_t\right)\right) - e^{-x_t} r V\right) \mathrm{d}t + e^{-rT}\left(1+e^{-x_T}\right)(V-k) \\
&= e^{-x_0} V - e^{-rT} e^{-x_T} V + \int_0^T e^{-rt}\left(1+e^{-x_t}\right)\left(w_t - c\left(u_t\right) - \frac{rV}{1+e^{x_t}}\right) \mathrm{d}t + e^{-rT}\left(1+e^{-x_T}\right)(V-k),
\end{aligned}$$

so that as $r \to 0$ (and defining v as $rV \to v$) we obtain

$$\left(1+e^{-x_0}\right)\left(V_0 - V\right) = \int_0^T \left(1+e^{-x_t}\right)\left(w_t - c\left(u_t\right) - \frac{v}{1+e^{x_t}}\right) \mathrm{d}t - k\left(1+e^{-x_T}\right).$$

Similarly, one can show the social planner's payoff is given by

$$\left(1+e^{-x_0}\right)\left(V_0 - V\right) - e^{-x_0} + k = -\int_0^T \left(1+e^{-x_t}\right)\left(c\left(u_t\right) + \frac{v}{1+e^{x_t}}\right) \mathrm{d}t - (1+k)\, e^{-x_T}.$$

A Proofs for Section 2

Proof of Lemma 2.1. In both the linear and convex cases, existence and uniqueness of a solution follow as special case of Lemma 3.1, when $w = 0$ identically (the transversality condition must be adjusted). To see that the social planner's problem is equivalent to this, note that (whether the cost is convex or linear), the "revenue" term of the social planner's objective satisfies

$$\int_0^T (1 + e^{-x_t}) \frac{\lambda + u_t}{1 + e^{x_t}} \mathrm{d}t = \int_0^T \dot{x}_t e^{-x_t} \mathrm{d}t = e^{-x^0} - e^{-x_T},$$

and so this revenue only affects the necessary conditions through the transversality condition at T.

Let us start with the linear case. The social planner maximizes

$$\int_0^T (1 + e^{-x_t}) \left(\frac{\lambda + u_t}{1 + e^{x_t}} - \alpha u_t - v \right) \mathrm{d}t - k e^{-x_T}, \text{ s.t. } \dot{x}_t = \lambda + u_t.$$

We note that the maximization problem cannot be abnormal, since there is no restriction on the terminal value of the state variable. See Note 5, Ch. 2, Seierstad and Sydsæter (1987). The same holds for all later optimization problems.

It will be understood from now on that statements about derivatives only hold almost everywhere.

Let γ_t be the costate variable. The Hamiltonian for this problem is

$$H(x, u, \gamma, t) = e^{-x_t}(\lambda + u_t) - (1 + e^{-x_t})(v + \alpha u_t) + \gamma_t (\lambda + u_t).$$

Define $\phi_t := \partial H / \partial u_t = (1 - \alpha) e^{-x_t} - \alpha + \gamma_t$. Note that given x_t and γ_t, the value of ϕ_t does not depend on u_t. Pontryagin's principle applies, and yields

$$u_t = \bar{u} \ \ (u_t = 0) \Leftrightarrow \phi_t := \frac{\partial H}{\partial u_t} = (1 - \alpha) e^{-x_t} - \alpha + \gamma_t > (<) 0,$$

as well as

$$\dot{\gamma}_t = e^{-x_t}(\lambda - v + (1 - \alpha) u_t), \gamma_T = k e^{-x_T}.$$

Differentiating ϕ_t with respect to time, and using the last equation gives

$$\dot{\phi}_t = e^{-x_t}(\alpha \lambda - v), \phi_T = (1 + k - \alpha) e^{-x_T} - \alpha.$$

Note that ϕ is either increasing or decreasing depending on the sign of $\alpha\lambda - v$. Therefore, the planner's solution is either maximum effort–no effort, or no effort–maximum effort, depending on the sign of this expression. Finally, the marginal product $p(\lambda + u)$ is decreasing if effort maximum–zero. If effort is zero–maximum, the marginal product is decreasing, jumps up, and then decreases again.

Consider now the convex case. Applying Pontryagin's theorem (and replacing the revenue term by its expression in terms of x_t and x^0, as explained above) yields as necessary conditions

$$\dot{\gamma}_t = -e^{-x}(c(u) + v), \gamma_t = (1 + e^{-x_t}) c'(u_t),$$

where γ_t is the co-state variable, as before. Differentiate the second expression with respect to time, and use the first one to obtain

$$\dot{u} = \frac{(\lambda+u)\,c'(u) - c(u) - v}{c''(u)\,(1+e^x)}, \tag{13}$$

in addition to $\dot{x} = \lambda + u$ (time subscripts will often be dropped for brevity). Let

$$\phi(u) := (\lambda+u)\,c'(u) - c(u) - v.$$

Note that $\phi(0) = -v < 0$, and $\phi'(u) = (\lambda+u)\,c''(u) > 0$, and so ϕ is strictly increasing and convex. Let $u^* \geq 0$ be the unique solution to

$$\phi(u^*) = 0,$$

and so ϕ is negative on $[0, u^*]$ and positive on $[u^*, \infty)$. Accordingly, $u < u^* \Longrightarrow \dot{u} < 0$, $u = u^* \Longrightarrow \dot{u} = 0$ and $u > u^* \Longrightarrow \dot{u} > 0$. Given the transversality condition

$$(1+e^{x_T})\,c'(u_T) = 1 + k,$$

we can then define $x_T\left(x^0\right)$ by

$$x_T(x^0) = \frac{1}{\lambda+u^*}\left[\ln\left(\frac{1+k}{c'(u^*)} - 1\right) - x^0\right],$$

and so effort is decreasing throughout if $x_T > x_T(x^0)$, increasing throughout if $x_T < x_T(x^0)$, and equal to u^* throughout otherwise. The conclusion then follows from the proof of Lemma 2.3, which establishes that the belief x_T at the deadline is increasing in T.

We now turn to the marginal product $p(\lambda+u)$. In terms of x, the marginal product is given by

$$\begin{aligned} w(x) &:= \frac{\lambda+u(x)}{1+e^x}, \text{ and so} \\ w'(x) &= \frac{u'(x)}{1+e^x} - w(x)\,\frac{e^x}{(1+e^x)}, \end{aligned}$$

so that $w'(x) = 0$ is equivalent to

$$u'(x) = w(x)\,e^x.$$

Notice that $u'(x) \leq 0$ implies $w'(x) < 0$. Conversely, if $u'(x) > 0$, consider the second derivative of $w(x)$. We have

$$w''(x) = -\frac{e^x}{1+e^x}w'(x) + \frac{1}{1+e^x}\left(u''(x) - w(x)\,e^x - u'(x)\,e^x\right),$$

so that when $w'(x) = 0$ we have

$$w''(x) = \frac{u''(x) - u'(x)}{1+e^x}.$$

From equation (13) we obtain an expression for the derivative of u with respect to x:

$$u'(x) = \frac{(\lambda+u)\,c'(u) - c(u) - v}{c''(u)\,(1+e^x)\,(\lambda+u)}.$$

Let $g(u) = v + c(u) - (\lambda + u)c'(u)$ and study $u''(x)$ when $w'(x) = 0$. We have

$$\begin{aligned} u''(x) &= \frac{u'}{1+e^x} + \frac{g\left(\left(1+e^x\right)\left(c'' + (\lambda+u)c'''\right)u'(x) + e^x c''(\lambda+u)\right)}{\left(c''(u)\left(1+e^x\right)(\lambda+u)\right)^2} \\ &= \frac{c''(\lambda+u)u'(x)}{c''\left(1+e^x\right)(\lambda+u)} - \frac{u'(x)\left(\left(1+e^x\right)\left(c'' + (\lambda+u)c'''\right)u'(x) + e^x c''(\lambda+u)\right)}{c''\left(1+e^x\right)(\lambda+u)} \\ &= -\frac{u'(x)\left(\left(1+e^x\right)\left(c'' + (\lambda+u)c'''\right)u'(x) + e^x c''(\lambda+u) - c''(\lambda+u)\right)}{c''\left(1+e^x\right)(\lambda+u)} \\ &= -\frac{u'(x)\left(\left(2c'' + (\lambda+u)c'''\right)e^x(\lambda+u) - c''(\lambda+u)\right)}{c''\left(1+e^x\right)(\lambda+u)}. \end{aligned}$$

We therefore consider the quantity

$$\begin{aligned} u''(x) - u'(x) &= -\frac{u'(x)\left(\left(2c'' + (\lambda+u)c'''\right)e^x - c'' + c''\left(1+e^x\right)\right)}{c''\left(1+e^x\right)} \\ &= -\frac{u'(x)\left(3c'' + (\lambda+u)c'''\right)e^x}{c''\left(1+e^x\right)} < 0, \end{aligned}$$

if as we have assumed, $c'' + (\lambda+u)c''' > 0$. Therefore, we have a single-peaked (at most increasing then decreasing) marginal product. □

Proof of Lemma 2.3. We shall use the necessary conditions obtained in the previous proof. Part (1) is almost immediate. Note that in both the linear and convex case, the necessary conditions define a vector field $(\dot{u}, \dot{x})$, with trajectories that only define on the time left before the deadline and the current belief. Because trajectories do not cross (in the plane $(-\tau, x)$, where τ is time-to-go and x is the belief), and belief x can only increase with time, if we compare two trajectories starting at the same level x^0, the one that involves a longer deadline must necessarily involve as high a terminal belief x as the other (as the deadline expires).

(2) In the linear case, it is straightforward to solve for the switching time (or switching belief) under Assumption 2.2. For all terminal beliefs $x_T > x^*$, for which no effort is exerted at the deadline, the switching belief between equilibrium phases is determined by

$$(1 + k - \alpha)e^{-x_T} - \alpha = \int_x^{x_T} e^{-s}\frac{\alpha\lambda - v}{\lambda}\mathrm{d}s,$$

which gives as value of x (as a function of t)

$$x(t) = \ln\left((1 + k - v/\lambda)e^{-\lambda(T-t)} - (\alpha - v/\lambda)\right) - \ln\alpha.$$

This represents a frontier in (t, x) space that the equilibrium path will cross from below for sufficiently long deadlines. Consistent with the fact that, in the optimum, a switch to zero effort is irreversible, when $u_t = 0$ and $\dot{x}_t = \lambda$, the path leaves this locus (i.e., it holds that $x'(t) < \lambda$).

The switching belief $x(t)$ decreases in T: the longer the deadline, the longer maximum effort will be exerted (recall that x measures pessimism). This belief decreases in α and increases in v and k: the higher the cost of

failing, or the lower the cost of effort, the longer effort is exerted. These are the comparative statics mentioned in the text before Lemma 2.3.

Furthermore, by differentiating, the boundary $x(\cdot)$ satisfies $x'(t) < 0$ (resp. > 0) if and only if $1 + k < v/\lambda$. In that case, total effort increases with T: considering the plane $(-\tau, x)$, where τ is time-to-go and x is the belief, increasing the deadline is equivalent to increasing τ, i.e. shifting the initial point to the left; if $x' < 0$, it means that the range of beliefs over which high effort is exerted (which is 1-to-1 with time spent exerting maximum effort, given that $\dot{x} = \lambda + \bar{u}$) increases. If instead x' is positive, total effort decreases with T, by the same argument.

Consider now the convex case. Note that

$$\frac{1}{x'(u)} = \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\dot{u}}{\dot{x}} = \frac{(\lambda + u)\, c'(u) - c(u) - v}{c''(u)\,(1 + e^x)\,(\lambda + u)},$$

along with

$$(1 + e^{x_T})\, c'(u_T) = 1 + k,$$

which is the transversality condition, can be integrated to

$$\phi(u) = \frac{(k+1)\phi(u_T)}{1 + k - c'(u_T)} \frac{1}{1 + e^{-x}}.$$

Note also that, defining $g(u) := \frac{\phi(u)}{1+k-c'(u)}$,

$$g'(u) = \frac{(\lambda + u)\, c''(u)}{1 + k - c'(u)} + \frac{\phi(u)}{(1 + k - c'(u))^2} c''(u),$$

which is of the sign of

$$\psi(u) := (\lambda + u)(1 + k) - c(u) - v,$$

which is strictly concave, negative at ∞, and positive for u small enough if and only if $1 + k > v/\lambda$.

Note that increasing T is equivalent to increasing x_T, which in turn is equivalent to decreasing u_T, because the transversality condition yields

$$\frac{\mathrm{d}u_T}{\mathrm{d}x_T} = -\frac{c'(u)\, e^x}{(1 + e^x)\, c''(u)} < 0.$$

Because ϕ is increasing, u increases when u_T decreases if and only if ψ is decreasing at u.

So if $\max_u[(\lambda + u)(1 + k) - c(u)] < v$, ψ is negative for all u, and it follows that u increases for fixed x; in addition to all values of x that are visited in the interval $[x^0, x_T]$, as T increases, additional effort accrues at time T; overall, it is then unambiguous: total effort increases.

On the other hand, if $1 + k > v/\lambda$, then if the deadline is long enough for effort to be small throughout, effort at $x < x_T$ decreases as T increases, but since an additional increment of effort is produced at time T, it is unclear. A simple numerical example shows that total effort can then decrease. □

B Proofs for Section 3

Proof of Lemma 3.1. We address the two claims in turn.

Existence: Note that the state equation is linear in the control u, while the objective's integrand is concave in u. Hence the set $N(x,U,t)$ is convex (see Thm. 8, Ch. 2 of Seierstad and Sydsæter, 1987). Therefore, the Filippov-Cesari existence theorem applies.

Uniqueness: We can write the objective as, up to constant terms,

$$\int_0^T (1+e^{-x_t})(w_t - v - c(u_t))\mathrm{d}t - ke^{-x_T},$$

or, using the likelihood ratio $l_t := p_t/(1-p_t) > 0$,

$$J(l) := \int_0^T (1+l_t)(w_t - v - c(u_t))\ \mathrm{d}t - kl_T.$$

Consider the linear case. Letting $g_t := w_t - v + \alpha\lambda$, we rewrite the objective in terms of the likelihood ratio as

$$\int_0^T l_t g_t \mathrm{d}t - (k-\alpha)l_T + \alpha \ln l_T + \text{ Constant}.$$

Because the first two terms are linear in l while the last is strictly concave, it follows that there exists a unique optimal terminal odds ratio $l_T^* := l_T$. Suppose that there exists two optimal trajectories l_1, l_2 that differ. Because $l_{1,0} = l_{2,0} = p^0/(1-p^0)$ and $l_{1,T} = l_{2,T} = l_T^*$, yet the objective is linear in l_t, it follows that every feasible trajectory l with $l_t \in [\min\{l_{1,t}, l_{2,t}\}, \max\{l_{1,t}, l_{2,t}\}]$ is optimal as well.[36] Consider any interval $[a,b] \subset [0,T]$ for which $t \in [a,b] \implies \min\{l_{1,t}, l_{2,t}\} < \max\{l_{1,t}, l_{2,t}\}$. Consider any feasible trajectory l with $l_t \in [\min\{l_{1,t}, l_{2,t}\}, \max\{l_{1,t}, l_{2,t}\}]$ for all t, $l_t \in (\min\{l_{1,t}, l_{2,t}\}, \max\{l_{1,t}, l_{2,t}\})$ for $t \in [a,b]$ and associated control such that $u_t \in (0, \bar{u})$ for $t \in [a,b]$. Because there is an open set of variations of u that must be optimal in $[a,b]$, it follows from Lemma 2.4.ii of Cesari (1983) that $g_t = 0$ (a.e.) on $[a,b]$.

Consider now the convex case. Suppose that there are two distinct optimal trajectories l_1 and l_2, with associated controls u_1 and u_2. Assume without loss of generality that

$$l_{1,t} < l_{2,t} \text{ for all } t \in (0,T].$$

We analyze the modified objective function

$$\tilde{J}(l) := \int_0^T (1+l_t)(w_t - v - \tilde{c}_t(u_t))\mathrm{d}t - kl_T,$$

in which we replace the cost function $c(u_t)$ with

$$\tilde{c}_t(u) := \begin{cases} \alpha_t u & \text{if} \quad u \in [\min\{u_{1,t}, u_{2,t}\}, \max\{u_{1,t}, u_{2,t}\}] \\ c(u) & \text{if} \quad u \notin [\min\{u_{1,t}, u_{2,t}\}, \max\{u_{1,t}, u_{2,t}\}], \end{cases}$$

[36] Feasibility means that $\dot{l}_t \in [l_t\lambda, l_t(\lambda + \bar{u})]$ for all t.

where

$$\alpha_t := \frac{\max\{c(u_{1,t}), c(u_{2,t})\} - \min\{c(u_{1,t}), c(u_{2,t})\}}{\max\{u_{1,t}, u_{2,t}\} - \min\{u_{1,t}, u_{2,t}\}}.$$

(If $u_{1,t} = u_{2,t} =: u_t$ for some t, set α_t equal to $c'(u_t)$). Because $\tilde{c}_t(u) \geq c(u)$ for all t, u, the two optimal trajectories l_1 and l_2, with associated controls u_1 and u_2, are optimal for the modified objective $\tilde{J}(l)$ as well. Furthermore, $\tilde{J}(l_1) = J(l_1)$ and $\tilde{J}(l_2) = J(l_2)$.

We will construct a feasible path l_t and its associated control $u_t \in [\min\{u_{1,t}, u_{2,t}\}, \max\{u_{1,t}, u_{2,t}\}]$ which attains a higher payoff $\tilde{J}(l)$ and therefore a strictly higher payoff $J(l)$. Suppose $u_t \in [u_{1,t}, u_{2,t}]$ for all t. Letting $g_t := w_t - v + \alpha\lambda - \dot{\alpha}_t$, we rewrite the modified objective as

$$\int_0^T l_t g_t \mathrm{d}t - \int_0^T \dot{\alpha}_t \ln l_t \mathrm{d}t - (k - \alpha_T) l_T + \alpha_T \ln l_T + \text{ Constant.}$$

We now consider a continuous function $\varepsilon_t \geq 0$ and two associated variations on the paths l_1 and l_2,

$$\begin{aligned} l'_{1,t} &:= (1 - \varepsilon_t) l_{1,t} + \varepsilon_t l_{2,t} \\ l'_{2,t} &:= (1 - \varepsilon_t) l_{2,t} + \varepsilon_t l_{1,t}. \end{aligned}$$

Because l_1 and l_2 are optimal, for any ε_t it must be the case that

$$\begin{aligned} \tilde{J}(l_1) - \tilde{J}(l'_1) &\geq 0 \\ \tilde{J}(l_2) - \tilde{J}(l'_2) &\geq 0. \end{aligned}$$

We can write these payoff differences as

$$\begin{aligned} \int_0^T \varepsilon_t (l_{1,t} - l_{2,t}) g_t \mathrm{d}t + \int_0^T \dot{\alpha}_t \varepsilon_t \frac{l_{2,t} - l_{1,t}}{l_{1,t}} \mathrm{d}t - (k - \alpha_T)\varepsilon_T (l_{1,T} - l_{2,T}) - \alpha_T \varepsilon_T \frac{l_{2,T} - l_{1,T}}{l_{1,T}} + o(\|\varepsilon\|) &\geq 0 \\ \int_0^T \varepsilon_t (l_{2,t} - l_{1,t}) g_t \mathrm{d}t + \int_0^T \dot{\alpha}_t \varepsilon_t \frac{l_{1,t} - l_{2,t}}{l_{2,t}} \mathrm{d}t - (k - \alpha_T)\varepsilon_T (l_{2,T} - l_{1,T}) - \alpha_T \varepsilon_T \frac{l_{1,T} - l_{2,T}}{l_{2,T}} + o(\|\varepsilon\|) &\geq 0. \end{aligned}$$

Letting

$$\rho_t : l_{1,t}/l_{2,t} < 1 \text{ for all } t > 0,$$

we can sum the previous two conditions (up to the second order term). Finally, integrating by parts, we obtain the following condition,

$$\int_0^T \left[\frac{\dot{\varepsilon}_t}{\varepsilon_t} \left(2 - \rho_t - \frac{1}{\rho_t} \right) + \dot{\rho}_t \frac{1 - \rho_t^2}{\rho_t^2} \right] \alpha_t \varepsilon_t \mathrm{d}t \geq 0,$$

which must hold for all ε_t. Using the fact that $\dot{\rho} = \rho(u_2 - u_1)$ we have

$$\int_0^T \left[-\frac{\dot{\varepsilon}_t}{\varepsilon_t} (1 - \rho_t) + (u_{2,t} - u_{1,t})(1 + \rho_t) \right] \alpha_t \varepsilon_t \frac{1 - \rho_t}{\rho_t} \mathrm{d}t \geq 0. \tag{14}$$

We now identify bounds on the function ε_t so that both variations l'_1 and l'_2 are feasible and their associated controls lie in $[\min\{u_{1,t}, u_{2,t}\}, \max\{u_{1,t}, u_{2,t}\}]$ for all t. Consider the following identities

$$\begin{aligned} \dot{l}'_1 &= -l'_{1,t}(\lambda + u_t) \equiv \dot{\varepsilon}_t (l_{2,t} - l_{1,t}) - \lambda l'_{1,t} - (1 - \varepsilon_t) u_{1,t} l_{1,t} - \varepsilon_t u_{2,t} l_{2,t} \\ \dot{l}'_2 &= -l'_{2,t}(\lambda + u_t) \equiv \dot{\varepsilon}_t (l_{1,t} - l_{2,t}) - \lambda l'_{2,t} - \varepsilon_t u_{1,t} l_{1,t} - (1 - \varepsilon_t) u_{2,t} l_{2,t}. \end{aligned}$$

We therefore have the following expressions for the function $\dot{\varepsilon}/\varepsilon$ associated with each variation

$$\frac{\dot{\varepsilon}_t}{\varepsilon_t} = \frac{(u_{1,t}-u_t)\frac{1-\varepsilon_t}{\varepsilon_t}l_{1,t}+l_{2,t}(u_{2,t}-u_t)}{l_{2,t}-l_{1,t}}, \tag{15}$$

$$\frac{\dot{\varepsilon}_t}{\varepsilon_t} = \frac{(u_{1,t}-u_t)l_{1,t}+\frac{1-\varepsilon_t}{\varepsilon_t}l_{2,t}(u_{2,t}-u_t)}{l_{1,t}-l_{2,t}}. \tag{16}$$

In particular, whenever $u_{2,t} > u_{1,t}$ the condition

$$\frac{\dot{\varepsilon}_t}{\varepsilon_t} \in \left[-\frac{1-\varepsilon_t}{\varepsilon_t}\frac{l_{2,t}(u_{2,t}-u_{1,t})}{l_{2,t}-l_{1,t}}, \frac{l_{1,t}(u_{2,t}-u_{1,t})}{l_{2,t}-l_{1,t}}\right]$$

ensures the existence of two effort levels $u_t \in [u_{1,t}, u_{2,t}]$ that satisfy conditions (15) and (16) above. Similarly, whenever $u_{1,t} > u_{2,t}$ we have the bound

$$\frac{\dot{\varepsilon}_t}{\varepsilon_t} \in \left[-\frac{l_{1,t}(u_{1,t}-u_{2,t})}{l_{2,t}-l_{1,t}}, \frac{1-\varepsilon_t}{\varepsilon_t}\frac{l_{2,t}(u_{2,t}-u_{1,t})}{l_{2,t}-l_{1,t}}\right].$$

Note that $\dot{\varepsilon}_t/\varepsilon_t = 0$ is always contained in both intervals.

