Local Incentive Compatibility in Moral Hazard Problems: A Unifying Approach*

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Abstract

I suggest a unifying new approach to moral hazard. Once local incentive compatibility (L-IC) is satisfied, the problem of verifying global incentive compatibility (G-IC) is shown to be isomorphic to the well-understood problem of comparing two classes of distribution functions. This observation makes it trivial to derive, for a given action, succinct necessary and sufficient conditions for L-IC to imply G-IC among all monotonic and monotonic and concave contracts. The sufficient conditions for the validity of the first-order approach (FOA) provided by Rogerson and Jewitt, respectively, follow as corollaries. Their conditions are analogous to first and second order stochastic dominance, respectively. New and economically meaningful conditions, in the spirit of third order stochastic dominance, are presented. Even when the standard FOA is not valid, a modified FOA may be valid on the set of implementable actions. The modified FOA applies to settings where Grossman and Hart's spanning condition is satisfied and can be used to resolve Mirrlees' famous "counterexample". Extensions to multi-signal principal-agent problems are also considered.

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1 Introduction

The principal-agent model of moral hazard is among the core models of microeconomic theory and central to the economics of information. The problem is conceptually simple; a principal must design a contract to induce the agent to take the desired action. From the agent's point of view the intended action must be made preferable to all other actions. Thus, a multitude of incentive compatibility constraints must be satisfied. Unfortunately, it is generally difficult to determine which constraints bind and to make robust predictions about the structure of optimal contracts.

In response, much of the literature has focused on environments where the only binding constraint is the "local" incentive compatibility constraint (L-IC). In such cases, ensuring the agent has no incentive to deviate marginally from the intended action guarantees global incentive compatibility (G-IC), i.e. larger deviations can be ruled out too. Indeed, the classic first-order approach (FOA) simply uses the agent's first-order condition to summarize G-IC. The optimal contract is then easily derived. The FOA has a long history, dating back to Holmström (1979) and Mirrlees (1976, 1999). Rogerson (1985) and Jewitt (1988) have provided sufficient conditions under which the FOA is valid. However, although there are similarities in the structure of their proofs, the techniques they use are quite different. Moreover, despite criticizing the stringency of his assumptions, most textbooks on the topic prove Rogerson's result, but none even state Jewitt's.[1] In short, Jewitt's result may be underappreciated and there is little in the current literature to unify the two results.

With these observations in mind, the objective of this paper is to propose an accessible and unifying approach to the moral hazard problem. Indeed, I will argue that Jewitt's (1988) contribution can be seen as a natural progression from Rogerson's (1985) seminal work.[2] The next step in that progression is identified here.

The approach relies on "translating" the problem of verifying global incentive compatibility into a problem that is familiar to, and well-understood by, any economist. In particular, I will show that checking G-IC (once L-IC is satisfied) is isomor-

[1] While Mas-Colell et al (1995) make only passing reference to Rogerson's conditions, Wolfstetter (1999), Bergin (2005), Bolton and Dewatripont (2005), and Laffont and Martimort (2002) all prove his result. Only the last two mention that Jewitt has provided alternative sufficient conditions, with Laffont and Martimort (2002) providing an example. For future reference, the first three sources all contain a comprehensive treatment of stochastic dominance and choice under uncertainty.

[2] Jewitt's (1988) original proof is made complicated by the fact that it relies on results in an unpublished working paper. The full proof is published in Conlon (2009b). In the existing literature, Conlon (2009a) comes closest to methodologically unifying Rogerson's and Jewitt's results. Specifically, Conlon (2009a, footnote 7) observes that Rogerson's proof relies on integration by parts, and that a second round of integration by part can be used to prove Jewitt's result. He does not ask, for instance, what can be obtained from further rounds of integration by parts.

phic to the problem of comparing two classes of risky prospects, or two classes of distribution functions. Once this equivalence has been established, the main results follow immediately by simply calling upon well-known results from the literature on stochastic dominance. Hence, when attention is restricted to certain interesting classes of contracts, it turns out to be trivial to identify succinct necessary and sufficient conditions for L-IC to imply G-IC for any fixed action. Stated differently, I characterize the subset of actions for which L-IC is guaranteed to be sufficient for G-IC (among different subsets of contracts). Rogerson and Jewitt provide conditions under which all actions belongs to this set. Since the cost of implementation and the structure of the optimal contract is determined by the binding constraint(s), findings of this nature are not only of methodological, but also of economic, significance.