Finally, because $\rho_0 = 1$ and $\rho_t < 1$ for all $t > 0$, we must have $u_{1,t} > u_{2,t}$ for $t \in [0, t^*)$ with $t^* > 0$. Therefore, we can construct a path ε_t that satisfies

$$(u_{2,t}-u_{1,t})\frac{1+\rho_t}{1-\rho_t} < \frac{\dot{\varepsilon}_t}{\varepsilon_t} < 0 \quad \forall t \in [0, t^*),$$

with $\varepsilon_0 > 0$, and $\varepsilon_t \equiv 0$ for all $t \geq t^*$. Substituting into condition (14) immediately yields a contradiction. □

Proof of Proposition 3.3. Consider the convex case. Applying Pontryagin's theorem yields eqn. (7). It also follows that the effort and belief (x, u) trajectories satisfy

$$c''(u)(1+e^x)\dot{u} = (\lambda+u)c'(u) - c(u) + w_t - v \tag{17}$$

$$\dot{x} = \lambda + u \tag{18}$$

with boundary conditions

$$x_0 = x^0 \tag{19}$$

$$ke^{-x_T} = \left(1+e^{-x_T}\right)c'(u_T). \tag{20}$$

Differentiating (17) further, we obtain

$$\begin{aligned}\left(c''(u)(1+e^x)\right)^2 u_t'' = {} & \left((\lambda+u)c''(u)u_t' + w_t'\right)c''(u)(1+e^x) \\ & - \left((\lambda+u)c'(u) + w_t - c(u) - v\right)\left(c'''(u)u_t'(1+e^x) + e^x(\lambda+u)c''(u)\right).\end{aligned}$$

So that when $u_t' = 0$ we obtain

$$c''(u)(1+e^x)u_t'' = w_t'.$$

This immediately implies the first conclusion.

In the linear case, mimicking the proof of Lemma 2.1, Pontryagin's principle applies, and yields the existence of an absolutely continuous function $\gamma : [0, T] \to \mathbb{R}$ such that

$$\gamma_t - \alpha \left(1 + e^{-x_t}\right) > (<)0 \Rightarrow u_t = \bar{u} \ \ (u_t = 0) .$$

as well as

$$\dot{\gamma}_t = e^{-x_t}(w_t - \alpha u_t - v), \gamma_T = ke^{-x_T}.$$

Define ϕ by $\phi_t := \gamma_t - \alpha(1 + e^{-x_t})$. Note that $\phi_t > 0$ (resp. < 0) $\Rightarrow u_t = \bar{u}$ (resp. $= 0$). Differentiating ϕ_t with respect to time, and using the last equation gives

$$\dot{\phi}_t = e^{-x_t}(\alpha\lambda + w_t - v), \phi_T = (k - \alpha)e^{-x_T} - \alpha.$$

(This is the formal derivation of eqn. (8).) Observe now that if w is monotone, so is $\alpha\lambda + w_t - v$, and hence $\dot{\phi}$ changes signs only once. Conclusion 1 follows for the linear case. If it is strictly monotone, ϕ is equal to zero at most at one date t, and so the optimal strategy is extremal, yielding the second conclusion of the lemma. □

Proof of Proposition 3.4. Recall from the proof of Lemma 2.3 that

$$\frac{1}{x'(u)} = \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\dot{u}}{\dot{x}} = \frac{(\lambda + u)\, c'(u) - c(u) - v}{c''(u)\,(1 + e^x)\,(\lambda + u)}$$

must hold for the optimal trajectory (in the (x, u)-plane) for the social planner. Denote this trajectory x^{FB}. The corresponding law of motion for the agent's optimum trajectory x^* given w is

$$\frac{1}{x'(u)} = \frac{(\lambda + u)\, c'(u) - c(u) + w_t - v}{c''(u)\,(1 + e^x)\,(\lambda + u)}.$$

(Note that, not surprisingly, time matters). This implies that (in the (x, u)-plane) the trajectories x^{FB} and x^* can only cross one way, if at all, with x^* being the flatter one. Yet the (decreasing) transversality curve of the social planner, implicitly given by

$$(1 + e^{x_T})\, c'(u_T) = 1 + k,$$

lies above the (decreasing) transversality curve of the agent, which is defined by

$$(1 + e^{x_T})\, c'(u_T) = k.$$

Suppose now that the trajectory x^{FB} ends (on the transversality curve) at a lower belief x_T^{FB} than x^*: then it must be that effort u was higher throughout along that trajectory than along x^* (since the latter is flatter, x^{FB} must have remained above x^* throughout). But since the end value of the belief x is simply $x^0 + \int_0^T u_s \mathrm{d}s$, this contradicts $x_T^{FB} < x_T^*$.

It follows that for a given x, the effort level u is higher for the social planner.

The same reasoning implies the second conclusion: if $x_T^{FB} = x_T^*$, so that total effort is the same, yet the trajectories can only cross one way (with x^* being flatter), it follows that x^* involves lower effort first, and then larger effort, i.e. the agent backloads effort. □

C Proofs for Section 4

We shall start with Proposition 4.3 before turning to Theorem 4.2.

Proof of Proposition 4.3. We prove the following:

1. If there exists $t \in (0,T)$ such that $\phi_t > 0$, then there exists $t' \in [t,T]$ such that $u_s = \bar{u}$ for $s \in [t,t']$, $u_s = 0$ for $s \in (t',T]$.
2. If there exists $t \in (0,T)$ such that $\phi_t < 0$, then either $u_s = 0$ for all $s \in [t,T]$ or $u_s = 0$ for all $s \in [0,t]$,

which implies the desired decomposition. For the first part, note that either $u_s = \bar{u}$ for all $s > t$, or there exists t'' such that both $\phi_{t''} > 0$ (so in particular $u_{t''} = \bar{u}$) and $\dot{\phi}_{t''} < 0$. Because p_t decreases over time, and $u_s \leq u_{t''}$ for all $s > t''$, it follows that $w_s < w_{t''}$, and so $\dot{\phi}_s < \dot{\phi}_{t''} < 0$. Hence ϕ can cross 0 only once for values above t, establishing the result. For the second part, note that either $u_s = 0$ for all $s \geq t$, or there exists $t'' \geq t$ such that $\phi_{t''} < 0$ (so in particular $u_{t''} = 0$) and $\dot{\phi}_{t''} > 0$. Because p_t decreases over time, and $u_s \geq u_{t''}$ for all $s < t''$, it follows that $w_s \geq w_{t''}$, and so $\dot{\phi}_s > \dot{\phi}_{t''} > 0$. For all $s < t''$, $\phi_s < 0$ and $\dot{\phi}_s > 0$. Hence, $u_s = 0$ for all $s \in [0,t]$. □

Proof of Theorem 4.2. We study the linear and convex cases in turn.

Proof of Theorem 4.2 (Linear case). We start by establishing uniqueness.

Uniqueness: Assume an equilibrium exists, and note that, given a final belief x_T, the pair of differential equations for ϕ and x (along with the transversality condition) admit a unique solution, pinning down, in particular, the effort exerted by, and the wage received by the agent. Therefore, if two (or more) equilibria existed for some values (x_0, T), it would have to be the case that each of them is associated with a different terminal belief x_T. However, we shall show that, for any x_0, the time it takes to reach a terminal belief x_T is a continuous, strictly increasing function $T(x_T)$; therefore, no two different terminal beliefs can be reached in the same time T.

We start with a very optimistic initial belief $x_0 < x_1$, as this allows for the richest paths (the other cases are subsets of these).

Clearly, we have $T(x_0) = 0$. As long as $x_0 < x^*$, we have a first range for x_T over which full effort is always exerted. For these terminal beliefs, we have $T(x_T) = (x_T - x_0)/(\lambda + \bar{u})$, increasing. If for all $x_T \leq x^*$ the following expression is strictly positive

$$(k-\alpha)\, e^{-x_T} - \alpha - \int_{x_0}^{x_T} e^{-x} \left(\frac{1}{1+e^x} - \frac{v - \alpha\lambda}{\lambda + \bar{u}} \right) \mathrm{d}x, \tag{21}$$

then we always have full effort, until $x_T = x^*$. If so, go to the section "Long Terminal Beliefs." Otherwise, go to the section "Short Terminal Beliefs."

Short Terminal Beliefs

For these beliefs, we have a full effort phase at the end. We assume $x_0 < x_1 < x^*$, as the other cases are subsets of those discussed here. Full effort is exerted at the end typically for short deadlines. If $x_T < x^*$ then the full effort region is given by $[x_2, x_T]$, where x_2 solves

$$(k-\alpha)\, e^{-x_T} - \alpha - \int_{x_2}^{x_T} e^{-x} \left(\frac{1}{1+e^x} - \frac{v-\alpha\lambda}{\lambda+\bar{u}} \right) \mathrm{d}x = 0.$$

Therefore, we have

$$\frac{\mathrm{d}x_2}{\mathrm{d}x_T} = \left(\frac{1}{1+e^{x_2}} - \frac{v-\alpha\lambda}{\lambda+\bar{u}} \right)^{-1} \left(k - \alpha + \frac{1}{1+e^{x_T}} - \frac{v-\alpha\lambda}{\lambda+\bar{u}} \right) e^{x_2 - x_T}.$$

The denominator is positive by construction ($\phi(x)$ hits zero going backwards).

1. Suppose $x_2 > x_1$. Then the time to get to x_T is given by

$$T(x_T) = \frac{x_T - x_2}{\lambda+\bar{u}} + \int_{x_1}^{x_2} \frac{\mathrm{d}x}{\lambda + u(x)} + \frac{x_1 - x_0}{\lambda}.$$

Using the formula for interior effort,

$$u(x) = (v-\alpha\lambda)(1+e^x) - \lambda,$$

we can write

$$\begin{aligned} T'(x_T) &= \frac{1}{\lambda+\bar{u}} + \frac{\mathrm{d}x_2}{\mathrm{d}x_T} \frac{\bar{u} - u(x_2)}{(\lambda+\bar{u})(\lambda+u(x_2))} \\ &\propto \lambda + u(x_2) + \frac{\mathrm{d}x_2}{\mathrm{d}x_T}(\bar{u} - u(x_2)) \\ &= (v-\alpha\lambda)(1+e^{x_2}) + (\bar{u} - u(x_2)) \frac{\mathrm{d}x_2}{\mathrm{d}x_T}. \end{aligned}$$

We want to show $T'(x_T) > 0$. Clearly, if $\mathrm{d}x_2/\mathrm{d}x_T > 0$, we are done. If not, then we have

$$\begin{aligned} T'(x_T) &> (v-\alpha\lambda)(1+e^{x_2}) + (\lambda + \bar{u} - (v-\alpha\lambda)(1+e^{x_2})) \frac{\mathrm{d}x_2}{\mathrm{d}x_T} e^{-(x_2 - x_T)} \\ &= (v-\alpha\lambda)(1+e^{x_2}) + (1+e^{x_2})(\lambda+u) \left(k - \alpha + \frac{1}{1+e^{x_T}} - \frac{v-\alpha\lambda}{\lambda+\bar{u}} \right) \\ &\propto k - \alpha + \frac{1}{1+e^{x_T}} > 0. \end{aligned}$$

2. Now suppose $x_0 < x_2 < x_1$, and so no effort is exerted on $[x_0, x_2]$. Notice that if $x_2(x_T) \leq x_0$ then $T(x_T)$ is clearly increasing, in x_T (since we have full effort throughout). If $x_2(x_T) > x_0$, the time necessary to reach the terminal belief is given by

$$T(x_T) = \frac{x_T - x_2}{\lambda+\bar{u}} + \frac{x_2 - x_0}{\lambda}.$$

Therefore,

$$\lambda\left(\lambda+\bar{u}\right)T'\left(x_T\right)=\lambda+\bar{u}\frac{\mathrm{d}x_2}{\mathrm{d}x_T}.$$

It is immediate that if x_2 is increasing in x_T then $T'(\cdot)>0$. If not, then we have

$$\begin{aligned}T'\left(x_T\right) &\propto \lambda+\bar{u}\frac{\mathrm{d}x_2}{\mathrm{d}x_T}>\lambda+\bar{u}\frac{\mathrm{d}x_2}{\mathrm{d}x_T}e^{-(x_2-x_T)}\\ &\propto \lambda\left(\frac{1}{1+e^{x_2}}-\frac{v-\alpha\lambda}{\lambda+\bar{u}}\right)+\bar{u}\left(k-\alpha+\frac{1}{1+e^{x_T}}-\frac{v-\alpha\lambda}{\lambda+\bar{u}}\right).\end{aligned}$$

We also know $e^{x_2}<e^{x_1}=\lambda/\left(v-\alpha\lambda\right)-1$, and thus

$$\begin{aligned}T'\left(x_T\right) &> \lambda\left(\frac{v-\alpha\lambda}{\lambda}-\frac{v-\alpha\lambda}{\lambda+\bar{u}}\right)+\bar{u}\left(k-\alpha+\frac{1}{1+e^{x_T}}-\frac{v-\alpha\lambda}{\lambda+\bar{u}}\right)\\ &= \bar{u}\left(k-\alpha+\frac{1}{1+e^{x_T}}\right)>0.\end{aligned}$$

Longer Terminal Beliefs

For $x_T>x^*$ we can have four possible patterns: never work (in which case the time to x_T is clearly increasing), zero-interior-zero, zero-interior-full-zero, or zero-full-zero. We now show that $T\left(x_T\right)$ is increasing under any of these patterns. In addition the times at which the equilibrium path switches between the various effort regions are continuous functions of x_T, so it suffices to establish $T'\left(x_T\right)$ in each of these cases separately.

Zero and Interior Effort Phases

We again consider the time necessary to reach a given terminal belief x_T. We consider beliefs $x_T>x^*$, for which the agent does not work at the end. If there is no full effort phase, the agent works at a rate

$$u\left(x\right)=\left(v-\alpha\lambda\right)\left(1+e^{x}\right)-\lambda$$

until the switching belief x_3, then stops until x_T. The two thresholds are linked by the equation

$$\left(k-\alpha\right)e^{-x_T}-\alpha-\int_{x_3}^{x_T}e^{-x}\left(\frac{1}{1+e^{x}}+\alpha-\frac{v}{\lambda}\right)\mathrm{d}x=0.$$

From the state equation, we know beliefs increase at rate $\lambda+u\left(x\right)$ in the first phase, and at rate λ afterwards. The time to x_T is therefore given by

$$T\left(x_T\right)=\int_{x_1}^{x_T}\frac{1}{\lambda+u\left(x\right)}\mathrm{d}x=\int_{x_1}^{x_3(x_T)}\frac{1}{\left(v-\alpha\lambda\right)\left(1+e^{x}\right)}\mathrm{d}x+\frac{x_T-x_3\left(x_T\right)}{\lambda}.$$

Consider the derivative of T with respect to x_T,

$$\lambda T'\left(x_T\right)=1+\left(\frac{\lambda}{\lambda+u\left(x_3\right)}-1\right)\frac{\mathrm{d}x_3}{\mathrm{d}x_T},$$

where $\mathrm{d}x_3/\mathrm{d}x_T$ is given by

$$\frac{\mathrm{d}x_3}{\mathrm{d}x_T}=\left(\frac{1}{1+e^{x_3}}-\frac{v}{\lambda}+\alpha\right)^{-1}\left(k+\frac{1}{1+e^{x_T}}-\frac{v}{\lambda}\right)e^{x_3-x_T}. \tag{22}$$

Now, we know $(1+e^{x_3})^{-1}+\alpha-v/\lambda<0$ for all $x>x_1$. Therefore, if $k\geq v/\lambda$ (or more generally if $(1+e^{x_3})^{-1}+k-v/\lambda>0$), then $\mathrm{d}x_3/\mathrm{d}x_T<0$, the whole expression is positive and we are done.
Conversely, suppose that $(1+e^{x_3})^{-1}+k-v/\lambda<0$. We then check whether $T'(x_T)$ can be negative. We obtain

$$\begin{aligned}\lambda T'(x_T) &= 1-e^{x_3-x_T}\frac{u(x_3)}{\lambda+u(x_3)}\left(\frac{v}{\lambda}-\frac{1}{1+e^{x_T}}-k\right)\Big/\left(\frac{v}{\lambda}-\frac{1}{1+e^{x_3}}-\alpha\right)\\ &> 1-\frac{u(x_3)}{\lambda+u(x_3)}\left(\frac{v}{\lambda}-\frac{1}{1+e^{x_T}}-k\right)\Big/\left(\frac{v}{\lambda}-\frac{1}{1+e^{x_3}}-\alpha\right).\end{aligned}$$

Now plug in the expression for $u(x_3)$, notice that the x_3 drops out, and obtain

$$\lambda T'(x_T)>\lambda\frac{k-\alpha}{v-\alpha\lambda}>0.$$

Full and Interior Effort Phases

Now suppose the path involves interior effort on $[x_1,x_2]$, full effort on $[x_2,x_3]$ and zero effort on $[x_3,x_T]$. The time it takes to reach x_T is then given by

$$\lambda T(x_T)=\int_{x_1}^{x_2(x_T)}\frac{\lambda}{(v-\alpha\lambda)(1+e^x)}\mathrm{d}x+\frac{\lambda}{\lambda+\bar{u}}(x_3(x_T)-x_2(x_T))+x_T-x_3(x_T).$$

Hence

$$\lambda T'(x_T)=1-\frac{\bar{u}}{\lambda+\bar{u}}\frac{\mathrm{d}x_3}{\mathrm{d}x_T}+\frac{\mathrm{d}x_2}{\mathrm{d}x_T}\left(\frac{\lambda}{\lambda+u(x_2)}-\frac{\lambda}{\lambda+\bar{u}}\right).$$

Notice that x_2 is the solution to

$$\int_{x_2}^{x_3}e^{-x}\left(\frac{1}{1+e^x}+\frac{\alpha\lambda-v}{\lambda+\bar{u}}\right)\mathrm{d}x=0. \tag{23}$$

We then have

$$\frac{\mathrm{d}x_2}{\mathrm{d}x_T}=-\frac{e^{-x_3}\left(\frac{1}{1+e^{x_3}}+\frac{\alpha\lambda-v}{\lambda+\bar{u}}\right)}{e^{-x_2}\left(\frac{1}{1+e^{x_2}}+\frac{\alpha\lambda-v}{\lambda+\bar{u}}\right)}\frac{\mathrm{d}x_3}{\mathrm{d}x_T}, \tag{24}$$

and so

$$\lambda T'(x_T)=1-\frac{\mathrm{d}x_3}{\mathrm{d}x_T}\frac{\bar{u}}{\lambda+\bar{u}}+\frac{\mathrm{d}x_2}{\mathrm{d}x_T}\left(\frac{\lambda}{\lambda+u(x_2)}-\frac{\lambda}{\lambda+\bar{u}}\right).$$

Clearly, $k\geq v/\lambda$ implies $\mathrm{d}x_3/\mathrm{d}x_T<0$, $\mathrm{d}x_2/\mathrm{d}x_T>0$ and $T'(x_T)>0$.
Conversely, suppose $k<v/\lambda$. Plug in the explicit formula for $u(x_2)$ and for $\mathrm{d}x_3/\mathrm{d}x_T$ to obtain the following expression for $\lambda T'(x_T)$:

$$\frac{(e^{x_T}(v-k\lambda)-(k+1)\lambda+v)(-\lambda e^{x_2}(\bar{u}+e^{x_3}(\alpha\lambda-v)-v+\alpha\lambda+\lambda)-\bar{u}(e^{x_3}+1)e^{x_3}(v-\alpha\lambda))}{e^{x_T}(\bar{u}+\lambda)(e^{x_T}+1)(v-\alpha\lambda)(e^{x_3}(v-\alpha\lambda)+v-(\alpha+1)\lambda)}+1.$$

To simplify, let $V=\lambda/(v-\alpha\lambda)-1, U=(\lambda+\bar{u})/(v-\alpha\lambda)-1$, $k=\alpha(K+1)$, and $X_i=e^{x_i}$ to get

$$1-\frac{(K(V+1)(X_3+1)\alpha+V-X_T)\left(U\left(VX_2+X_2+X_3^2+X_3\right)-X_3(V(X_2+X_3+1)+X_2)\right)}{(U+1)X_T(X_T+1)(V-X_3)}.$$

The constraints are: $0 < V < X_2 < U < X_3 < X_T$, $0 < K < X_T$, and $\alpha > 0$. Note that the conditions $v > \alpha\lambda$ and $\bar{u} > 0$ follow from $U > V > 0$. The condition $\alpha < k < \alpha(X_T + 1)$ is captured by $0 < K < X_T$. Finally, note that if $v > k\lambda$ (which is equivalent to $\alpha K < (1+V)^{-1}$) then this expression is positive, as it is linear in $A = K(1+V)\alpha$, and it is positive both for $A = 0, 1$.[37]

Full Effort Phase Only

In this case, the incentives to exert effort hit zero when beliefs are at a level that does not allow interior effort, or $x_2 < x_1$. The candidate equilibrium involves zero-full-zero effort. The time required is then given by

$$\begin{aligned} \lambda T(x_T) &= x_T - x_3 + (x_3 - x_2)\frac{\lambda}{\lambda + \bar{u}} + x_2 - x_0 \\ &= x_T - (x_3 - x_2)\frac{\bar{u}}{\lambda + \bar{u}} - x_0, \end{aligned}$$

where x_3 and x_2 solve the same equations as before. Therefore,

$$\lambda T'(x_T) = 1 - \frac{\bar{u}}{\lambda + \bar{u}}\left(\frac{\mathrm{d}x_3}{\mathrm{d}x_T} - \frac{\mathrm{d}x_2}{\mathrm{d}x_T}\right),$$

where the last two terms are given by equations (22) and (24) respectively.
If $k \geq v/\lambda$, then $\mathrm{d}x_3/\mathrm{d}x_T < 0$, $\mathrm{d}x_2/\mathrm{d}x_T > 0$, and we are done by the same argument as before.

[37] This requires a little bit of work. Consider the case $A = 0$. The derivative w.r.t. U of the expression is

$$-\frac{(1+V)(X_3+1)(X_3+X_2)(X_T-V)}{(1+U)^2(X_3-V)X_T(1+X_t)} < 0,$$

so the expression is minimized by choosing U as high as possible given the constraints, i.e. $U = X_3$, in which case the expression simplifies to

$$\frac{X_3 V + X_T(1 + X_T - X_3)}{X_T(1+X_T)} > 0.$$

Consider now $A = 1$. Similarly, the derivative w.r.t. U does not depend on U itself, so the expression is minimized at one of the extreme values of U; if $U = X_3$, it is equal to

$$\frac{X_3(1 + X_3 + V) + X_T(X_T - X_3 + 1)}{X_T(1+X_T)} > 0;$$

if $U = X_2$, the resulting expression's derivative w.r.t. X_2 is independent of X_2, so we can again plug in one of the two extreme cases, $X_2 = X_3$ or $X_2 = V$; the values are then, respectively,

$$\frac{X_3(1 + V + X_3) + X_T(X_T - X_3 + 1)}{X_T(1+X_T)} > 0$$

and

$$\frac{X_T(1 + X_T + V) - V(1 + V + X_3)}{X_T(1+X_T)} \geq \frac{X_3(X_3+1) - V(V+1)}{X_T(1+X_T)} \geq 0.$$

If $v > k\lambda$, then $\mathrm{d}x_3/\mathrm{d}x_T > 0$ and we proceed as follows. Substituting the expressions in (22) and (24), and using the same change of variable as before, we want to show that

$$1 - \frac{X_2\left(X_2+1\right)\left(U-V\right)\left(U-X_3\right)\left(X_T(K(V+1)\alpha-1)+K(V+1)\alpha+V\right)}{(U+1)X_T\left(X_T+1\right)\left(U-X_2\right)\left(V-X_3\right)} > 0.$$

To establish this inequality, it is simpler to bound α. Setting the expression to zero, this is equivalent to requiring that

$$\alpha K < \frac{(U+1)X_T\left(U-X_2\right)\left(V-X_3\right)}{(V+1)X_2\left(X_2+1\right)\left(U-V\right)\left(U-X_3\right)} - \frac{1}{X_T+1} + \frac{1}{V+1},$$

a sufficient condition for this is that $\alpha K < (1+V)^{-1}$, which is equivalent to $v > k\lambda$.

Existence: We have established that the time necessary to reach the terminal belief is a continuous and strictly increasing function. Therefore, the terminal belief reached in equilibrium is itself given by a strictly increasing function

$$x_T\left(T\right) : \mathbb{R}_+ \to [x_0, \infty).$$

Since there exists a unique path consistent with optimality for each terminal belief, given a deadline T we can establish existence by constructing the associated equilibrium outcome, and in particular, the equilibrium wage path. Existence and uniqueness of an optimal strategy for the worker, after any (on or off-path) history, follow then from Lemma 3.1.

Proof of Theorem 4.2 (Convex case). We proceed as in the linear case.
Uniqueness: Fix T. The two differential equations obeyed by the (x,u)-trajectory are

$$\begin{aligned} \dot{x} &= \lambda + u \\ \dot{u} &= \frac{\left(\lambda+u\right)c'\left(u\right) - c\left(u\right) + \frac{\lambda+u}{1+e^x} - v}{c''\left(u\right)\left(1+e^x\right)}. \end{aligned}$$

We have, using that $\mathrm{d}x = \left(\lambda+u\right)\mathrm{d}t$,

$$\frac{\mathrm{d}u}{\mathrm{d}x} =: f\left(u,x\right) = \frac{\left(\lambda+u\right)c'\left(u\right) - c\left(u\right) + \frac{\lambda+u}{1+e^x} - v}{\left(\lambda+u\right)c''\left(u\right)\left(1+e^x\right)}. \tag{25}$$

Recall also that the transversality curve is given by

$$\left(1+e^x\right)c'\left(u\right) = k,$$

and so

$$\frac{\mathrm{d}u_T}{\mathrm{d}x_T} = -\frac{c'\left(u\right)e^x}{\left(1+e^x\right)c''\left(u\right)} < 0.$$

Note that the slope of $u\left(x\right)$ at the deadline T is at most first positive then negative. To see this, differentiate the numerator in (25) and impose (25) equal to zero. We obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x_T}\frac{\mathrm{d}u\left(x_T\right)}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x_T}\left[\left(\lambda+u\left(x_T\right)\right)c'\left(u\left(x_T\right)\right) - c\left(u\left(x_T\right)\right) + \frac{\lambda+u\left(x_T\right)}{1+e^{x_T}}\right] \\ &= \left(\left(\lambda+u\left(x_T\right)\right)c''\left(u\left(x_T\right)\right) + \frac{1}{1+e^{x_T}}\right)\frac{\mathrm{d}u_T}{\mathrm{d}x_T} - \frac{\left(\lambda+u\left(x_T\right)\right)e^{x_T}}{\left(1+e^{x_T}\right)^2} < 0. \end{aligned}$$

Suppose now we had

$$u'(x) > \frac{\mathrm{d}u_T}{\mathrm{d}x_T}, \text{ at } x = x_T,$$

so that the trajectory does not cross the transversality line from above. Then we would be done. A path leading to a higher x_T lies below one leading to the lower x_T so later beliefs take longer to reach.

Denote the difference in the slopes of the effort and transversality lines by

$$\Delta(u,x) := \frac{(\lambda+u)c'(u) - c(u) + \frac{\lambda+u}{1+e^x} - v}{(\lambda+u)c''(u)(1+e^x)} + \frac{c'(u)e^x}{(1+e^x)c''(u)}.$$

Note that $\Delta = 0 \Rightarrow \Delta'(x_T) < 0$ so our trajectory crosses transversality (at most) first from below then from above.

More generally, the time required to reach terminal belief x_T is given by

$$T = \int_{x_0}^{x_T} \frac{1}{\lambda + u(x)}\mathrm{d}x = \int_{x_0}^{x_T} \left(\lambda + u(x_T) - \int_x^{x_T} u'(x)\,\mathrm{d}x\right)^{-1} \mathrm{d}x.$$

Differentiating with respect to x_T we obtain

$$\begin{aligned}\frac{\mathrm{d}T}{\mathrm{d}x_T} &= \frac{1}{\lambda+u(x_T)} + \left(u'(x_T) - \frac{\mathrm{d}u_T}{\mathrm{d}x_T}\right)\int_{x_0}^{x_T} \frac{1}{(\lambda+u(x))^2}\mathrm{d}x \\ &= \frac{1}{\lambda+u(x_T)} + \Delta(u_T,x_T)\int_{x_0}^{x_T} \frac{1}{(\lambda+u(x))^2}\mathrm{d}x.\end{aligned}$$

Clearly, if the function $u(x)$ crosses the transversality line from below ($\Delta > 0$) then we are done: a path leading to a higher x_T lies below one leading to the lower x_T so later beliefs take longer to reach. Imposing transversality and simplifying we obtain that a necessary condition for $\Delta > 0$ for all x_T is

$$k \geq v/\lambda.$$

Because we do not wish to assume that, note that the function $f(u,x)$ in (25) has the following properties

$$\begin{aligned} f(u,x) &\leq 0 \Rightarrow f_u(u,x) > 0 \\ f(u,x) &\geq 0 \Rightarrow f_x(u,x) < 0. \end{aligned}$$

In words, a trajectory at $(x, u+\mathrm{d}u)$ comes down not as fast as a trajectory at (x,u) if (x,u) is such that $\dot{u} \leq 0$. Conversely, a trajectory at $(x-\mathrm{d}x, u)$ climbs faster than a trajectory at (x,u) if (x,u) is such that $\dot{u} \geq 0$.

Therefore, consider a trajectory $u(x)$ such that $\Delta(x_T) < 0$. As we increase x_T, the new trajectory lies everywhere above the original one. For a small increase in x_T, because the trajectory changes continuously, the two properties of $f(u,x)$ ensure that the vertical distance between the two trajectories is maximized at x_T.

We then have the condition

$$\frac{\mathrm{d}T}{\mathrm{d}x_T} > \frac{1}{\lambda+u(x_T)} + (x_T - x_0)\frac{\Delta(u_T,x_T)}{(\lambda+u(x_T))^2}.$$

Using transversality and rewriting $\Delta(u_T, x_T)$ we obtain the condition

$$c''(u(x_T))(\lambda + u(x_T)) \geq \frac{x_T - x_0}{1 + e^{x_T}} \left(\frac{c(u(x_T)) + v}{\lambda + u(x_T)} - k - \frac{1}{1 + e^{x_T}} \right). \tag{26}$$

Note that (26) clearly holds at $x_T = x_0$. Furthermore, if $c''(0) > 0$, (26) also holds in the limit for $x_T \to \infty$. Finally, since $c''(u)(\lambda + u)$ was assumed increasing, a sufficient condition for (26) to be satisfied is given by

$$c''(0) > \frac{1}{\lambda} \left(\frac{v}{\lambda} - k \right) h(x_0) \geq \frac{1}{\lambda} \left(\frac{v}{\lambda} - k \right) e^{-x_0},$$

which is the condition for uniqueness. Existence is established as in the linear case.

Single-peakedness: Single-peakedness of effort is almost immediate. Substituting the equilibrium expression $w_t = (\lambda + u_t) / (1 + e^{x_t})$ in the boundary value problem (17). Differentiating u_t' further, we obtain

$$u_t' = 0 \Rightarrow c''(u) \left(1 + e^{-x}\right) u_t'' = -(w_t)^2,$$

which implies that the function u is at most first increasing then decreasing.

We now argue that the wage is single-peaked. In terms of x, the wage is given by

$$\begin{aligned} w(x) &= \frac{\lambda + u(x)}{1 + e^x}, \text{ and so} \\ w'(x) &= \frac{u'(x)}{1 + e^x} - \frac{\lambda + u(x)}{(1 + e^x)^2} e^x, \end{aligned}$$

so that $w'(x) = 0$ is equivalent to

$$u'(x) = w(x) e^x.$$

As in the proof of Lemma 2.1, when $w'(x) = 0$ we have

$$w''(x) = \frac{u''(x) - u'(x)}{1 + e^x}.$$

Furthermore, we know that

$$u'(x) = \frac{(\lambda + u) c'(u) - c(u) + \frac{\lambda + u}{1 + e^x} - v}{c''(u)(1 + e^x)(\lambda + u)}.$$

Mimicking the proof of Lemma 2.1, we conclude that $w'(x) = 0$ implies

$$u''(x) - u'(x) = -\frac{u'(x)(3c'' + (\lambda + u) c''') e^x}{c''(1 + e^x)} < 0,$$

if as we have assumed, $c'' + (\lambda + u) c''' > 0$. Therefore, we also have single-peaked (at most increasing then decreasing) wages. (More generally, if $c''' < 0$ but $3c'' + (\lambda + u) c'''$ is increasing in u then the wage can be increasing on at most one interval.) □

Proof of Proposition 4.4. An important distinction is whether a full effort region occurs right before the terminal belief $x_T = x^*$. This depends on the sign of

$$\phi'(x^* \mid \bar{u}) := \frac{1}{1 + e^{x^*}} - \frac{v - \alpha\lambda}{\lambda + \bar{u}} = \frac{\alpha}{k} - \frac{v - \alpha\lambda}{\lambda + \bar{u}} \lessgtr 0.$$

(1.) Fix a terminal belief $x_T = x_3 + \lambda(T-t)$ and consider the equation defining the no effort frontier, which is given in (51). The left hand side of (51) is decreasing in x_3, because fixing x_T the derivative is simply given by $\phi'(x_3 \mid u = 0)$, which is negative by construction. In addition, it is immediate to show that the left hand side of (51) is increasing in k and v and decreasing in α and λ, which establishes the result.

(2.) We analyze the cases of $x_T \leq x^*$ and $x_T > x^*$ separately.

Fix a terminal belief $x_T \leq x^*$ and consider the definition of the full effort frontier, which is obtained by setting $x_0 = x_2$ in equation (21). The left hand side of (21) is increasing in x_2, because fixing x_T the derivative is simply given by $\phi'(x_2 \mid u = \bar{u})$, which is positive by construction. In addition, it is immediate to show that the left hand side of (21) is increasing in k and v and decreasing in α, λ, and $\bar{u}$,which establishes the result.

Fix a terminal belief $x_T > x^*$ and consider the equation defining the full effort frontier, which in this case is given in (23) and depends on $x_3(x_T)$ as well. The left hand side of (23) is increasing in x_2, because fixing x_T and hence $x_3(x_T)$ the derivative is simply given by $\phi'(x_2 \mid u = \bar{u})$, which is positive by construction. In addition, it is immediate to show that the integrand in (21) is increasing in α, λ, and $\bar{u}$, and decreasing in v. Finally, the left hand side of (21) is decreasing in x_3 (the derivative is given by $\phi'(x_3 \mid u = \bar{u}) < 0$). Combining these facts with the comparative statics of x_3 from part (1.) establishes the result. □

D Proofs for Section 5

D.1 Proofs for Subsection 5.1

Proof of Theorem 5.1. The proof is divided in several steps. Consider the maximization program $\mathcal{P}$ in the text: we begin by conjecturing a full-zero (or "FO") solution, i.e. a solution in which the agent first exerts maximum effort, then no effort; we show this solution solves a relaxed program; and finally we verify that it also solves the original program.

D.1.1 Candidate solution

Consider the following compensation scheme: pay a wage $w_t = 0$ for $t \in [0, t_0] \cup [t_1, T]$, a constant wage $w_t = v - \alpha\lambda$ for $t \in [t_0, t_1]$, and a lump-sum L at $t = T$. The agent exerts maximal effort for $t \leq t_1$ and zero thereafter. Furthermore, the agent is indifferent among all effort levels for $t \in [t_0, t_1]$.

For short enough deadlines, there exists a payment scheme of this form that induces full effort throughout, *i.e.* $t_0 > 0$ and $t_1 = T$, and leaves the agent indifferent between effort levels at T. Whenever this is the case, we take this to be our candidate solution. The conditions that pin down this solution are given by indifference at T and by zero profits at $t = 0$. Recall the definition of ϕ_t from the proof of Proposition 3.3. The conditions are

then given by

$$\phi_T = (k-\alpha-L)\,e^{-x_T} - \alpha \;=\; 0, \tag{27}$$

$$\int_0^{t_0} \left(1+e^{-x_s}\right)\frac{\lambda+u}{1+e^{x_s}}\mathrm{d}s + \int_{t_0}^{T}\left(1+e^{-x_s}\right)\left(\frac{\lambda+\bar{u}}{1+e^{x_s}} - v + \alpha\lambda\right)\mathrm{d}s - \left(1+e^{-x_T}\right)L \;=\; 0. \tag{28}$$

As T increases, $t_0 \to 0$. Let T^* denote the longest deadline for which this solution induces full effort throughout. The threshold T^* is the unique solution to (27) and (28) with $x_T = x_0 + (\lambda+\bar{u})\,T$ and $t_0 = 0$.

Lemma D.1 *The candidate solution is the unique compensation scheme that induces full effort on* $[0, T^*]$.