Any contract translates into a distribution of wages (where the distribution is determined in part by the agent's action). For brevity, I will refer to a contract as nondecreasing or monotonic if the agent's utility is nondecreasing in the outcome or state. A contract is concave if the agent's utility is concave in the state. With this terminology, Rogerson's (1985) and Jewitt's (1988) proofs can be decomposed into two concise parts. In Rogerson's case, the first part is to identify conditions under which any monotonic and L-IC contract is also G-IC. The second part is then to identify additional conditions under which the candidate contract is in fact monotonic. In Jewitt's case, contracts are both monotonic and concave.

The two first columns in the top row of Table 1 summarize the conclusions in step 1 of Rogerson and Jewitt, respectively. For future reference, the third column identifies a natural extension. In comparison, the second row summarizes the notions of first, second, and third order stochastic dominance (FOSD, SOSD, and TOSD, respectively) between two lotteries, G and H.[3] Note that Jewitt weakens Rogerson's assumption on the distribution function, but in exchange has to strengthen the assumptions imposed on the shape of the contract. This trade-off is remarkably similar to the one encountered when first and second order stochastic dominance are compared. As mentioned, I will make the point that this is no coincide, and that much can be gained from exploring the relationship between the two rows in Table 1. As the third column in Table 1 reveals, once the pattern is identified it becomes easy to develop a third set of conditions to validate the FOA.[4] Obviously, extensions to higher order stochastic dominance are also possible.

[3] See Hadar and Russell (1969), Rothschild and Stiglitz (1970), Whitmore (1970), and Menezes et al (1980). For textbook treatments, see the texts mentioned in footnote 1.

[4] To make the second step in the proof work, the assumptions in Table 1 must be accompanied by assumptions on the agent's utility function and on the likelihood ratio. It turns out that there is an appealing pattern in those assumptions as well. See Section 4.

[TABLE 1 ABOUT HERE (SEE THE LAST PAGE)]

To complete the comparison in Table 1, I will provide "if and only if" versions of the first step in Rogerson's and Jewitt's proofs.[5] It follows that Rogerson's and Jewitt's conditions are in fact the weakest conditions that can be imposed to justify the FOA when only monotonicity or monotonicity and concavity are exploited.[6]

The analysis is based on comparing the agent's real problem with an auxiliary problem. The auxiliary problem is constructed using counter-factual distributions chosen in such a manner that the agent's expected payoff is independent of his action whenever he faces an L-IC contract. Moreover, the constant payoff coincides with the payoff the agent would achieve in the real problem, if he chooses the intended action. It follows that L-IC implies G-IC if and only if payoff in the auxiliary problem is never below payoff in the real problem, for any action. To learn which problem give the agent greater utility, it suffices to compare the actual distributions with the counter-factual distributions. Thus, stochastic dominance results can be invoked.

In general, the set of actions for which L-IC implies G-IC is a subset of the set of implementable actions. A secondary contribution of the paper is to identify a model where the two sets coincide. Here, it is valid to apply the FOA on the "feasible set" of implementable actions. Specifically, this method of analysis is valid whenever Grossman and Hart's (1983) spanning condition is satisfied. Though this simple model was proposed three decades ago, no complete analysis has been offered until now. As in the first part of the paper, the crucial step is the exploration of the link between L-IC and G-IC, which ultimately leads to the modified FOA. As a special case, the modified FOA is valid in textbook settings with two states (but a continuum of actions). The method also resolves Mirrlees' (1999) original "counterexample", the purpose of which was to demonstrate that the standard FOA may fail. Finally, the leading example in Araujo and Moreira (2001) also succumbs to the modified FOA.