Proof of Lemma D.1. Delaying any payment from t to t' would induce the agent to shirk at t' because he is now indifferent for $t \leq t_1$. Anticipating payments while preserving zero profits ex ante would lead the agent to shirk at t. To see this, notice that, if the firm wants to hold the ex-ante profit level constant and shift wages across time periods, it can do so by setting

$$\Delta w_1 = -\frac{1+e^{-x_2}}{1+e^{-x_1}}\Delta w_2.$$

Then by construction,

$$\Delta w_1 + \Delta w_2 = -\left(e^{-x_1}\Delta w_1 + e^{-x_2}\Delta w_2\right).$$

Therefore, by delaying payments (in a profit-neutral way, and without affecting effort), incentives at time t can be increased. Consider the function

$$\phi_t = \phi_T - \int_t^T e^{-x_s}\left(w_s - v + \alpha\lambda\right)\mathrm{d}s$$

and two times t_1 and t_2. Indeed, if $\Delta w_2 > 0$, then $\Delta w_1 < 0$ and $\Delta w_1 + \Delta w_2 > 0$, which increases ϕ_1. Conversely, anticipating payments reduces incentives ϕ_1. □

For $T > T^*$, we cannot obtain full effort throughout. Our candidate solution is then characterized by $t_0 = 0$, $t_1 < T$, indifference at $t = T$, and zero profits at $t = 0$. The final belief is given by $x_T = x_t + \lambda\,(T-t) + \bar{u}\,(t_1 - t)$. It is useful to rewrite our three conditions in beliefs space. We have

$$(k-\alpha-L)\,e^{-x_T} - \alpha + (v/\lambda - \alpha)\left(e^{-x_1} - e^{-x_T}\right) \;=\; 0, \tag{29}$$

$$e^{-x_0} - e^{-x_T} - \frac{v-\alpha\lambda}{\lambda+\bar{u}}\left(e^{-x_0} - e^{-x_1} + x_1 - x_0\right) - \left(1+e^{-x_T}\right)L \;=\; 0, \tag{30}$$

$$\frac{x_T - x_1}{\lambda} + \frac{x_1 - x_0}{\lambda+\bar{u}} - T \;=\; 0, \tag{31}$$

that determine the three variables (L, x_1, x_T) as a function of x_0 and T. In order to compute the solution, we can solve the second one for L and the third for x_T and obtain one equation in one unknown for x_1.

We can now compute the agent's payoff under this compensation scheme

$$\begin{aligned}\tilde{W}\left(x_0, T\right) &= \int_0^{t_1}\left(1+e^{-x_s}\right)\left(v - \alpha\lambda - \alpha\bar{u} - v\right)\mathrm{d}s - \int_{t_1}^{T}\left(1+e^{-x_s}\right)v\mathrm{d}s + \left(1+e^{-x_T}\right)(L-k)\\ &= -\int_{x_0}^{x_1}\left(1+e^{-x}\right)\alpha\mathrm{d}x - \int_{x_1}^{x_T}\left(1+e^{-x}\right)\frac{v}{\lambda}\mathrm{d}x + \left(1+e^{-x_T}\right)(L-k),\end{aligned}$$

where (L, x_1, x_T) are the solution to (29)–(31) given (x_0, T). Plugging in the value of L from (28), we can rewrite payoffs as

$$\tilde{W}(x_0, T) = -\int_{x_0}^{x_1} \left(\frac{v - \alpha\lambda}{\lambda + \bar{u}} + e^{-x} \left(\frac{v + \bar{u}\alpha}{\bar{u} + \lambda} - 1 \right) \right) \mathrm{d}x - \int_{x_1}^{x_T} \left(\frac{v}{\lambda} + e^{-x} \frac{v - \lambda}{\lambda} \right) \mathrm{d}x - \left(1 + e^{-x_T}\right) k.$$

Now fix x_0 and T. We denote by $J(x)$ the payoff under an offer that follows our candidate solution to an agent who holds belief x. This requires solving the system (29)–(31) as a function of the current belief and the residual time. In particular, we have $J(x) = \tilde{W}\left(x, T - \frac{x - x_0}{\lambda + \bar{u}}\right)$ when $x < x_1(x_0, T)$ and $J(x) = \tilde{W}\left(x, T - \frac{x_1 - x_0}{\lambda + \bar{u}} - \frac{x - x_1}{\lambda}\right)$ when $x \geq x_1(x_0, T)$.

Finally, we denote by $Y(x)$ the agent's continuation payoff at x under the original scheme. Notice that the bound in (12) ensures that

$$\frac{\lambda + \bar{u}}{1 + e^{x_t}} \geq v - \alpha\lambda,$$

for all $t \leq t_1$ and for all T. This means the firm is running a positive flow profit when paying $v - \alpha\lambda$ during full a effort phase, hence effort at t contributes positively to the lump sum L. In other words, the firm does not obtain positive profits when the agent's continuation value is $Y(x)$. We show how to derive this bound in Section D.1.5.

D.1.2 Original and relaxed programs

Consider the original program $\mathcal{P}$, and rewrite it in terms of the log-likelihood ratios x_t, up to constant terms.

$$W(t, x_t) = \max_{w,u} \int_t^T \left(1 + e^{-x_s}\right)(w_s - v - \alpha u_s)\,\mathrm{d}s - k e^{-x_T}, \tag{32}$$

$$\text{s.t. } u = \arg\max_u \int_t^T \left(1 + e^{-x_s}\right)(w_s - v - \alpha u_s)\,\mathrm{d}s - k e^{-x_T},$$

$$\forall \tau \geq t : \int_\tau^T \left(1 + e^{-x_s}\right)(w_s - v - \alpha u_s)\,\mathrm{d}s - k e^{-x_T} \geq W(\tau, x_\tau), \tag{33}$$

$$0 \leq \int_0^T \left(1 + e^{-x_t}\right)\left(\frac{\lambda + u_t}{1 + e^{x_t}} - w_t\right)\mathrm{d}t. \tag{34}$$

We first argue that the non negative profit constraint (34) will be binding. This is immediate if we observe that constraint (33) implies the firm cannot make positive profits on any interval $[t, T]$, $t \geq 0$. If it did, the worker could be poached by a competitor that offers, for example, the same wage plus a signing bonus. We now consider a relaxed problem in which we substitute (33) and (34) with the non positive profit constraint (35).

$$\begin{aligned} W(t, x_t) &= \max_{w,u} \int_0^T \left(1 + e^{-x_t}\right)(w_t - v - \alpha u_t)\,\mathrm{d}t - k e^{-x_T}, \\ \text{s.t. } u &= \arg\max_u \int_0^T \left(1 + e^{-x_t}\right)(w_t - v - \alpha u_t)\,\mathrm{d}t - k e^{-x_T}, \\ 0 &\geq \int_\tau^T \left(1 + e^{-x_t}\right)\left(\frac{\lambda + u_t}{1 + e^{x_t}} - w_t\right)\mathrm{d}t \text{ for all } \tau \leq T. \end{aligned} \tag{35}$$

We then use the following result to further relax this program.

Lemma D.2 *Let $T > T^*$ and consider our candidate solution described in (29)–(31). If another contract generates a strictly higher surplus $W(0, x_0)$, then it must yield a strictly higher x_T.*

Proof of Lemma D.2. We use the fact that our solution specifies maximal frontloading of effort, given x_T. Notice that we can rewrite the social surplus (which is equal to the agent's payoff at time 0) as

$$-(1+k-\alpha)\,e^{-x_T} - \alpha x_T - \int_0^T \left(1+e^{-x_t}\right)(v-\alpha\lambda)\,dt + \text{Constant}. \tag{36}$$

Therefore, for a given x_T, surplus is maximized by choosing the highest path for x_t, which is obtained by frontloading effort. Furthermore, (36) is strictly concave in x_T. Because $T > T^*$, we know from Proposition 3.4 that, under any non negative payment function w, the agent works strictly less than the social planner. Since the agent receives the entire surplus, his ex ante payoff is then strictly increasing in x_T. □

We therefore consider the even more relaxed problem $\mathcal{P}'$ which is given by

$$\begin{aligned} & \max_{w,u}\ x_T \\ \text{s.t. } u &= \arg\max_u \int_0^T \left(1+e^{-x_t}\right)(w_t - v - \alpha u_t)\,\mathrm{d}t - ke^{-x_T} \\ 0 &\geq \int_\tau^T \left(1+e^{-x_t}\right)\left(\frac{\lambda+u_t}{1+e^{x_t}} - w_t\right)\mathrm{d}t \text{ for all } \tau \leq T. \end{aligned}$$

We will prove that our candidate solves the relaxed program $\mathcal{P}'$. We then show that under our candidate "FO" solution, constraint (33) in the original program never binds (except at $t = 0$), and hence that we have found a solution to the original program $\mathcal{P}$.

D.1.3 Solving the relaxed program

We argue that our candidate "FO" contract solves the relaxed program $\mathcal{P}'$ in four steps:

1. Showing that gaps in effort provision should be achieved with zero wages and lump sums.
2. Ruling out final zero-full-zero ("OFO") phases (that is, a structure in which no effort is followed by maximum effort and then by no effort until T).
3. Ruling out an overall zero-full-zero phase (that is, an OFO phase beginning at 0 and extending to T).
4. Ruling out interior effort (that is, showing that $u_t \in \{0, \bar{u}\}$ a.e.).

For this part, as mentioned in the text, we also need the following technical assumption:

$$v \geq \lambda(1+k). \tag{37}$$

Zero wages and lump sums. Suppose the agent exerts zero effort on $[t, t']$ and consider his incentives to work at t as measured by the function ϕ_t. The firm can then backload all wages w_s owed to the agent at times $s \in [t, t']$,

and pay a single lump-sum $L_{t'}$ that keeps ϕ_t constant. Since the agent's incentives have not changed, the value of x_T also remains constant. However, by the argument in proof of Lemma D.1, we know the firm now obtains lower profits on $[t, t']$, hence the non positive continuation profits constraint (35) is relaxed.

Ruling out final OFO. Consider a final "OFO" phase. We must have non-positive profits at the beginning of the first O phase, *i.e.* $\pi_0 \leq 0$. Let x_0 denote the belief at the beginning of this phase. In addition, if the OFO does not begin at $t = 0$, we must have $\phi_{x_0} = 0$. Conversely, since OFO is final, at the end of the second O phase, we have ϕ_T as given by transversality. We now hold both ϕ_0 and π_0 constant, and we consider shrinking the initial O phase by varying the belief x_1 at the end of the first O.

Without loss, we consider O phases achieved through lump sums and zero wages. In particular let M and L denote the intermediate and final lump sums respectively. Let x_2 denote the second switching belief F→O, and consider the equations characterizing the endogenous variables.

$$\begin{aligned}
\phi_T - Le^{-x_T} + (v/\lambda - \alpha)\left(e^{-x_2} - e^{-x_T}\right) &= 0 \\
-Me^{-x_1} + (v/\lambda - \alpha)\left(e^{-x_0} - e^{-x_1}\right) &= \phi_0 = 0 \\
e^{-x_0} - e^{-x_T} - M\left(1 + e^{-x_1}\right) - \frac{v - \alpha\lambda}{\lambda + \bar{u}}\left(e^{-x_1} - e^{-x_2} + x_2 - x_1\right) - L\left(1 + e^{-x_T}\right) &= \pi_0 \\
x_0 + \frac{x_2 - x_1}{\lambda + \bar{u}}\bar{u} + \lambda T &= x_T \\
(k - \alpha)\, e^{-x_T} - \alpha &= \phi_T
\end{aligned}$$

Note that this includes as a special case the "OF" structure in which $x_2 = x_T$. We then wish to show that x_T is decreasing in x_1, or equivalently that

$$\partial x_2 / \partial x_1 < 1.$$

Therefore, solve for L and M and substitute into the first equation, letting $b := x_2 - x_1$. We obtain

$$\begin{aligned}
0 \;=\; & \phi_T + (v/\lambda - \alpha)\left(e^{-x_1 - b} - e^{-x_0 - \frac{b}{\lambda+\bar{u}}\bar{u} - \lambda T}\right) \\
& - \frac{e^{-x_0} - \pi_0 - (v/\lambda - \alpha)\left(e^{-x_0} - e^{-x_1}\right)\left(1 + e^{x_1}\right) - e^{-x_0 - \frac{b}{\lambda+\bar{u}}\bar{u} - \lambda T} - \frac{v - \alpha\lambda}{\lambda + \bar{u}}\left(e^{-x_1} - e^{-x_1 - b} + b\right)}{1 + e^{x_0 + \frac{b}{\lambda+\bar{u}}\bar{u} + \lambda T}}.
\end{aligned}$$

Multiplying by $(1 + e^{x_0 + \frac{b}{\lambda+\bar{u}}\bar{u} + \lambda T})e^{x_1}$ and collecting the terms in e^{x_1}, we obtain

$$\begin{aligned}
0 \;=\; & e^{2x_1}(v/\lambda - \alpha)\, e^{-x_0} + (v/\lambda - \alpha)\left(e^{x_0 - \frac{\lambda}{\lambda+\bar{u}}b + \lambda T} + \bar{u}\frac{e^{-b} - 1}{\lambda + \bar{u}}\right) \\
& + e^{x_1}\left(\begin{array}{c}(k - v/\lambda + 1)\left(1 + e^{-x_0 - \frac{b}{\lambda+\bar{u}}\bar{u} - \lambda T}\right) - \alpha e^{x_0 + \frac{b}{\lambda+\bar{u}}\bar{u} + \lambda T} \\ -\left((1 + \alpha - v/\lambda)\, e^{-x_0} - \pi_0 + 1 + v/\lambda - \frac{v - \alpha\lambda}{\lambda + \bar{u}}b\right)\end{array}\right).
\end{aligned} \tag{38}$$

For $\bar{u}$ large enough (see Section D.1.5), this expression is decreasing in b. Furthermore, it is quadratic in e^{x_1} with a positive coefficient on e^{2x_1}. Therefore

$$\frac{\partial b}{\partial x_1} = -\frac{\partial\,[38]\,/\partial x_1}{\partial\,[38]\,/\partial b}$$

has the same sign as $\partial [38] / \partial x_1$, computed at the relevant root. Let

$$\begin{aligned} A &= (v/\lambda - \alpha) e^{-x_0}, \\ B &= (k - v/\lambda + 1) \left(1 + e^{-x_0 - \frac{b}{\lambda+\bar{u}}\bar{u} - \lambda T}\right) - \alpha e^{x_0 + \frac{b}{\lambda+\bar{u}}\bar{u} + \lambda T} \\ &- \left((1 + \alpha - v/\lambda) e^{-x_0} - \pi_0 + 1 + v/\lambda - \frac{v - \alpha\lambda}{\lambda + \bar{u}} b\right), \\ C &= (v/\lambda - \alpha) \left(e^{-b} e^{x_0 + \frac{b}{\lambda+\bar{u}}\bar{u} + \lambda T} + \bar{u} \frac{e^{-b} - 1}{\lambda + \bar{u}}\right), \end{aligned}$$

and consider the two roots

$$e^{x_1} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}.$$

Because $A > 0$, if the relevant solution to (38) is the left root, then $\partial [38] / \partial x_1 < 0$ (which is the desired result here).

Now consider profits at the beginning of the full effort phase.

$$\begin{aligned} \pi_1 &= e^{-x_1} - e^{-x_T} - \frac{v - \alpha\lambda}{\lambda + \bar{u}} \left(e^{-x_1} - e^{-x_1 - b} + b\right) - L \left(1 + e^{-x_T}\right) \\ &= e^{-x_1} - \left(e^{-x_0} - M \left(1 + e^{-x_1}\right)\right) \\ &= e^{-x_1} - e^{-x_0} + (1 + e^{x_1}) (v/\lambda - \alpha) \left(e^{-x_0} - e^{-x_1}\right) \\ &= \left(e^{-x_0} - e^{-x_1}\right) \frac{v - \lambda + v e^{x_1} - \alpha\lambda - \alpha\lambda e^{x_1}}{\lambda} \\ &\propto (e^{x_1} + 1) (v - \alpha\lambda) - \lambda. \end{aligned}$$

Note that profits are increasing in x_1 (actually, their sign). Impose the solution $e^{x_1} = \frac{-B + \sqrt{B^2 - 4AC}}{2A}$, and find

$$\begin{aligned} \pi_1 &= \left(\frac{-B + \sqrt{B^2 - 4AC}}{2A} + 1\right) (v - \alpha\lambda) - \lambda \\ &> \left(-\frac{B}{2A} + 1\right) (v - \alpha\lambda) - \lambda \\ &\propto 2 (v/\lambda - \alpha - 1) e^{-x_0} - (k - v/\lambda + 1) \left(1 + e^{-x_0 - \frac{b}{\lambda+\bar{u}}\bar{u} - \lambda T}\right) \\ &\quad + \alpha e^{x_0 + \frac{b}{\lambda+\bar{u}}\bar{u} + \lambda T} + (1 + \alpha - v/\lambda) e^{-x_0} - \pi_0 + 1 + v/\lambda - \frac{v - \alpha\lambda}{\lambda + \bar{u}} b. \end{aligned}$$

Since $b \geq 0$ we have

$$\pi_1 > 2e^{-x_0} (v/\lambda - \alpha - 1) + \alpha e^{x_0} + (1 + \alpha - v/\lambda) e^{-x_0} + 1 + v/\lambda - (k - v/\lambda + 1) \left(1 + e^{-x_0}\right) - \frac{v - \alpha\lambda}{\lambda + \bar{u}} b,$$

and for $\bar{u}$ large enough we can ignore the last term and obtain

$$\pi_1 > (e^{x_0} + 1) (v - k\lambda) + v e^{x_0} - \lambda + \alpha\lambda e^{2x_0}.$$

Under assumption (37), this expression is positive for all x_0.Therefore we can have $\pi_1 < 0$ only under the lower root. We then conclude that $\partial [38] / \partial x_1 < 0$ and $\partial x_T / \partial x_1 < 0$.

Ruling out overall OFO. Suppose the optimal contract induced a single OFO phase. Then we would have $\pi_0 = 0$ and $\phi_0 \leq 0$. In addition, it would be optimal for the firm to keep profits as low as possible at the beginning of the F phase, so we also have $\pi_1 = 0$. We now shrink the initial O phase while holding profits constant and equal to zero at the beginning of the F phase. At the end of the second O phase, the value of ϕ is pinned down by the transversality condition.

We have the following equations for the endogenous variables x_2, x_T and L:

$$\begin{aligned}
\phi_T - Le^{-x_T} + (v/\lambda - \alpha)\left(e^{-x_2} - e^{-x_T}\right) &= \phi_0 = 0 \\
e^{-x_1} - e^{-x_T} - \frac{v - \alpha\lambda}{\lambda + \bar{u}}\left(e^{-x_1} - e^{-x_2} + x_2 - x_1\right) - L\left(1 + e^{-x_T}\right) &= \pi_1 = 0 \\
x_0 + \frac{x_2 - x_1}{\lambda + \bar{u}}\bar{u} + \lambda T &= x_T \\
(k - \alpha)\, e^{-x_T} - \alpha &= \phi_T.
\end{aligned}$$

Note that an "OF" phase corresponds to the special case $x_2 = x_T$. We then wish to show that x_T is decreasing in x_1, or

$$\partial x_2 / \partial x_1 < 1.$$

Therefore, substitute the second and third into the first equation, and let $b := x_2 - x_1$. Collecting the terms with e^{-x_1}, we obtain

$$\begin{aligned}
0 &= \phi_T - \frac{e^{-x_1} - e^{-x_0 - \frac{b}{\lambda+\bar{u}}\bar{u} - \lambda T} - \frac{v-\alpha\lambda}{\lambda+\bar{u}}\left(e^{-x_1} - e^{-x_1 - b} + b\right)}{1 + e^{-x_0 - \frac{b}{\lambda+\bar{u}}\bar{u} - \lambda T}} e^{-x_0 - \frac{b}{\lambda+\bar{u}}\bar{u} - \lambda T} \\
&+ (v/\lambda - \alpha)\left(e^{-x_1 - b} - e^{-x_0 - \frac{b}{\lambda+\bar{u}}\bar{u} - \lambda T}\right) \\
&= e^{-x_1}\left((v/\lambda - \alpha)\, e^{-b} - \frac{1 - \frac{v-\alpha\lambda}{\lambda+\bar{u}}\left(1 - e^{-b}\right)}{1 + e^{-x_0 - \frac{b}{\lambda+\bar{u}}\bar{u} - \lambda T}} e^{-x_0 - \frac{b}{\lambda+\bar{u}}\bar{u} - \lambda T}\right) \\
&\quad + \phi_T + \left(\frac{e^{-x_0 - \frac{b}{\lambda+\bar{u}}\bar{u} - \lambda T} + \frac{v-\alpha\lambda}{\lambda+\bar{u}} b}{1 + e^{-x_0 - \frac{b}{\lambda+\bar{u}}\bar{u} - \lambda T}} - (v/\lambda - \alpha)\right) e^{-x_0 - \frac{b}{\lambda+\bar{u}}\bar{u} - \lambda T},
\end{aligned}$$

then solving for $e^{-x_1}(b)$ we get (just plug-in to verify)

$$\begin{aligned}
e^{-x_1} &= \frac{-\phi_T\left(1 + e^{x_0 + \frac{b}{\lambda+\bar{u}}\bar{u} + \lambda T}\right) + (v/\lambda - \alpha - 1)\, e^{-x_0 - \frac{b}{\lambda+\bar{u}}\bar{u} - \lambda T} + v/\lambda - \alpha - \frac{v-\alpha\lambda}{\lambda+\bar{u}} b}{e^{-b}\left((v/\lambda - \alpha)\left(1 + e^{x_0 + \frac{b}{\lambda+\bar{u}}\bar{u} + \lambda T}\right) - \frac{v-\alpha\lambda}{\lambda+\bar{u}}\right) - \left(1 - \frac{v-\alpha\lambda}{\lambda+\bar{u}}\right)} \\
&= \frac{\alpha e^{x_0 + \frac{b}{\lambda+\bar{u}}\bar{u} + \lambda T} + (v/\lambda - k - 1)\, e^{-x_0 - \frac{b}{\lambda+\bar{u}}\bar{u} - \lambda T} + v/\lambda - k + \alpha - \frac{v-\alpha\lambda}{\lambda+\bar{u}} b}{e^{-b}\left((v/\lambda - \alpha)\left(1 + e^{x_0 + \frac{b}{\lambda+\bar{u}}\bar{u} + \lambda T}\right) - \frac{v-\alpha\lambda}{\lambda+\bar{u}}\right) - \left(1 - \frac{v-\alpha\lambda}{\lambda+\bar{u}}\right)}.
\end{aligned}$$

For $\bar{u}$ high enough, the expression is increasing in b. Therefore, $x_1'(b) < 0$ and again we have

$$x_2'(x_1) = 1 + b'(x_1) = 1 + \frac{1}{x_1'(b(x_1))} < 1,$$

which is the desired result.