Section 2 introduces the model and offers some preliminary observations. Section 3 defines the auxiliary problems and explores the relationship between L-IC and G-IC for a specific action. Section 4 unifies the existing justifications for the FOA. New conditions that justify the FOA are also provided. Section 5 develops a modified FOA that is valid in some prominent examples where the standard FOA does not apply. Section 6 considers extensions to multi-signal models. Section 7 concludes.

[5] That is, the $\Downarrow$'s in Table 1 can be converted into $\Updownarrow$'s. Strictly speaking, these results apply to the set of interior actions only. Some modification is required to handle actions on the boundary.

[6] Araujo and Moreira (2001) propose a general Lagrangian approach to solve moral hazard problems. However, their method yields little economic intuition (see Section 5). Ke (2011) proposes a fixed-point method for justifying the FOA.

2 Model and preliminaries

A risk averse agent takes a costly action that is not verifiable to others. The set of possible actions is some closed and bounded interval, $[\underline{a}, \overline{a}]$. The agent's action determines the distribution over verifiable and one-dimensional outcomes or states, x. If the action is a, the cumulative distribution function is $F(x|a)$, where it is assumed that the domain, $[\underline{x}, \overline{x}]$, is compact and independent of a. It is assumed that $F(x|a)$ has no mass points and is continuously differentiable in x and a to the requisite degree, with $f(x|a) = F_x(x|a)$ denoting the density for fixed a. Assume that $f(x|a)$ is strictly positive. For now, $F_a(x|a) < 0$ for all $a \in [\underline{a}, \overline{a}]$ and all $x \in (\underline{x}, \overline{x})$.

The agent faces a contract that, to him, is fixed. He receives wage $w(x)$ if the outcome is x, in which case utility is $v(w(x)) - a$.[7] The agent's expected utility (assuming it exists) given action a is then

$$EU(a) = \int_{\underline{x}}^{\overline{x}} v(w(x))dF(x|a) - a. \tag{1}$$

Evidently, costs are assumed to be linear in the action. For instance, think of the agent's action, a, as being his choice of what cost of effort to incur. The linearity is convenient since it implies that only the first term in (1) has curvature, which simplifies the search for necessary and sufficient conditions. Incidentally, Rogerson (1985) chose this parameterization too, although he only pursued sufficient conditions. Conlon (2009a, footnote 3) also observes that curvature in the cost function can be important, and thus chooses the same parameterization.

In the first part of the paper, attention is restricted to wage schedules that are monotonic (i.e. nondecreasing). First, as Innes (1990) points out, this restriction arises naturally in situations where the agent could sabotage (i.e. lower) the outcome before it is observed by the principal. Secondly, the candidate solution using the FOA turns out to be a monotonic contract under standard assumptions. The second step in validating the FOA is then to verify that such a monotonic contract is in fact also globally incentive compatible. The monotonicity restriction is removed in Section 5.

The agent's utility function $v(w)$ is strictly increasing and continuous. The domain of the utility function is some interval which may or may not be the entire real line. Finally, utility is unbounded above and/or below. In this section and the next, no curvature or differentiability assumptions are needed.

[7] Additive separability is important. While it is a standard assumption in the literature, there are exceptions. Alvi (1997) and Fagart and Fluet (2012) provide conditions that justify the FOA without additive separability.

2.1 Incentive compatibility

By assumption, the composite function $v(w(\cdot))$ is monotonic on a compact interval, and hence of bounded variation. Since $F(x|a)$ is continuous in x, it follows that the (Riemann-Stieltjes) integral in (1) exists, and that integration by parts can be performed on (1) to yield the alternative expression

$$EU(a) = v(w(\overline{x})) - \int_{\underline{x}}^{\overline{x}} F(x|a)dv(w(x)) - a. \tag{2}$$

Moreover, it is permissible to differentiate (with respect to a) under the integral sign. Since the contract is monotonic, or $dv(w(x)) \geq 0$, $EU(a)$ is concave if Rogerson's condition that $F_{aa}(x|a) \geq 0$ for all x and all a is satisfied. More generally, it should now be obvious that the properties of $EU(a)$ depend critically on $F(x|a)$.

If the principal wishes to induce action $a^* \in [\underline{a}, \overline{a}]$, this action must provide the agent with higher expected utility than any other action, or

$$EU(a^*) \geq EU(a) \text{ for all } a \in [\underline{a}, \overline{a}], \tag{G-IC$_{a^*}$}$$

in which case the contract $w(x)$ is said to be globally incentive compatible. If $a^* \in (\underline{a}, \overline{a})$, a minimum requirement is that $EU(a)$ attains a stationary point at a^*, or

$$-\int_{\underline{x}}^{\overline{x}} F_a(x|a^*)dv(w(x)) - 1 = 0. \tag{L-IC$_{a^*}$}$$

Of course, the stationary point may in principle be a local minimum or a saddle-point. Nevertheless, I will refer to the condition $EU'(a^*) = 0$ as the local incentive compatibility condition. Thus, any contract that satisfies $EU'(a^*) = 0$ will be termed L-IC$_{a^*}$ and any contract that satisfies $EU(a^*) \geq EU(a)$ for all $a \in [\underline{a}, \overline{a}]$ is G-IC$_{a^*}$. The implementation of $\underline{a}$ and $\overline{a}$ is discussed in Section 4.

In practice, a contract may need to satisfy a number of other constraints as well. Examples includes participation constraints, monotonicity of the principal's rewards ($x - w(x)$, if x measures the monetary value of the state), minimum wages, and the like. In the next section, any such constraints are simply ignored. The reason is that the question of when L-IC$_{a^*}$ implies G-IC$_{a^*}$ has little to do with these other constraints. Note, in particular, that the next section is not directly concerned with the design of optimal contracts. The participation constraint is added in Section 4, when optimal contracts and the FOA is analyzed.

2.2 Some L-IC contracts and stochastic dominance

A *step contract* is one that delivers utility v_1 if x is below some critical value, $\widehat{x} \in (\underline{x}, \overline{x})$, and $v_2 > v_1$ otherwise. Of course, v_1 and v_2 must be in the range of $v(\cdot)$. In this case, L-IC$_{a^*}$ reduces to $-(v_2 - v_1)F_a(\widehat{x}|a^*) = 1$. Recall that $F_a(\widehat{x}|a^*) < 0$, by assumption. Since $v(\cdot)$ is unbounded above and/or below, it is possible to make $(v_2 - v_1)$ arbitrarily large and thus, by continuity, it is possible to satisfy the first order condition. The unboundedness of $v(\cdot)$ ensures that v_1 and v_2 can be chosen in such a way that both are in the range of $v(\cdot)$. In other words, for any $a^* \in (\underline{a}, \overline{a})$ and any $\widehat{x} \in (\underline{x}, \overline{x})$, there is a step contract which satisfies L-IC$_{a^*}$.