Ruling out interior effort. Consider a "FO" phase that generates profits π_0 and a terminal ϕ_T. This phase is characterized by the following equations:

$$\begin{aligned}
\phi_T - Le^{-x_T} + (v/\lambda - \alpha)\left(e^{-x_2} - e^{-x_T}\right) &= 0 \\
e^{-x_0} - e^{-x_T} - \frac{v - \alpha\lambda}{\lambda + \bar{u}}\left(e^{-x_0} - e^{-x_2} + x_2 - x_0\right) - L\left(1 + e^{-x_T}\right) &= \pi_0 \\
x_0 + \frac{x_2 - x_0}{\lambda + \bar{u}}\bar{u} + \lambda T &= x_T.
\end{aligned}$$

We ask whether we can improve the final x_T by choosing interior effort and generating the same revenue. We would then have

$$\begin{aligned}
\phi_T - Me^{-x_T} + (v/\lambda - \alpha)\left(e^{-x_1} - e^{-x_T}\right) &= 0 \\
e^{-x_0} - e^{-x_T} - \int_{x_0}^{x_1} \frac{v - \alpha\lambda}{\lambda + u(x)}\left(1 + e^{-x}\right)\mathrm{d}x - M\left(1 + e^{-x_T}\right) &= \pi_0 \\
x_0 + \int_{x_0}^{x_1} \frac{u(x)}{\lambda + u(x)}\mathrm{d}x + \lambda T &= x_T.
\end{aligned}$$

Now assume the agent is indifferent at the end of the phase, so that $\phi_T = 0$ (it is straightforward to extend the calculations to the case of a terminal ϕ_T pinned down by the transversality condition). We obtain

$$\begin{aligned}
\int_{x_1}^{x_T}\left(e^{-x} - \frac{v - \alpha\lambda}{\lambda}e^{-x}\left(1 + e^{x_T}\right)\right)\mathrm{d}x + \int_{x_0}^{x_1}\left(e^{-x} - \frac{v - \alpha\lambda}{\lambda + u(x)}\left(1 + e^{-x}\right)\right)\mathrm{d}x &= \pi_0 \\
x_0 + \int_{x_0}^{x_1} \frac{u(x)}{\lambda + u(x)}\mathrm{d}x + \lambda T &= x_T.
\end{aligned}$$

Now consider increasing u at x. We obtain

$$\frac{\mathrm{d}x_T}{\mathrm{d}u} = \frac{\lambda}{(\lambda + u(x))^2} + \frac{u(x_1)}{\lambda + u(x_1)}\frac{\partial x_1}{\partial u},$$

with

$$\frac{\partial x_1}{\partial u} = -\frac{\partial \pi / \partial u}{\partial \pi / \partial x_1} = -\frac{\frac{\partial \pi_0}{\partial x_T}\frac{\lambda}{(\lambda + u(x))^2} + \frac{v - \alpha\lambda}{(\lambda + u(x))^2}\left(1 + e^{-x}\right)}{\frac{\partial \pi_0}{\partial x_T}\frac{u(x_1)}{\lambda + u(x_1)} + (v/\lambda - \alpha)e^{-x_1}\left(1 + e^{x_T}\right) - \frac{v - \alpha\lambda}{\lambda + u(x_1)}\left(1 + e^{-x_1}\right)},$$

therefore

$$\frac{\mathrm{d}x_T}{\mathrm{d}u} = \frac{\lambda}{(\lambda + u(x))^2} - \frac{\frac{\partial \pi_0}{\partial x_T}\frac{\lambda}{(\lambda + u(x))^2} + \frac{v - \alpha\lambda}{(\lambda + u(x))^2}\left(1 + e^{-x}\right)}{\frac{\partial \pi_0}{\partial x_T}\frac{u(x_1)}{\lambda + u(x_1)} + (v/\lambda - \alpha)e^{-x_1}\left(1 + e^{x_T}\right) - \frac{v - \alpha\lambda}{\lambda + u(x_1)}\left(1 + e^{-x_1}\right)}\frac{u(x_1)}{\lambda + u(x_1)}.$$

We can compute the derivative $\partial \pi_0 / \partial x_T$ and rewrite $\partial x_T / \partial u$ up to constant terms. We have

$$\frac{\mathrm{d}x_T}{\mathrm{d}u} = \left(\frac{u(x_1)}{\lambda + u(x_1)}e^{-x_1} + e^{x_T - x_1} - \frac{\lambda}{\lambda + u(x_1)}\right) - \left(1 + e^{-x}\right)\frac{u(x_1)}{\lambda + u(x_1)}.$$

Because this expression is strictly monotone in x, it is optimal to ask the agent for zero (or maximal) effort, and we have characterized the optimal contract under extremal effort levels. □

D.1.4 Competing contracts

We know that our FO contract maximizes aggregate effort (and hence, by frontloading, social surplus) for each (x_0, T). We now show that the agent's continuation value under the original contract is higher than the value of the best FO contract offered at a later date. In particular, consider a full-zero competing contract offered when the agent's belief is x, and denote by $x_2(x)$ the new switching belief. The value of the new contract for the agent is given by

$$J(x) = \int_x^{x_2} \frac{1+e^{-x}}{\lambda+\bar{u}} \left(\frac{\lambda+\bar{u}}{1+e^x} - v - \alpha\bar{u} \right) \mathrm{d}x + \int_{x_2}^{x_T} \frac{1+e^{-x}}{\lambda} \left(\frac{\lambda}{1+e^x} - v \right) \mathrm{d}x - \left(1+e^{-x_T}\right) k,$$

with

$$x_T(x_2) = x_0 + \lambda T + \bar{u}\frac{x_2 - x_0}{\lambda+\bar{u}}.$$

We compare this to the continuation value under the original contract. We are led to analyze three cases, depending on the timing of payments in the two contracts.

Case 1: the original contract induces full effort at x.

$$\begin{aligned} Y(x) &= \int_0^{t_1} \left(1+e^{-x_s}\right)(v - \alpha\lambda - \alpha\bar{u} - v)\,\mathrm{d}s - \int_{t_1}^{T} \left(1+e^{-x_s}\right) v \mathrm{d}s + \left(1+e^{-x_T}\right)(L_0 - k) \\ &= -\int_x^{x_1} \left(1+e^{-x}\right) \alpha \mathrm{d}x - \int_{x_1}^{x_T} \left(1+e^{-x}\right) \frac{v}{\lambda} \mathrm{d}x + \left(1+e^{-x_T}\right)(L_0 - k), \end{aligned}$$

where (x_1, x_T, L_0) are the switching and terminal beliefs, and the lump sum, under the original contract. We then evaluate the difference in continuation payoffs:

$$\frac{\mathrm{d}\left(Y(x) - J(x)\right)}{\mathrm{d}x} = \alpha\left(1+e^{-x}\right) + e^{-x} - \frac{v+\alpha\bar{u}}{\lambda+\bar{u}}\left(1+e^{-x}\right) - \frac{\partial J}{\partial x_2}\frac{\partial x_2}{\partial x}.$$

Note that because $\mathrm{d}J/\mathrm{d}x_2 > 0$ and because, by assumption, we have $(\lambda+\bar{u})/(1+e^x) > v - \alpha\lambda$, a sufficient condition for the difference to increase is $\partial x_2/\partial x < 0$. Furthermore, notice that the social surplus under a competing contract evolves according to:

$$\begin{aligned} \frac{\mathrm{d}J}{\mathrm{d}x_2} &= -\frac{v+\alpha\bar{u}}{\lambda+\bar{u}}\left(1+e^{-x_2}\right) + \frac{v}{\lambda}\left(1+e^{-x_2}\right) + \frac{\bar{u}}{\lambda+\bar{u}}\left(-\frac{v}{\lambda}\left(1+e^{-x_T}\right) + (1+k)\,e^{-x_T}\right) \\ &= \frac{\bar{u}}{\lambda+\bar{u}}\left(\left(1+k-\frac{v}{\lambda}\right)e^{-x_T} - \frac{v}{\lambda} + \left(\frac{v}{\lambda} - \alpha\right)\left(1+e^{-x_2}\right)\right). \end{aligned} \tag{39}$$

Using equation (30), and substituting into (29), we also have an expression characterizing the switching x_2 as a function of x and $x_T(x_2)$ only:

$$\begin{aligned} & (k - v/\lambda)\left(1+e^{-x_T}\right) + \left((v/\lambda - \alpha)\,e^{-x_2} - \alpha\right)\left(1+e^{x_T}\right) + e^{-x_T} \\ + \quad & \frac{v-\alpha\lambda}{\lambda+\bar{u}}\left(-e^{-x_2} + x_2\right) - e^{-x} + \frac{v-\alpha\lambda}{\lambda+\bar{u}}\left(e^{-x} - x\right) = 0. \end{aligned} \tag{40}$$

Totally differentiating with respect to x yields

$$\frac{\partial x_2}{\partial x} = \frac{e^{-x} - \frac{v-\alpha\lambda}{\lambda+\bar{u}}\left(1+e^{-x}\right)}{\frac{\bar{u}}{\lambda+\bar{u}}e^{-x_T}\left(1+k-v/\lambda\right) + \left(v/\lambda - \alpha\right)\frac{\lambda}{\lambda+\bar{u}}e^{x_T}e^{-x_2} + \alpha\frac{\bar{u}}{\lambda+\bar{u}}e^{x_T} - \left(1+e^{-x_2}\right)\frac{v-\alpha\lambda}{\lambda+\bar{u}} + \left(v/\lambda - \alpha\right)e^{-x_2}}. \tag{41}$$

Therefore

$$\frac{\mathrm{d}\left(Y\left(x\right)-J\left(x\right)\right)}{\mathrm{d}x}=\left(e^{-x}-\frac{v-\alpha\lambda}{\lambda+\bar{u}}\left(1+e^{-x}\right)\right)\left(1-\frac{\partial J}{\partial x_2}\frac{\partial x_2}{\partial x}\right).$$

Computing the difference between (39) and the denominator of (41) we obtain

$$\frac{\left(e^{-x_2}e^{x_T}-1\right)\left(v-\alpha\lambda\right)+\alpha\bar{u}\left(1+e^{x_T}\right)}{\bar{u}+\lambda}>0. \tag{42}$$

Clearly, $Y(x_0)=J(x_0)$ and $Y'>J'$ imply the original contract offers a higher continuation value to the agent at all x.

Case 2: the original contract induces no effort at x. If there is no effort to be exerted under the competing contract, the comparison is immediate. Assuming there is still effort to be exerted under the competing contract, we have

$$\begin{aligned}
Y\left(x\right) &= -\int_x^{x_T}\left(1+e^{-x}\right)\frac{v}{\lambda}\mathrm{d}x+\left(1+e^{-x_T}\right)\left(L_0-k\right),\\
J\left(x\right) &= \int_x^{x_2}\frac{1+e^{-x}}{\lambda+\bar{u}}\left(\frac{\lambda+\bar{u}}{1+e^{x}}-v-\alpha\bar{u}\right)\mathrm{d}x+\int_{x_2}^{x_T}\frac{1+e^{-x}}{\lambda}\left(\frac{\lambda}{1+e^{x}}-v\right)\mathrm{d}x-\left(1+e^{-x_T}\right)k,
\end{aligned}$$

with

$$\begin{aligned}
x_T\left(x,x_2\right) &= x+\lambda\left(T-\frac{x_1-x_0}{\lambda+\bar{u}}-\frac{x-x_1}{\lambda}\right)+\bar{u}\frac{x_2-x}{\lambda+\bar{u}}\\
&= \lambda T+\frac{\bar{u}x_1+\lambda x_0}{\lambda+\bar{u}}+\bar{u}\frac{x_2-x}{\lambda+\bar{u}}.
\end{aligned} \tag{43}$$

Therefore,

$$\begin{aligned}
\frac{\mathrm{d}\left(Y\left(x\right)-J\left(x\right)\right)}{\mathrm{d}x} &= \frac{v}{\lambda}\left(1+e^{-x}\right)+e^{-x}-\frac{v+\alpha\bar{u}}{\lambda+\bar{u}}\left(1+e^{-x}\right)\\
&+ \frac{\bar{u}}{\lambda+\bar{u}}\left(e^{-x_T}\left(1+k\right)-\frac{v}{\lambda}\left(1+e^{-x_T}\right)\right)-\frac{\mathrm{d}J}{\mathrm{d}x_2}\frac{\mathrm{d}x_2}{\mathrm{d}x}.
\end{aligned}$$

Furthermore, notice that $\mathrm{d}J/\mathrm{d}x_2$ is still given by (39). Therefore we can write

$$\begin{aligned}
\frac{\mathrm{d}\left(Y\left(x\right)-J\left(x\right)\right)}{\mathrm{d}x} &= \left(\frac{v}{\lambda}-\alpha\right)\left(1+e^{-x}\right)+e^{-x}-\frac{v-\alpha\lambda}{\lambda+\bar{u}}\left(1+e^{-x}\right)\\
&+ \frac{\bar{u}}{\lambda+\bar{u}}\left(e^{-x_T}\left(1+k\right)-\frac{v}{\lambda}\left(1+e^{-x_T}\right)\right)-\frac{\mathrm{d}J}{\mathrm{d}x_2}\frac{\mathrm{d}x_2}{\mathrm{d}x},
\end{aligned}$$

and with the notation $J'\left(x\right):=\left[\mathrm{d}J\left(x_2\left(x\right)\right)/\mathrm{d}x_2\right]_{x_2=x}$, we obtain

$$\frac{\mathrm{d}\left(Y\left(x\right)-J\left(x\right)\right)}{\mathrm{d}x}=e^{-x}+J'\left(x\right)-J'\left(x_2\right)\frac{\mathrm{d}x_2}{\mathrm{d}x}.$$

Since $\mathrm{d}J/\mathrm{d}x_2$ is decreasing in x_2 it will be sufficient to show that

$$e^{-x}+J'\left(x_2\right)\left(1-\frac{\mathrm{d}x_2}{\mathrm{d}x}\right)>0.$$

Assume $\mathrm{d}x_2/\mathrm{d}x > 1$ otherwise the result is immediate. Now consider equation (40) for $x_2(x)$, but keep in mind now x_T is given by (43). Differentiating with respect to x yields

$$\frac{\mathrm{d}x_2}{\mathrm{d}x} = \frac{e^{-x} - \frac{v-\alpha\lambda}{\lambda+\bar{u}}\left(1+e^{-x}\right) - \frac{\bar{u}}{\lambda+\bar{u}}\left(-e^{-x_T}\left(1+k-v/\lambda\right) + e^{x_T}\left(\left(v/\lambda-\alpha\right)e^{-x_2}-\alpha\right)\right)}{e^{-x_2}\left(v/\lambda-\alpha\right)\left(1+e^{x_T}\right) - \left(1+e^{-x_2}\right)\frac{v-\alpha\lambda}{\lambda+\bar{u}} - \frac{\bar{u}}{\lambda+\bar{u}}\left(-e^{-x_T}\left(1+k-v/\lambda\right) + e^{x_T}\left(\left(v/\lambda-\alpha\right)e^{-x_2}-\alpha\right)\right)}.$$

Notice that the denominator (and the derivative $J'(x_2)$) are unchanged from the previous case. Therefore, if $\mathrm{d}x_2/\mathrm{d}x > 1$, we can use the result in (42) to conclude

$$\begin{aligned}
& e^{-x} + \frac{\mathrm{d}J}{\mathrm{d}x_2}\left(1 - \frac{\mathrm{d}x_2}{\mathrm{d}x}\right) \\
> \; & e^{-x} - \left(e^{-x} - \frac{v-\alpha\lambda}{\lambda+\bar{u}}\left(1+e^{-x}\right) - \left(e^{-x_2}\left(v/\lambda-\alpha\right)\left(1+e^{x_T}\right) - \left(1+e^{-x_2}\right)\frac{v-\alpha\lambda}{\lambda+\bar{u}}\right)\right) \\
= \; & \frac{v-\alpha\lambda}{\lambda+\bar{u}}\left(1+e^{-x}\right) + e^{-x_2}\left(v/\lambda-\alpha\right)\left(1+e^{x_T}\right) - \left(1+e^{-x_2}\right)\frac{v-\alpha\lambda}{\lambda+\bar{u}} \\
> \; & \frac{v-\alpha\lambda}{\lambda+\bar{u}}e^{-x} + e^{-x_2}\left(\frac{v-\alpha\lambda}{\lambda}\left(1+e^{x_T}\right) - \frac{v-\alpha\lambda}{\lambda+\bar{u}}\right) > 0.
\end{aligned}$$

Case 3: the competing contract induces full effort throughout. Consider the scenario in which the original contract induces full effort at x and the competing contract full effort throughout the remaining time. We obtain

$$\begin{aligned}
Y(x) &= -\int_x^{x_1}\left(1+e^{-x}\right)\alpha\mathrm{d}x - \int_{x_1}^{x_T}\left(1+e^{-x}\right)\frac{v}{\lambda}\mathrm{d}x + \left(1+e^{-x_T}\right)\left(L_0-k\right) \\
J(x) &= \int_x^{x_T}\frac{1+e^{-x}}{\lambda+\bar{u}}\left(\frac{\lambda+\bar{u}}{1+e^{x}} - v - \alpha\bar{u}\right)\mathrm{d}x - \left(1+e^{-x_T}\right)k,
\end{aligned}$$

with (for the competing contract)

$$x_T = x_0 + \left(\lambda+\bar{u}\right)T.$$

Therefore

$$Y'(x) - J'(x) = \alpha\left(1+e^{-x}\right) + e^{-x} - \frac{v+\alpha\bar{u}}{\lambda+\bar{u}}\left(1+e^{-x}\right) > 0$$

as before. If conversely the original contract induces zero effort for the remaining time, we have

$$\begin{aligned}
Y(x) &= -\int_x^{x_T}\left(1+e^{-x}\right)\frac{v}{\lambda}\mathrm{d}x + \left(1+e^{-x_T}\right)\left(L_0-k\right), \\
J(x) &= \int_x^{x_T}\frac{1+e^{-x}}{\lambda+\bar{u}}\left(\frac{\lambda+\bar{u}}{1+e^{x}} - v - \alpha\bar{u}\right)\mathrm{d}x - \left(1+e^{-x_T}\right)k,
\end{aligned}$$

and

$$x_T = x_0 + \left(\lambda+\bar{u}\right)T + \frac{\bar{u}}{\lambda}\left(x_1 - x\right).$$

Also remember it must be the case that

$$\phi_{x_T} := \left(k-\alpha\right)e^{-x_T} - \alpha > 0.$$

Therefore,

$$\begin{aligned} Y'(x) - J'(x) &= \left(1+e^{-x}\right)\frac{v}{\lambda} + \frac{1+e^{-x}}{\lambda+\bar{u}}\left(\frac{\lambda+\bar{u}}{1+e^{x}} - v - \alpha\bar{u}\right) + \frac{\bar{u}}{\lambda}\left(e^{-x_T} - (v+\alpha\bar{u})\frac{1+e^{-x_T}}{\lambda+\bar{u}} + ke^{-x_T}\right) \\ &> \left(1+e^{-x}\right)\frac{v}{\lambda} + \frac{1+e^{-x}}{\lambda+\bar{u}}\left(\frac{\lambda+\bar{u}}{1+e^{x}} - v - \alpha\bar{u}\right) + \frac{\bar{u}}{\lambda}\left(e^{-x_T} - (v-\alpha\lambda)\frac{1+e^{-x_T}}{\lambda+\bar{u}}\right) \\ &= e^{-x} + \frac{e^{-x}-e^{-x_T}}{\lambda+\bar{u}}\frac{\bar{u}}{\lambda}(v-\alpha\lambda) + \frac{\bar{u}}{\lambda}e^{-x_T} > 0. \end{aligned}$$

This ends this step of the proof.

D.1.5 Bound on $\bar{u}$

We now derive a lower bound on $\bar{u}$ that ensures

$$\frac{\lambda+\bar{u}}{1+e^{x}} \geq v - \alpha\lambda$$

over all beliefs x for which the agent exerts maximal effort. This clearly requires finding an upper bound on the range of such beliefs. Under the conjectured strategy, the switching belief and the lump sum payment are determined by the two equations (29) and (30). Solve (30) for L, and substitute into (29). We obtain

$$\left(k + \frac{1}{1+e^{x_T}} - v/\lambda\right)e^{-x_T} - \frac{e^{-x_0} - \frac{v-\alpha\lambda}{\lambda+\bar{u}}\left(e^{-x_0} - e^{-x_1} + x_1 - x_0\right)}{1+e^{x_T}} - \alpha + (v/\lambda - \alpha)\,e^{-x_1}. \tag{44}$$

Notice that as we let $x_T \to \infty$, x_1 must approach

$$\bar{x}_1 := \ln\left(v/\alpha\lambda - 1\right).$$

Furthermore, we have

$$\frac{\mathrm{d}x_1}{\mathrm{d}x_T} = \frac{\left((v/\lambda-\alpha)\,e^{-x_1} - \alpha\right)e^{x_T} - (k+1-v/\lambda)\,e^{-x_T}}{(v/\lambda-\alpha)\left(e^{x_T-x_1} - \frac{\lambda}{\lambda+\bar{u}} + \frac{\bar{u}}{\lambda(\bar{u}+\lambda)}e^{-x_1}\right)},$$

whose numerator is clearly positive.

If (37) holds, then $v/\lambda \geq 1+k$. Consider equation (44). Notice that the numerator of the second term is

$$e^{-x_0} - \frac{v-\alpha\lambda}{\lambda+\bar{u}}\left(e^{-x_0} - e^{-x_1} + x_1 - x_0\right) > \int_{x_0}^{x_1}\left(1+e^{-x}\right)\left(\frac{1}{1+e^{x}} - \frac{v-\alpha\lambda}{\lambda+\bar{u}}\right)\mathrm{d}x > 0.$$

Therefore the sum of the first two terms is negative, and hence the last two terms must be positive, which implies $x_1 < \bar{x}_1$.

Now suppose $v/\lambda \leq 1+k$. Notice that $x_1 > \bar{x}_1$ implies $\mathrm{d}x_1/\mathrm{d}x_T < 0$. Therefore, if $x_1\,(x_T)$ ever exceeds $\bar{x}_1$, it does so for all lower x_T compatible with work-shirk solutions. Now consider the equation characterizing the lowest such x_T:

$$\int_{x_0}^{x_T}\left(1+e^{-x}\right)\left(\frac{1}{1+e^{x}} - \frac{v-\alpha\lambda}{\lambda+\bar{u}}\right)\mathrm{d}x - \left(1+e^{-x_T}\right)\left(k - \alpha\left(1+e^{x_T}\right)\right) = 0.$$

As $x_0 \to -\infty$ the integrand is positive and grows without bound. This implies the solution to this equation x_T must also diverge to $-\infty$. But since we must have $x_1 \leq x_T$, this contradicts x_1 lying above $\bar{x}_1$ for some x_T.

It follows that $\bar{x}_1$ is a tight upper bound on x_1 independently of our assumption (37) on v, and that a lower bound on $\bar{u}$ is given by

$$\bar{u} \geq \left(\frac{v}{\alpha\lambda} - 1\right) v - \lambda.$$

D.2 Proofs for Subsection 5.2

Proof of Lemma 5.2. Suppose that the equilibrium effort is zero on some open set Ω. Consider the sets $\Omega_{t'} = \{(x,s) : s \in (t', T]\}$ such that the trajectory starting at (x,s) intersects Ω. Suppose that u is not identically zero on Ω_0 and let $\tau = \inf\{t' : \ u = 0 \ \text{on} \ \Omega_{t'}\}$. That is, for all $t' < \tau$, there exists $(x,s) \in \Omega_{t'}$ such that $u(x,s) > 0$. Suppose first that we take $(x,\tau) \in \Omega_\tau$. According to the definition of τ and Ω_τ, there exists $(x_k, k) \in \Omega$ such that the trajectory starting at (x,τ) intersects Ω at (x_k,k) and along the path the effort is zero. We can write the payoff

$$V(x,\tau) = \int_x^{x_k} \frac{1+e^{-s}}{1+e^{-x}} \left(\frac{\lambda}{1+e^s} - v\right) \frac{1}{\lambda} \mathrm{d}s + \frac{1+e^{-x_k}}{1+e^{-x}} V(x_k, k),$$

or, rearranging,

$$\left(1+e^{-x}\right) V(x,\tau) = -\left(e^{-x_k} - e^{-x}\right)\left(1 - \frac{v}{\lambda}\right) - \frac{v}{\lambda}(x_k - x) + \left(1+e^{-x_k}\right) V(x_k,k),$$

where $V(x_k,k)$ is differentiable. The Hamilton-Jacobi-Bellman (HJB) equation (a function of (x,τ)) can be derived from

$$\begin{aligned} V(x,\tau) \quad = \quad & \frac{\lambda+\hat{u}}{1+e^x}\mathrm{d}t - v\mathrm{d}t \\ & + \max_u \left[-\alpha u \mathrm{d}t + \left(1 - \frac{\lambda+u}{1+e^x}\mathrm{d}t + o(\mathrm{d}t)\right)\left(V(x,\tau) + V_x(x,\tau)(\lambda+u)\mathrm{d}t + V_t(x,\tau)\mathrm{d}t + o(\mathrm{d}t)\right)\right], \end{aligned}$$

which gives, taking limits,

$$0 = \frac{\lambda+\hat{u}}{1+e^x} - v + \max_{u\in[0,\bar{u}]}\left[-\alpha u - \frac{\lambda+u}{1+e^x} V(x,\tau) + V_x(x,\tau)(\lambda+u) + V_t(x,\tau)\right].$$

Therefore, if $u(x,\tau) > 0$,

$$-\frac{V(x,\tau)}{1+e^x} - \alpha + V_x(x,\tau) \geq 0, \text{ or } \left(1+e^{-x}\right) V_x(x,\tau) - e^{-x} V(x,\tau) \geq \alpha\left(1+e^{-x}\right),$$

or finally,

$$\frac{\partial}{\partial x}\left[\left(1+e^{-x}\right) V(x,\tau)\right] - \alpha\left(1+e^{-x}\right) \geq 0.$$

Notice, however, by direct computation, that, because low effort is exerted from (x,τ) to (x_k,k), for all points (x_s,s) on this trajectory, $s \in (\tau,k)$,

$$\frac{\partial}{\partial x}\left[\left(1+e^{-x_s}\right) V(x_s,s)\right] - \alpha\left(1+e^{-x_s}\right) = -e^{-x_s}\left(1+\alpha - \frac{v}{\lambda}\right) + \frac{v}{\lambda} - \alpha \leq 0,$$

so that, because $x < x_s$, and $1 + \alpha - v/\lambda > 0$,

$$\frac{\partial}{\partial x}\left[\left(1+e^{-x}\right)V(x,\tau)\right] - \alpha\left(1+e^{-x}\right) < 0,$$

a contradiction to $u(x,\tau) > 0$.

If instead $u(x,\tau) = 0$ for all $(x,\tau) \in \Omega_\tau$, then there exists $(x',t') \to (x,\tau) \in \Omega_\tau$, $u(x',t') > 0$. Because u is upper semi-continuous, for every $\varepsilon > 0$, there exists a neighborhood $\mathcal{N}$ of (x,τ) such that $u < \varepsilon$ on $\mathcal{N}$. Hence

$$\lim_{(x',t')\to(x,\tau)} \frac{\partial}{\partial x}\left[\left(1+e^{x'}\right)V(x',t')\right] - \alpha\left(1+e^{x'}\right) = \frac{\partial}{\partial x}\left[\left(1+e^{-x}\right)V(x,\tau)\right] - \alpha\left(1+e^{-x}\right) < 0,$$

a contradiction. □

Proof of Theorem 5.3. We start with (1.). That is, we show that $u(x,t) = \bar{u}$ for $x < \underline{x}_t$ in all equilibria. We first define $\underline{x}$ as the solution to the differential equation

$$(\lambda(1+\alpha) - v + (\lambda+\bar{u})\,\alpha e^{\underline{x}(t)} + \bar{u} - ((1+k)(\lambda+\bar{u}) - (v+\alpha\bar{u}))e^{-(\lambda+\bar{u})(T-t)})\left(\frac{\underline{x}'(t)}{\lambda+\bar{u}} - 1\right) = -\bar{u}, \tag{45}$$

subject to $\underline{x}(T) = x^*$. This defines a strictly increasing function of slope larger than $\lambda + \bar{u}$, for all $t \in (T - t^*, T]$, with $\lim_{t\uparrow t^*} \underline{x}(T-t) = -\infty$.[38] Given some equilibrium, and an initial value (x_t, t), let $u(\tau; x_\tau)$ denote the value at time $\tau \geq t$ along the equilibrium trajectory. For all t, let

$$\tilde{x}(t) := \sup\{x_t : \forall \tau \geq t : u(\tau; x_t) = \bar{u} \text{ in all equilibria}\},$$

with $\tilde{x}(t) = -\infty$ if no such x_t exists. By definition the function $\tilde{x}$ is increasing (in fact, for all $\tau \geq t$, $\tilde{x}(\tau) \geq \tilde{x}(t) + (\lambda+\bar{u})(\tau - t)$), and so it is a.e. differentiable (set $\tilde{x}'(t) = +\infty$ if x jumps at t). Whenever finite, let $s(t) = \tilde{x}'(t)/(\tilde{x}'(t) - \lambda) > 0$. Note that, from the transversality condition, $\tilde{x}(T) = x^*$. In an abuse of notation, we also write $\tilde{x}$ for the set function $t \to [\lim_{t'\uparrow t} \tilde{x}(t'), \lim_{t'\downarrow t} \tilde{x}(t')]$.

We first argue that the incentives to exert high effort decrease in x (when varying the value x of an initial condition (x,t) for a trajectory along which effort is exerted throughout). Indeed, recall that high effort is exerted iff

$$\frac{\partial}{\partial x}\left(V(x,t)\left(1+e^{-x}\right)\right) \geq \alpha\left(1+e^{-x}\right). \tag{46}$$

[38] The differential equation for $\underline{x}$ can be implicitly solved, which yields

$$\begin{aligned}\ln\frac{k-\alpha}{\alpha} &= (\underline{x}_t + (\lambda+\bar{u})(T-t)) + \frac{\bar{u}}{\lambda(1+\alpha)+\bar{u}-v}\ln(k-\alpha)\,\bar{u}\,(\lambda+\bar{u}) \\ &\quad - \frac{\bar{u}}{\lambda(1+\alpha)+\bar{u}-v}\ln\left(\begin{array}{c} e^{(\lambda+\bar{u})(T-t)}(\lambda(1+\alpha)+\bar{u}-v)(\lambda(1+\alpha)-v+\alpha(\lambda+\bar{u})e^{\underline{x}_t}) \\ -(\lambda(1+\alpha)-v)(\lambda(1+\alpha)+\bar{u}-v+(k-\alpha)(\lambda+\bar{u})) \end{array}\right).\end{aligned}$$

The value $V^H(x,t)$ obtained from always exerting (and being paid for) high effort is given by

$$\begin{aligned}(1+e^{-x})V^H(x,t) &= \int_t^T (1+e^{-x_s})\left[\frac{\lambda+\bar{u}}{1+e^{x_s}} - v - \alpha\bar{u}\right]\mathrm{d}s - k(1+e^{-x_T}) \\ &= \left(e^{-x}-e^{-x_T}\right)\left(1-\frac{v+\alpha\bar{u}}{\lambda+\bar{u}}\right) - (T-t)(v+\alpha\bar{u}) - k(1+e^{-x_T}) \qquad (47)\end{aligned}$$

where $x_T = x + (\lambda+\bar{u})(T-t)$. Therefore, using (46), high effort is exerted if and only if

$$k - \left(1+k-\frac{v+\alpha\bar{u}}{\lambda+\bar{u}}\right)\left(1-e^{-(\lambda+\bar{u})(T-t)}\right) \geq \alpha(1+e^{x}).$$

Note that the left-hand side is independent of x, while the right-hand side is increasing in x. Therefore, if high effort is exerted throughout from (x,t) onward, then it is also from (x',t) for all $x' < x$.

Fix an equilibrium and a state (x_0,t_0) such that $x_0 + (\lambda+\bar{u})(T-t_0) < x^*$. Note that the equilibrium trajectory must eventually intersect some state $(\tilde{x}_t,t)$. We start again from the formula for the payoff

$$\begin{aligned}\left(1+e^{-x_0}\right)V(x_0,t_0) &= \int_{t_0}^{t}\left[e^{-x_s}(\lambda+u(x_s,s)) - \left(1+e^{-x_s}\right)(v+\alpha u(x_s,s))\right]\mathrm{d}s \\ &\quad + \left(1+e^{-\tilde{x}_t}\right)V^H(\tilde{x}_t,t).\end{aligned}$$

Let $W(\tilde{x}_t) = V^H(\tilde{x}_t,t)$ (since $\tilde{x}$ is strictly increasing, it is well-defined). Differentiating with respect to x_0, and taking limits as $(x_0,t_0)\to(\tilde{x}_t,t)$, we obtain

$$\begin{aligned}&\lim_{(x_0,t_0)\to(\tilde{x}_t,t)} \frac{\partial\left(1+e^{-x_0}\right)V(x_0,t_0)}{\partial x_0} \\ &= \left[e^{-\tilde{x}_t}\lambda - \left(1+e^{-\tilde{x}_t}\right)v\right]\frac{s(\tilde{x}_t)-1}{\lambda} + s(\tilde{x}_t)\left[W'(\tilde{x}_t)\left(1+e^{-\tilde{x}_t}\right) - W(\tilde{x}_t)e^{-\tilde{x}_t}\right].\end{aligned}$$

If less than maximal effort can be sustained arbitrarily close to, but before the state $(\tilde{x}_t,t)$ is reached, it must be that this expression is no more than $\alpha\left(1+e^{-\tilde{x}_t}\right)$ in some equilibrium, by (46). Rearranging, this means that

$$\left(1-W(x)+(1+e^x)\left(W'(x)-\frac{v}{\lambda}\right)\right)s(x) + \left(\frac{v}{\lambda}-\alpha\right)e^x \leq 1+\alpha-\frac{v}{\lambda},$$

for $x=\tilde{x}_t$. Given the explicit formula for W (see (47)), and since $s(\tilde{x}_t) = \tilde{x}_t'/(\tilde{x}_t'-\lambda)$, we can rearrange this to obtain an inequality for $\tilde{x}_t$. The derivative $\tilde{x}_t'$ is smallest, and thus the solution $\tilde{x}_t$ is highest, when this inequality binds for all t. The resulting ordinary differential equation is precisely (45).

Next, we turn to (2.). That is, we show that $u(x,t)=0$ for $x > \bar{x}_t$ in all equilibria. We define $\bar{x}$ by

$$\bar{x}_t = \ln\left[k-\alpha+\left(\frac{v+\bar{u}\alpha}{\lambda+\bar{u}}-(1+k)\right)\left(1-e^{-(\lambda+\bar{u})(T-t)}\right)\right] - \ln\alpha, \qquad (48)$$

which is well-defined as long as $k-\alpha+\left(\frac{v+\bar{u}\alpha}{\lambda+\bar{u}}-(1+k)\right)\left(1-e^{-(\lambda+\bar{u})(T-t)}\right) > 0$. This defines a minimum time $T-t^*$ mentioned above, which coincides with the asymptote of $\underline{x}$ (as can be seen from (45)). It is immediate to check that $\bar{x}$ is continuous and strictly increasing on $[T-t^*,T]$, with $\lim_{t\uparrow t^*}\bar{x}_{T-t} = -\infty$, $x_T = x^*$, and for all $t\in(T-t^*,T)$, $\bar{x}_t' > \lambda+\bar{u}$.

Let us define $W(x,t) = (1+e^{-x})V(x,t)$, and re-derive the HJB equation. The payoff can be written as

$$W(x,t) = \left[(\lambda + u(x,t))\, e^{-x} - \left(1+e^{-x}\right)(v+\alpha u)\right] \mathrm{d}t + W(x+\mathrm{d}x, t+\mathrm{d}t),$$

which gives

$$0 = (\lambda + u(x,t))\, e^{-x} - v\left(1+e^{-x}\right) + W_t(x,t) + \lambda W_x(x,t) + \max_{u\in[0,\bar{u}]} \left(W_x(x,t) - \alpha\left(1+e^{-x}\right)\right) u.$$

As we already know (see (46)), effort is maximum or minimum depending on $W_x(x,t) \lessgtr \alpha(1+e^{-x})$. Let us rewrite the previous equation as

$$\begin{aligned} & v - \alpha\lambda - W_t(x,t) \\ = \;& ((1+\alpha)\lambda - v + u(x,t))\, e^{-x} + \lambda\left(W_x(x,t) - \alpha\left(1+e^{-x}\right)\right) + \left(W_x(x,t) - \alpha\left(1+e^{-x}\right)\right)^+ \bar{u}. \end{aligned}$$

Given W_x, W_t is maximized when effort $u(x,t)$ is minimized: the lower $u(x,t)$, the higher $W_t(x,t)$, and hence the lower $W(x,t-\mathrm{d}t) = W(x,t) - W_t(x,t)\,\mathrm{d}t$. Note also that, along any equilibrium trajectory, no effort is never strictly optimal (by (iv)). Hence, $W_x(x,t) \geq \alpha(1+e^{-x})$, and therefore, again $u(x,t)$ (or $W(x,t-\mathrm{d}t)$) is minimized when $W_x(x,t) = \alpha(1+e^{-x})$: to minimize $u(x,t)$, while preserving incentives to exert effort, it is best to be indifferent whenever possible.

Hence, integrating over the equilibrium trajectory starting at (x,t),

$$\begin{aligned} & (v-\alpha\lambda)(T-t) + k\left(1+e^{-x_T}\right) + W(x,t) \\ = \;& \int_t^T u(x_s,s)\, e^{-x_s}\mathrm{d}s + \int_t^T \left[((1+\alpha)\lambda - v)\, e^{-x_s} + (\lambda+\bar{u})\left(W_x(x_s,s) - \alpha\left(1+e^{-x_s}\right)\right)^+\right] \mathrm{d}s. \end{aligned}$$

We shall construct an equilibrium in which $W_x(x_s,s) = \alpha(1+e^{-x_s})$ for all $x > \underline{x}_t$. Hence, this equilibrium minimizes

$$\int_t^T u(x_s,s)\, e^{-x_s}\mathrm{d}s,$$

along the trajectory, and since this is true from any point of the trajectory onward, it also minimizes $u(x_s,s)$, $s\in[t,T]$; the resulting $u(x,t)$ will be shown to be increasing in x, and equal to $\bar{u}$ at $\bar{x}_t$.

Let us construct this interior effort equilibrium. Integrating (46) over any domain with non-empty interior, we obtain that

$$(1+e^x)V(x,t) = e^x(\alpha x + c(t)) - \alpha, \tag{49}$$

for some function $c(t)$. Because the trajectories starting at (x,t) must cross $\underline{x}$ (whose slope is larger than $\lambda+\bar{u}$), value matching must hold at the boundary, which means that

$$(1+e^{\underline{x}_t})V^H(\underline{x}_t,t) = e^{\underline{x}_t}(\alpha\underline{x}_t + c(t)) - \alpha,$$

which gives $c(t)$ (for $t \geq T - t^*$). From (49), we then back out $V(x,t)$. The HJB equation then reduces to

$$v - \alpha\lambda = \frac{\lambda + u(x,t)}{1+e^x} + V_t(x,t),$$

which can now be solved for $u(x,t)$. That is, the effort is given by

$$\begin{aligned}\lambda + u(x,t) &= (1+e^x)(v-\alpha\lambda) - \frac{\partial}{\partial t}\left[(1+e^x)V(x,t)\right] \\ &= (1+e^x)(v-\alpha\lambda) - e^x c'(t).\end{aligned}$$

It follows from simple algebra (c' is detailed below) that $u(x,t)$ is increasing in x. Therefore, the upper end $\bar{x}_t$ cannot exceed the solution to

$$\lambda + \bar{u} = \left(1+e^{\bar{x}}\right)(v-\alpha\lambda) - e^{\bar{x}}c'(t),$$

and so we can solve for

$$e^{\bar{x}} = \frac{\lambda(1+\alpha) - v + \bar{u}}{v - \alpha\lambda - c'(t)}.$$

Note that, from totally differentiating the equation that defines $c(t)$,

$$\begin{aligned}c'(t) &= \underline{x}'(t)\, e^{-\underline{x}(t)}\left[\left(W'(\underline{x}(t)) - \alpha\right)\left(e^{\underline{x}(t)}+1\right) - W(\underline{x}(t))\right] \\ &= v - \alpha\lambda + e^{-\underline{x}(t)}(v - (1+\alpha)\lambda),\end{aligned}$$

where we recall that $\underline{x}$ is the lower boundary below which effort must be maximal, and $W(\underline{x}) = V^H(\underline{x}_t, t)$. This gives

$$e^{\bar{x}} = e^{\underline{x}}\frac{\lambda(1+\alpha) - v + \bar{u}}{\lambda(1+\alpha) - v}, \text{ or } e^{\underline{x}} = \frac{\lambda(1+\alpha) - v}{\lambda(1+\alpha) - v + \bar{u}}e^{\bar{x}}.$$

Because (45) is a differential equation characterizing $\underline{x}$, we may substitute for $\bar{x}$ from the last equation to obtain a differential equation characterizing $\bar{x}$, namely

$$\begin{aligned}&\frac{\bar{u}}{1 - \frac{\bar{x}'(t)}{\lambda+\bar{u}}} + \left((1+k)(\lambda+\bar{u}) - (v+\alpha\bar{u})\right)e^{-(\lambda+\bar{u})(T-t)} \\ =\ &\lambda(1+\alpha) + \bar{u} - v + \frac{\alpha(\lambda+\bar{u})(\lambda(1+\alpha) - v)}{\lambda(1+\alpha) - v + \bar{u}}e^{\bar{x}},\end{aligned}$$

with boundary condition $\bar{x}(T) = x^*$. It is simplest to plug in the formula given by (48) and verify that it is indeed the solution of this differential equation.

Finally, we prove (3.). The same procedure applies to both, so let us consider $\bar{\sigma}$, the strategy that exerts high effort as long as $x < \bar{x}_t$, (and no effort above). We shall do so by "verification." Given our closed-form expression for $V^H(x,t)$ (see (47)), we immediately verify that it satisfies the (46) constraint for all $x \leq \bar{x}_t$ (remarkably, $\bar{x}_t$ is *precisely* the boundary at which the constraint binds; it is strictly satisfied at $\underline{x}_t$, when considering $\underline{\sigma}$). Because this function $V^H(x,t)$ is differentiable in the set $\{(x,t) : x < \bar{x}_t\}$, and satisfies the HJB equation, as well as the boundary condition $V^H(x,T) = 0$, it is a solution to the optimal control problem in this region (remember that the set $\{(x,t) : x < \bar{x}_t\}$ cannot be left under any feasible strategy, so that no further boundary condition needs to be verified). We can now consider the optimal control problem with exit region $\Omega := \{(x,t) : x = \bar{x}_t\} \cup \{(x,t) : t = T\}$ and salvage value $V^H(\bar{x}_t, t)$ or 0, depending on the exit point. Again, the strategy of exerting no effort satisfies the HJB equation, gives a differentiable value on $\mathbb{R} \times [0,T] \setminus \Omega$, and satisfies the boundary conditions. Therefore, it is a solution to the optimal control problem. □

Proof of Proposition 5.4 The results can be obtained directly by differentiating expression (48) for the frontier $\bar{x}(t)$. □

Proof of Proposition 5.5 (1.) The equation defining the full effort frontier in the unobservable case $x_2(t)$ is given by

$$(k-\alpha)\,e^{-x_2-(\lambda+u)(T-t)}-\alpha-\int_{x_2}^{x_2+(\lambda+u)(T-t)}e^{-x}\left(\frac{1}{1+e^x}-\frac{v-\alpha\lambda}{\lambda+\bar{u}}\right)\mathrm{d}x. \tag{50}$$

Plug the expression for $\bar{x}(t)$ given by (48) into (50) and notice that (50) cannot be equal to zero unless $\bar{x}(t)=x^*$ and $t=T$, or $\bar{x}(t)\to-\infty$. Therefore, the two frontiers cannot cross before the deadline T, but they have the same vertical asymptote.

Now suppose that $\phi'(x^*\mid\bar{u})>0$ so that the frontier $x_2(t)$ goes through (T,x^*). Consider the slopes of $x_2(t)$ and $\bar{x}(t)$ evaluated at (T,x^*). We obtain

$$[\bar{x}'(t)-x_2'(t)]_{t=T}\propto(\bar{u}+\lambda)(k-\alpha)>0,$$

so the unobservable frontier lies above the observable one for all t.

Next, suppose $\phi'(x^*\mid\bar{u})<0$, so there is no mixing at x^* and the frontier $x_2(t)$ does not go through (T,x^*). In this case, we still know the two cannot cross, and we also know a point on $x_2(t)$ is the pre-image of (T,x^*) under full effort. Since we also know the slope $\bar{x}'(t)>\lambda+\bar{u}$, we again conclude that the unobservable frontier $x_2(t)$ lies above $\bar{x}(t)$.

Finally, consider the equation defining the no effort frontier $x_3(t)$,

$$(k-\alpha)\,e^{-x_3-\lambda(T-t)}-\alpha-\int_{x_3}^{x_3+\lambda(T-t)}e^{-x}\left(\frac{1}{1+e^x}-\frac{v-\alpha\lambda}{\lambda}\right)\mathrm{d}x=0. \tag{51}$$

Totally differentiating with respect to t shows that $x_3'(t)<\lambda$ (might be negative). Therefore, the no effort region does not intersect the full effort region defined by $\bar{x}(t)$ in the observable case.

(2.) To compare the effort regions in the unobservable case and the full effort region in the social optimum, consider the planner's frontier $x_P(t)$, which is given by

$$x_P(t)=\ln\Big((1+k-v/\lambda)e^{-\lambda(T-t)}-(\alpha-v/\lambda)\Big)-\ln\alpha.$$

The slope of the planner's frontier is given by

$$x_P'(t)=\lambda\frac{(1+k-v/\lambda)e^{-\lambda(T-t)}}{(1+k-v/\lambda)e^{-\lambda(T-t)}+v/\lambda-\alpha}\in[0,\lambda].$$

In the equilibrium with unobservable effort, all effort ceases above the frontier $x_3(t)$ defined in (51) above, which has the following slope

$$x_3'(t)=\lambda\frac{\left(\left(1+e^{x_3+\lambda(T-t)}\right)^{-1}+k-v/\lambda\right)e^{-\lambda(T-t)}}{\left(\left(1+e^{x_3+\lambda(T-t)}\right)^{-1}+k-v/\lambda\right)e^{-\lambda(T-t)}+v/\lambda-\alpha-\left(1+e^{x_3}\right)^{-1}}.$$

We also know $x_3(T) = x^*$ and $x_P(T) = \ln((1+k-\alpha)/\alpha) > x^*$. Now suppose towards a contradiction that the two frontiers crossed at a point (t, x). Plugging in the expression for $x_P(t)$ in both slopes, we obtain

$$x_3'(t) = \left(1 + \frac{v/\lambda - \alpha - s(t)}{(1+k-v/\lambda+(1-s(t)))e^{-\lambda(T-t)}}\right)^{-1} > \left(1 + \frac{v/\lambda - \alpha}{(1+k-v/\lambda)e^{-\lambda(T-t)}}\right)^{-1} = x_P'(t),$$

with

$$s(t) = 1/\left(1 + e^{x_P(t)}\right) \in [0,1],$$

meaning the unobservable frontier would have to cross from below, a contradiction. □

D.3 Proofs for Subsection 5.3

This subsection starts by proving Proposition 5.6 in several steps.

If the agent is indifferent between continuing and stopping, then the flow expected payoff must be zero. His overall equilibrium payoff is given by

$$\int_0^T \left(e^{-x_t}((\lambda + u_t) - v) - \left(1 + e^{-x_t}\right)c(u_t)\right)\mathrm{d}t - ke^{-x_T}.$$

[Recall that, to analyze deadlines, we must take into account that v is premultiplied by the belief p_t in the original problem, see footnote 4.] When the agent stops, we also know the transversality condition

$$ke^{-x_T} = \left(1 + e^{-x_T}\right)c'(u_T). \tag{52}$$

Therefore, the terminal belief x_T must satisfy the following equation

$$(1+k)\frac{\lambda + u_T}{1+e^{x_T}} - \frac{v}{1+e^{x_T}} - c(u_T) = 0,$$

where u_T is given above. Now, remember the boundary-value problem

$$c''(u)(1+e^x)u' = (\lambda + u)c'(u) - c(u) + \frac{\lambda + u_t}{1+e^{x_t}} - v.$$

We ask how effort behaves at the quitting belief. We have

$$\begin{aligned} u_T' &\propto (\lambda+u)c'(u) + \frac{\lambda+u_t}{1+e^{x_t}} - c(u) - v \\ &= (\lambda+u)c'(u) - k\frac{\lambda+u_T}{1+e^{x_T}} + \frac{v}{1+e^{x_T}} - v \\ &= -\frac{e^{x_T}}{1+e^{x_T}}v < 0. \end{aligned}$$

This means wages and effort are decreasing at the deadline. Finally, compare the slope of the trajectory with the transversality curve at the stopping point.

Lemma D.3 *The equilibrium effort trajectory hits the locus in* (52) *from above if and only if* $v \geq k$.

Indeed, we have

$$\begin{aligned} & -\frac{\frac{e^{x_T}}{1+e^{x_T}}v}{c''(u)(1+e^x)} + \frac{c'(u)e^x}{(1+e^x)c''(u)} \\ \propto & -\frac{1}{1+e^{x_T}}v + c' = -v + k. \end{aligned}$$

Therefore $\Delta(x_T^*) < 0$ iff $v > k$.

Finally, note that the planner quits when

$$(1+k)\frac{\lambda + u_T}{1+e^{x_T}} - \frac{v}{1+e^{x_T}} - c(u_T) = 0 \tag{53}$$

He exerts effort given by

$$(1+k)e^{-x_T} = \left(1+e^{-x_T}\right)c'(u_T).$$

Now solve (53) for $1+e^x$ and consider the function

$$g_S(u) := \frac{(1+k)(\lambda+u) - v}{c(u)}.$$

The optimal deadline is characterized by the intersection of $g_S(u)$ with the transversality conditions, which may be expressed as

$$\begin{aligned} 1+e^x &= g_A(u) := k/c'(u) \\ 1+e^x &= g_P(u) := (1+k)/c'(u) \end{aligned}$$

for the agent and the planner respectively. Clearly $0 < g_A < g_P$ and both are strictly decreasing. Furthermore, notice that $g_S(u)$ is strictly quasi-concave whenever positive. Finally, it is immediate to check that $g_S'(u) = 0$ when intersecting $g_P(u)$. Therefore, it must be that $g_P(u_P^*) > g_P(u_P^*)$, hence the planner's terminal belief p_T is lower.

The planner's effort slope at the deadline is given by

$$u_T' \propto (\lambda+u)c'(u) - c(u) - v.$$

Therefore,

$$\begin{aligned} u_T' &= \frac{(\lambda+u)}{1+e^x}(1+k) - c(u) - v \\ &= -\frac{e^{x_T}}{1+e^{x_T}}v < 0. \end{aligned}$$

So it is efficient to have decreasing effort at the deadline. Of course, the planner's effort is decreasing throughout.

D.3.1 Convex Cost, with Commitment

Suppose that the agent commits to a deadline. With commitment, the first-order condition is different, as the agent takes into account the effect of x_T on all previous effort levels. Therefore we have

$$\begin{aligned} V(x_T) &= \int_0^T \left(e^{-x_t}\left((\lambda+u_t)-v\right)-\left(1+e^{-x_t}\right)c(u_t)\right)\mathrm{d}t - ke^{-x_T} \\ &= \int_{x_0}^{x_T}\left(e^{-x}\left(1-\frac{v}{\lambda+u(x)}\right)-\frac{1+e^{-x}}{\lambda+u(x)}c(u(x))\right)\mathrm{d}x - ke^{-x_T} \\ &= -\int_{x_0}^{x_T}\frac{e^{-x}v+(1+e^{-x})c(u(x))}{\lambda+u(x)}\mathrm{d}x-(1+k)e^{-x_T}, \end{aligned}$$

and so

$$\begin{aligned} V'(x_T) &= e^{-x_T}\left(1+k-\frac{v}{\lambda+u(x_T)}\right)-\frac{1+e^{-x_T}}{\lambda+u(x_T)}c(u(x_T)) \\ &\quad -\int_{x_0}^{x_T}\frac{d}{dx_T}\frac{e^{-x}v+(1+e^{-x})c(u(x))}{\lambda+u(x)}\mathrm{d}x, \end{aligned}$$

where

$$\begin{aligned} u(x) &= u(x_T)-\int_x^{x_T}u'(z)\,\mathrm{d}z, \\ \mathrm{d}u/\mathrm{d}x_T &= \mathrm{d}u_T/\mathrm{d}x_T - u'(x_T) = -\Delta(x_T), \end{aligned}$$

and therefore

$$\begin{aligned} V'(x_T) &= e^{-x_T}\left(1+k-\frac{v+c(u(x_T))}{\lambda+u(x_T)}\right)-\frac{c(u(x_T))}{\lambda+u(x_T)} \\ &\quad -\Delta(x_T)\int_{x_0}^{x_T}\frac{e^{-x}v+(1+e^{-x})\left(c(u(x))-(\lambda+u(x))c'(u(x))\right)}{(\lambda+u(x))^2}\mathrm{d}x. \end{aligned}$$

We know the first term is zero at the optimal deadline without commitment. Let us now study the numerator of the second term *i.e.*

$$\left(1+e^{-x}\right)\left(v+c(u(x))-(\lambda+u(x))c'(u(x))\right)-v.$$

Its derivative with respect to x is

$$\begin{aligned} &-\left(1+e^{-x}\right)(\lambda+u(x))u'(x)c''(u(x))-e^{-x}\left(c(u(x))-(\lambda+u(x))c'(u(x))\right)-ve^{-x} \\ =\ & e^{-x}\left(-(1+e^{x})(\lambda+u(x))c''(u(x))u'(x)+(\lambda+u(x))c'(u(x))-c(u(x))-v\right). \end{aligned}$$

Note that the slope of our trajectory is given by

$$(\lambda+u)c''(u)(1+e^x)u'(x)=(\lambda+u)c'(u)-c(u)+\frac{\lambda+u}{1+e^x}-v,$$

Thus the derivative of the numerator is given by

$$-\frac{\lambda+u(x)}{1+e^x}e^{-x}<0.$$

Its value at x_T is proportional to

$$c\left(u\left(x_T\right)\right)-\left(\lambda+u\left(x_T\right)\right)c'\left(u\left(x_T\right)\right)+\frac{v}{1+e^{x_T}}. \tag{54}$$

Now let us focus on the value of the numerator at x_T as we change x_T. We obtain

$$\begin{aligned}
& -\left(\lambda+u\left(x_T\right)\right)c''\left(u\left(x_T\right)\right)\frac{\mathrm{d}u_T}{\mathrm{d}x_T}-\frac{ve^{x_T}}{\left(1+e^{x_T}\right)^2} \\
= & \frac{c'\left(u\right)e^x}{\left(1+e^x\right)c''\left(u\right)}\left(\lambda+u\left(x_T\right)\right)c''\left(u\left(x_T\right)\right)-\frac{ve^{x_T}}{\left(1+e^{x_T}\right)^2} \\
= & c'\left(u\right)\left(\lambda+u\left(x_T\right)\right)-\frac{v}{1+e^{x_T}} \\
= & k\left(\lambda+u\left(x_T\right)\right)-v \\
\leq & k\left(\lambda+u_T\left(x_0\right)\right)-v.
\end{aligned}$$

We then impose the following condition

$$k\left(\lambda+u_T\left(x_0\right)\right)-v\leq 0 \tag{55}$$

which corresponds to the condition on the cost function

$$\begin{aligned}
v & \geq & k\left(\lambda+\zeta\left(\frac{k}{1+e^{x_0}}\right)\right), \\
\text{where } \zeta\left(v\right) & := & \left(c'\right)^{-1}\left(u\right).
\end{aligned}$$

Therefore, under this sufficient condition, the derivative of (54) is negative, and hence this expression is positive for all x_T (notice that it goes to zero from above as $x_T\to\infty$). Evaluating $V'(x_T)$ at the optimal deadline *without* commitment yields the following result.

Lemma D.4 *Under condition (55), the optimal deadline with commitment is longer than the one without commitment if and only if $v>k$.*

This is the intuitive result that says if the trajectory obtained by moving past the non commitment deadline lies above the previous one then keep going.

D.3.2 The Role of Observable Effort

We now adapt our results to the linear model, in order to assess the role of observable effort. As before, given an equilibrium deadline T, we fix the off-equilibrium beliefs to specify $\hat{u}_t=\bar{u}$ if $x_t<x^*$, and $\hat{u}_t=0$ otherwise. In other words, the market does not react to a failure to quit, anticipates the agent quitting immediately afterwards and expects instantaneous effort to be determined as if $x=x_T$ were the terminal belief.

In the linear model, the agent's payoff can be written as a function of the terminal belief as

$$V\left(x_T\right)=\int_{x_0}^{x_T}\frac{1+e^{-x}}{\lambda+u\left(x\right)}\left(\frac{\lambda+u\left(x\right)}{1+e^x}-\alpha u\left(x\right)-\frac{v}{1+e^x}\right)\mathrm{d}x-ke^{-x_T},$$

and its derivative is given by

$$V'(x_T) = \left(1 + k - \frac{v}{\lambda + u(x_T)}\right) e^{-x_T} - \left(1 + e^{-x_T}\right) \alpha \frac{u(x_T)}{\lambda + u(x_T)}.$$

Consider the unobservable case first.

If the agent is exerting full effort at x_T, his payoff is increasing if and only if $x_T \leq \hat{x}$, which is defined as the unique solution to equation (56) below.

$$(\lambda + \bar{u})(1 + k) - v - \alpha \bar{u}(1 + e^{x_T}) = 0. \tag{56}$$

Note that $\hat{x} \leq x^*$ if and only if

$$v \geq \lambda(1 + k) + \bar{u},$$

and that $\hat{x} < -\infty$ if and only if

$$v < \lambda(1 + k) + \bar{u}(1 + k - \alpha).$$

If the agent does not work at x_T, we obtain

$$V'(x_T) = \left(1 + k - \frac{v}{\lambda}\right) e^{-x_T},$$

which means that, while not working, the agent will quit immediately or never, depending on the value of $v - \lambda(1 + k)$.

To summarize, we have the following characterization in terms of the final beliefs x_T.

Lemma D.5 *If effort is unobservable, the optimal deadline in the absence of commitment is given by*

$$x_{T^*} = \begin{cases} \max\{x_0, \hat{x}\} & \text{if} \quad v > \lambda(1 + k) + \bar{u}, \\ \max\{x_0, x^*\} & \text{if} \quad v \in [\lambda(1 + k), \lambda(1 + k) + \bar{u}], \\ \infty & \text{if} \quad v \leq \lambda(1 + k). \end{cases}$$

Proof of Lemma D.5 Suppose the agent quits before x^*. Then he must quit while exerting maximal effort. This can only occur at $x_T = \hat{x} < x^*$. If $\hat{x} \geq x^*$ then the agent can quit at x^*. This requires $V'(x^*) < 0$, where the payoff is computed assuming the market expects zero effort (for $x > x^*$) and the agent does not work going forward. Therefore, if $v > \lambda(1 + k)$ the agent quits immediately at x^*. For $v \leq \lambda(1 + k)$, he never does. □

We now turn to the case of observable effort.

Lemma D.6 *If effort is observable, the optimal deadline in the absence of commitment is given by*

$$x_{T^*} = \begin{cases} \tilde{x} \leq \hat{x} & \text{if} \quad v > \lambda(1 + k) + \bar{u}, \\ \in [x_0, x^*] & \text{if} \quad v \in [\lambda(1 + k), \lambda(1 + k) + \bar{u}], \\ \infty & \text{if} \quad v \leq \lambda(1 + k). \end{cases}$$

Furthermore, $V(\hat{x}) < V(x_0)$ for v sufficiently close to $\lambda(1 + k) + \bar{u}$.

Proof of Lemma D.6 If $v < \lambda(1+k)$ the market never pays a wage corresponding to zero effort, and the worker chooses the optimal deadline as in the unobservable case, *i.e.* he never quits.

If $v \in [\lambda(1+k), \lambda(1+k)+\bar{u}]$ and the agent were paid a wage corresponding to full effort, he would never stop as long as $x_T < x^*$. However, he would stop immediately if he were expected to quit at $x_T > x^*$ and were paid a wage corresponding to zero effort. We therefore construct a mixed strategy equilibrium in which the agent randomizes at each point in time between the following strategies: (a) exerting full effort and quitting, and (b) exerting zero effort and staying in the game. Denote by μ the instantaneous probability of quitting. The equations characterizing this equilibrium are given by agent indifference and zero profits, or

$$\begin{aligned}(w_t - p_t v - \alpha\bar{u})\,\mathrm{d}t - (1 - p_t(\lambda+\bar{u})\,\mathrm{d}t)\,k &= (w_t - p_t v)\,\mathrm{d}t + (1 - p_t\lambda\mathrm{d}t)\,V_{t+\mathrm{d}t} \\ \text{with } V_{t+\mathrm{d}t} &= (w_{t+\mathrm{d}t} - p_{t+\mathrm{d}t} v - \alpha\bar{u})\,\mathrm{d}t - (1 - p_{t+\mathrm{d}t}(\lambda+\bar{u})\,\mathrm{d}t)\,k, \\ \text{and } w_t &= p_t(\lambda + \mu_t\bar{u}).\end{aligned}$$

Deleting terms of order higher than $\mathrm{d}t$ (notice that terms of order 1 cancel), we obtain

$$w_t = p_t(v - \lambda k)$$

and hence

$$\mu \equiv \frac{v - \lambda(1+k)}{\bar{u}}.$$

Therefore, as v approaches $\lambda(1+k)$, μ vanishes. In particular, when $v = \lambda(1+k)$ the agent is indifferent between stopping immediately and never quitting, and our equilibrium places a mass point at $T = 0$. Conversely, as v approaches $\lambda(1+k)+\bar{u}$, μ goes to one, and the agent quits immediately. Finally, we need to verify that the agent's incentives to exert full and zero effort are satisfied. For all $x_T < x^*$ the transversality condition implies the agent exerts full effort at the deadline. Because the agent quits at rate μ, his strategy assigns positive probability to stopping at x^*. (Notice that its support cannot exceed x^*, as the agent would not exert effort at the quitting time then.) It follows that the agent is indifferent between stopping and at x^* he is indifferent between effort levels. We know from the analysis with a fixed deadline and observable effort that, when expected not to work, the agent would not exert effort. In this case, he is expected to exert a constant amount of work ($\mu\bar{u}$), independent of t and x. As the agent cannot alter is wage (beyond what he can do for a fixed deadline) he has strict incentives to shirk in this case too.

Finally, if $v \geq \lambda(1+k)+\bar{u}$, we have $\hat{x} < x^*$. The agent's payoff is increasing in the deadline as long as he is exerting full effort, receiving the maximum wage, and $x_T \leq \hat{x}$. Conversely, it is decreasing in x_T if the agent is receiving the minimum wage. We now construct a backward induction equilibrium. In the continuation game starting at x, for x close enough to $\hat{x}$, the agent is expected to quit at $\hat{x}$ and to exert effort throughout. For lower values of x, he is expected to quit at $\hat{x}$ and shirk initially. Denote by $\{x_j\}_{j=1,\ldots J}$ a sequence of critical values. Let x_1 denote the belief x that leaves the agent indifferent between quitting and continuing until $\hat{x}$. This belief is well defined because as we decrease x the agent's payoff from continuing is first decreasing then increasing without bound. The agent is expected to quit at x_1. Therefore, he is expected to exert maximal effort for lower values of

x, close enough to x_1. For even lower x he will shirk, then work, and then quit at x_1. Let x_2 denote the belief at which he is indifferent between quitting and continuing until x_1. Recursively define x_{j+1}. We can repeat this construction. Clearly, the value of x^0 determines the equilibrium terminal belief, and the resulting effort pattern. □

An immediate consequence of Lemmas D.5 and D.6 is that the total amount of effort exerted in equilibrium is weakly higher in the unobservable case. Thus, the comparison result carries over to the case of endogenous termination of the relationship. In the unobservable case, the effort patterns can then be traced back to x_0. In particular, when the agent quits at $\hat{x}$, the equilibrium phases are interior-full (because then $v > \lambda(1+\alpha)$); when he quits at x^* the phases are interior-full or always interior effort; and when he never quits, the equilibrium can have all four phases.

What about the social planner, in the non-commitment case? She follows exactly the same behavior, except she has a lower threshold $\hat{v}^P := \lambda(1+k) + \alpha\bar{u}$, above which the planner chooses an interior stopping point with full effort at the end. This follows from the fact that the planner can work at full speed for a larger set of parameters. Note that, when quitting is inefficient, the agent takes the efficient quitting decision (he never does).

D.3.3 Deadlines with commitment but competition

The agent's payoff may be written as (up to constants)

$$V(x_T) = -(1+k)e^{-x_T} - \int_{x_0}^{x_1} g(\bar{u}, x)\,dx - \int_{x_1}^{x_T} g(0,x)\,dx,$$

where

$$g(u,x) := e^{-x}\frac{v}{\lambda+u} + \left(1+e^{-x}\right)\frac{\alpha u}{\lambda+u}.$$

Its derivative is given by

$$\begin{aligned} V'(x_T) &= (1+k-v/\lambda)e^{-x_T} + \frac{\left((v/\lambda-\alpha)e^{-x_1}-\alpha\right)e^{x_T} - (1+k-v/\lambda)e^{-x_T}}{(v-\alpha\lambda)\left(\frac{1}{\lambda}e^{x_T-x_1} - \frac{1}{\lambda+\bar{u}}\right) + \frac{\bar{u}(v-\alpha\lambda)}{\lambda(\bar{u}+\lambda)}e^{-x_1}}\left(g(0,x_1) - g(\bar{u},x_1)\right) \\ &= (1+k-v/\lambda)e^{-x_T} + \frac{\left((v/\lambda-\alpha)e^{-x_1}-\alpha\right)e^{x_T} - (1+k-v/\lambda)e^{-x_T}}{(v-\alpha\lambda)\left(\frac{1}{\lambda}e^{x_T-x_1} - \frac{1}{\lambda+\bar{u}}\right) + \frac{\bar{u}(v-\alpha\lambda)}{\lambda(\bar{u}+\lambda)}e^{-x_1}}\left(e^{-x_1}\left(\frac{v}{\lambda} - \frac{v}{\lambda+\bar{u}}\right) - \frac{(1+e^{-x_1})\alpha\bar{u}}{\lambda+\bar{u}}\right). \end{aligned}$$

Note that

$$g_u(u,x) \propto \alpha\lambda(1+e^x) - v.$$

Because we know from our bound that

$$e^{x_1} < e^{\bar{x}_1} = v/\alpha\lambda - 1.$$

We also know

$$\alpha\lambda(1+e^x) - v < 0.$$

Assume now $v/\lambda \leq 1+k$. Then we can write $V'(x_T)$ as

$$\begin{aligned}
V'(x_T) &\propto (1+k-v/\lambda)\, e^{-x_T}\left((v-\alpha\lambda)\left(\frac{1}{\lambda}e^{x_T-x_1}-\frac{1}{\lambda+\bar{u}}\right)+\frac{\bar{u}\,(v-\alpha\lambda)}{\lambda\,(\bar{u}+\lambda)}e^{-x_1}\right)\\
&\quad -(1+k-v/\lambda)\, e^{-x_T}\left(e^{-x_1}\left(\frac{v}{\lambda}-\frac{v}{\lambda+\bar{u}}\right)-\left(1+e^{-x_1}\right)\frac{\alpha\bar{u}}{\lambda+\bar{u}}\right)\\
&\quad +\left((v/\lambda-\alpha)\, e^{-x_1}-\alpha\right)e^{x_T}\left(e^{-x_1}\left(\frac{v}{\lambda}-\frac{v}{\lambda+\bar{u}}\right)-\left(1+e^{-x_1}\right)\frac{\alpha\bar{u}}{\lambda+\bar{u}}\right)\\
&= (1+k-v/\lambda)\, e^{-x_T}\left((v-\alpha\lambda)\left(\frac{1}{\lambda}e^{x_T-x_1}-\frac{1}{\lambda+\bar{u}}\right)+\bar{u}\frac{\alpha}{\bar{u}+\lambda}\right)\\
&\quad +e^{x_T}\frac{\bar{u}}{\bar{u}+\lambda}\left((v/\lambda-\alpha)\, e^{-x_1}-\alpha\right)^2,
\end{aligned}$$

which is positive as all terms in this expression are positive, and so the optimal deadline is infinite.

Conversely, if $v/\lambda \geq 1+k$ we know $\mathrm{d}x_1/\mathrm{d}x_T > 0$ and so

$$\begin{aligned}
V'(x_T) &= (1+k-v/\lambda)\, e^{-x_T}\left((v/\lambda-\alpha)\, e^{x_T-x_1}-\frac{v}{\lambda+\bar{u}}+\alpha\right)+e^{x_T}\frac{\bar{u}}{\bar{u}+\lambda}\left((v/\lambda-\alpha)\, e^{-x_1}-\alpha\right)^2\\
&= (1+k-v/\lambda)\, e^{-x_1}(v/\lambda-\alpha)+\left(\alpha-\frac{v}{\lambda+\bar{u}}\right)(1+k-v/\lambda)\, e^{-x_T}\\
&\quad +e^{x_T}\frac{\bar{u}}{\bar{u}+\lambda}\left((v/\lambda-\alpha)\, e^{-x_1}-\alpha\right)^2.
\end{aligned}$$

This expression is negative for x_T large enough (because x_1 converges to $\bar{x}_1$ at rate x_T and hence the last term vanishes). Therefore, if $v/\lambda > 1+k$, the optimal deadline is finite. Finally, plugging in $x_1 = x_T$ we obtain

$$\begin{aligned}
V'(x_T) &= (1+k-v/\lambda)\, v\frac{\bar{u}}{\lambda\,(\bar{u}+\lambda)}e^{-x_T}+e^{x_T}\frac{\bar{u}}{\bar{u}+\lambda}\left((v/\lambda-\alpha)\, e^{-x_1}-\alpha\right)^2\\
&\propto (1+k-v/\lambda)\,\frac{v}{\lambda}+\left(v/\lambda-\alpha\,(1+e^x)\right)^2\\
&= (1+k-2\alpha\,(1+e^x))\,\frac{v}{\lambda}+\alpha^2\,(1+e^x)^2. \qquad (57)
\end{aligned}$$

This is clearly positive for x_0 low enough (because we know in that case the lowest x_T yielding work-shirk is arbitrarily low). Now consider the equation determining the lowest x_T. We know this expression is increasing in x_0 (because the integrand is positive), and it is decreasing in x_T (all the terms go in the same direction). Therefore, we know $\mathrm{d}x_T/\mathrm{d}x_0 > 0$. We therefore look for the highest x_0 that allows for work-shirk, which is given by

$$x_0 = x^* = \ln\frac{k-\alpha}{\alpha}.$$

We then have condition (57), which is positive as $x \to -\infty$ and as $x = x^*$ (just plug in). Furthermore, it is quadratic in $\alpha\,(1+e^x)$ and decreasing at x^*. Therefore, it is everywhere positive.

Now consider the derivative of the social planner's payoff when exerting maximal effort throughout. We have

$$V'(x_T) = ((1+k)\,(\lambda+\bar{u})-v)\, e^{-x_T}-\left(1+e^{-x_T}\right)\alpha\bar{u}.$$

Therefore, we have

$$V'(x^*) \propto (1+k)\lambda + \bar{u} - v.$$

We conclude with the following result.

Lemma D.7 *The socially optimal deadline is finite if and only if $v/\lambda \geq 1+k$.*

Furthermore, if $v/\lambda \leq 1+k+\bar{u}/\lambda$ the optimal deadline induces full then zero effort.

For values of v exceeding the upper bound, the optimal deadline is finite and may induce either full or full, then zero effort.

Planner's optimal deadline: Contrast this with the planner's optimal deadline under full commitment. We again have

$$V'(x_T) = (1+k-v/\lambda)\, e^{-x_T} + \frac{\mathrm{d}x_1}{\mathrm{d}x_T}\left(g(0,x_1) - g(\bar{u},x_1)\right),$$

with

$$(1+k-\alpha)\, e^{-x_T} - \alpha + (v/\lambda - \alpha)\left(e^{-x_1} - e^{-x_T}\right) = 0, \tag{58}$$

and so

$$\begin{aligned} e^{-x_1} &= e^{-x_T}\left(1 - \frac{1+k-\alpha}{v/\lambda-\alpha}\right) + \frac{\alpha}{v/\lambda-\alpha}, \\ \frac{\mathrm{d}x_1}{\mathrm{d}x_T} &= \frac{e^{-x_T}\left(1-\frac{1+k-\alpha}{v/\lambda-\alpha}\right)}{e^{-x_T}\left(1-\frac{1+k-\alpha}{v/\lambda-\alpha}\right) + \frac{\alpha}{v/\lambda-\alpha}}. \end{aligned}$$

Therefore we can rewrite V' as

$$\begin{aligned} V'(x_T) &= (1+k-v/\lambda)\, e^{-x_T} + \frac{\mathrm{d}x_1}{\mathrm{d}x_T}\left(e^{-x_T}\left(1-\frac{1+k-\alpha}{v/\lambda-\alpha}\right) + \frac{\alpha}{v/\lambda-\alpha}\right)\left(\frac{v}{\lambda} - \frac{v}{\lambda+\bar{u}}\right) \\ &\quad - \frac{\mathrm{d}x_1}{\mathrm{d}x_T}\left(1 + e^{-x_T}\left(1-\frac{1+k-\alpha}{v/\lambda-\alpha}\right) + \frac{\alpha}{v/\lambda-\alpha}\right)\frac{\alpha\bar{u}}{\lambda+\bar{u}} \\ &= (1+k-v/\lambda)\, e^{-x_T} - \frac{e^{-x_T}\left(1-\frac{1+k-\alpha}{v/\lambda-\alpha}\right)}{e^{-x_T}\left(1-\frac{1+k-\alpha}{v/\lambda-\alpha}\right) + \frac{\alpha}{v/\lambda-\alpha}} \frac{\bar{u}}{\lambda}\frac{e^{-x_T}}{\bar{u}+\lambda}\left(\lambda - v + k\lambda\right) \\ &\propto e^{-x_T}\left(1-\frac{1+k-\alpha}{v/\lambda-\alpha}\right)\frac{\lambda}{\bar{u}+\lambda} + \frac{\alpha}{v/\lambda-\alpha} \\ &\propto (1+k-v/\lambda)\left(\alpha - e^{-x_T}(1+k-v/\lambda)\frac{\lambda}{\bar{u}+\lambda}\right). \end{aligned}$$

Notice that the second term is positive, because

$$\alpha - e^{-x_T}(1+k-v/\lambda)\frac{\lambda}{\bar{u}+\lambda} \geq \alpha\left(1 - \frac{1+k-v/\lambda}{1+k-\alpha}\frac{\lambda}{\bar{u}+\lambda}\right) > 0$$

which follows from plugging $x_1 = x_T$ into (58). Therefore, $v \leq \lambda(1+k)$ is necessary and sufficient for the planner's problem to be increasing in the deadline, and we have the following characterization.

Lemma D.8 *If $v \in [\lambda(1+k), \lambda(1+k)+\alpha\bar{u}]$, the planner's optimal deadline is $x_P^* = \ln\left(\frac{1+k}{\alpha} - 1\right)$.*

If $v > \lambda(1+k)+\alpha\bar{u}$, the optimal deadline is $\hat{x} < x_P^*$.

If $v < \lambda(1+k)$, the optimal deadline is infinite.

To summarize, when it is inefficient to stop the relationship, the socially optimal deadline is infinite, as is the planner's. However, when under full commitment (contractible output) the planner finds it optimal to work at full speed and stop, the socially optimal deadline typically includes shirking (speculating, because the payment of a lump sum depresses incentives so much at the end that a bit of shirking is nevertheless beneficial –without shirking the highest attainable x_T is very low).

D.3.4 Finishing Lines

Proof of Proposition 5.7 Let $\hat{x}$ denote the stopping belief, fixed exogenously for now. The payoff to be maximized is

$$\begin{aligned} & \int_x^{\hat{x}} \frac{(\lambda + u(x))e^{-x} - (1+e^{-x})(v+\alpha u)}{\lambda+u} \mathrm{d}x - k\left(1+e^{-\hat{x}}\right) + v\int_0^{\hat{x}} \frac{\mathrm{d}x}{\lambda+u} \\ = & \int_x^{\hat{x}} \frac{(\lambda - v + u(x))e^{-x} - (1+e^{-x})\alpha u}{\lambda+u} \mathrm{d}x - k\left(1+e^{-\hat{x}}\right), \end{aligned}$$

where $u(x)$ is the expected effort given state x and u is the control variable (equal to $u(x)$ at x in equilibrium). [The last term on the first line corresponds to the term vT discussed in footnote 5.] Transversality requires that $u = u(\hat{x})$ maximizes

$$\frac{(\lambda - v + u(\hat{x}))e^{-\hat{x}} - \left(1+e^{-\hat{x}}\right)\alpha u}{\lambda+u},$$

whose derivative w.r.t. u is proportional to

$$v - u(\hat{x}) - (1+\alpha)\lambda - \alpha\lambda e^{\hat{x}}.$$

Hence,

$$u(\hat{x}) = \begin{cases} \bar{u} \text{ if } e^{\hat{x}} < \frac{v-\bar{u}-(1+\alpha)\lambda}{\alpha\lambda}, \\ u \in (0,\bar{u}) \text{ if } e^{\hat{x}} = \frac{v-u-(1+\alpha)\lambda}{\alpha\lambda}, \\ 0 \text{ if } e^{\hat{x}} > \frac{v-(1+\alpha)\lambda}{\alpha\lambda}. \end{cases}$$

The intuition is straightforward: if $\hat{x}$ is high enough, there is no reason to work: the wage might be low, but then again the outside option v is not hurting, as it is unlikely to be collected.

More generally, the objective is maximized pointwise by setting:

$$u(x) = \begin{cases} \bar{u} \text{ for } e^x > \frac{v-\bar{u}-(1+\alpha)\lambda}{\alpha\lambda}, \\ v - (1+\alpha)\lambda - \alpha\lambda e^x \text{ if } e^x \in \left[\frac{v-\bar{u}-(1+\alpha)\lambda}{\alpha\lambda}, \frac{v-(1+\alpha)\lambda}{\alpha\lambda}\right], \\ 0 \text{ if } e^x > \frac{v-(1+\alpha)\lambda}{\alpha\lambda}, \end{cases}$$

for all relevant values of x (i.e., values such that $x < \hat{x}$). Note now that the derivative of the objective with respect to the finishing line is simply

$$\frac{(\lambda - v + u(\hat{x}))\, e^{-\hat{x}} - \left(1 + e^{-\hat{x}}\right) \alpha u(\hat{x})}{\lambda + u} + k e^{-\hat{x}},$$

which, given the formula for $u(\hat{x})$ is non-increasing in $\hat{x}$. When $1 + k > v/\lambda$, then this derivative is positive for all values of $\hat{x}$: the optimal finishing line is infinite in that case. When $1 + k \in [(v - \bar{u})/\lambda, v/\lambda]$, the optimal finishing line solves

$$e^{\hat{x}} = \frac{k - \alpha}{\alpha},$$

i.e. $\hat{x} = x^*$, and effort is interior at the finishing line. Finally, if $1 + k < (v - \bar{u})/\lambda$, the optimal finishing line solves

$$1 + e^{\hat{x}} = \frac{(1 + k)(\lambda + \bar{u}) - v}{\alpha \bar{u}},$$

with maximum effort throughout. Note that this finishing line coincides with the (belief at the) optimal deadline in the absence of commitment under non-observability (See Lemma D.3). □