In the following, a *concave contract* refers to a wage schedule for which $v(w(\cdot))$ is concave, though $w(\cdot)$ itself need not be. In fact, it may be helpful to think of the principal as designing the composite function, using $w(\cdot)$ only as a means to an end. For future reference, consider a concave contract that delivers utility of $\min\{\beta x + \gamma, \beta\widehat{x} + \gamma\}$, where $\widehat{x} \in (\underline{x}, \overline{x})$ and $\beta > 0$. As before, β and γ must be chosen to ensure that $\min\{\beta x + \gamma, \beta\widehat{x} + \gamma\}$ is in the range of $v(\cdot)$. The (finite and strictly positive) value of β is uniquely determined by L-IC$_{a^*}$. Since v is unbounded above and/or below, it is then easy to pick γ in such a manner that $\min\{\beta x + \gamma, \beta\widehat{x} + \gamma\}$ is in the range of $v(\cdot)$. Thus, for any $a^* \in (\underline{a}, \overline{a})$ and any $\widehat{x} \in (\underline{x}, \overline{x})$, there is a L-IC$_{a^*}$ contract of this form. Using similar reasoning, there is also a L-IC$_{a^*}$ contract where utility is $\gamma - \beta\left(\max\{\widehat{x} - x, 0\}\right)^2$, with $\beta > 0$. In the latter contract, utility is differentiable in x and this derivative is convex. Such contracts are said to be *positively skewed*. In fact, the contract in the last example is monotonic, concave, and positively skewed. In the following, the set of all monotonic contracts is denoted $\mathcal{C}_1$. The set of monotonic and concave contracts is $\mathcal{C}_2$, while $\mathcal{C}_3$ denotes the set of contracts that are monotonic, concave, and positively skewed. For ease of exposition, an element of $\mathcal{C}_i$ will occasionally be referred to simply as a $\mathcal{C}_i$ contract. Note that $\mathcal{C}_3 \subseteq \mathcal{C}_2 \subseteq \mathcal{C}_1$.

The primary objective of the first part of the paper is to identify actions for which *any* $\mathcal{C}_i$ and L-IC$_{a^*}$ contract is G-IC$_{a^*}$, $i = 1, 2, 3$. Let $A_i^{L \Longrightarrow G} \subseteq (\underline{a}, \overline{a})$ denote the set of such (interior) actions. Rogerson and Jewitt essentially propose sufficient conditions for $A_1^{L \Longrightarrow G} = (\underline{a}, \overline{a})$ and $A_2^{L \Longrightarrow G} = (\underline{a}, \overline{a})$, respectively.

Note that the three contracts described above take the same form as the utility functions used in standard proofs of properties of first, second, and third order stochastic dominance, respectively (see the references in the introduction). Thus, the existence of these three types of contracts are established with the sole purpose of ensuring that it is possible to call upon the standard stochastic dominance arguments. These arguments will be outlined next. Indeed, the proof of the main result in the next section quite simply invokes these results.

Consider two lotteries, G and H, with the same set of possible outcomes, $[\underline{x}, \overline{x}]$. The agent's utility function is $u(x)$. If $u(x)$ is monotonic, then integration by part, as in (2), immediately yields the conclusion that the lottery G is preferred to H if $G(x) \leq H(x)$ for all x. Conversely, if G is preferred to H for all monotonic utility functions then it must in particular be preferred for a utility function that have the "step" structure described above, with a jump at $\widehat{x}$. Evaluating expected utility from the two lotteries implies $G(\widehat{x}) \leq H(\widehat{x})$. Since this argument must hold for all $\widehat{x}$, it follows that if G is preferred to H for all monotonic utility functions then it must be the case that $G(x) \leq H(x)$ for all x. This establishes the equivalence between the two notions of first order stochastic dominance in Table 1. The proofs involving second and third order stochastic dominance are essentially the same, except integration by parts must be used repeatedly in the one direction, while utility functions $\min\{\beta x + \gamma, \beta \widehat{x} + \gamma\}$ or $\gamma - \beta \left(\max\{\widehat{x} - x, 0\}\right)^2$, respectively, can be used in the other direction.[8]

3 From local to global incentive compatibility

The first step in Rogerson (1985) and Jewitt (1988) can be thought of as finding sufficient conditions for L-IC$_{a^*}$ to imply G-IC$_{a^*}$ for *any* a^* in the class of monotonic or monotonic and concave contracts, respectively. The starting point of the current paper is slightly different. *Fixing* $a^* \in (\underline{a}, \overline{a})$, the first question addressed here is: When is *any* contract that is both monotonic (or monotonic and concave) and L-IC$_{a^*}$ also G-IC$_{a^*}$? Any action in this set can be implemented "cheaply" with a monotonic contract, in the sense that only the local incentive compatibility constraint is relevant.

To begin, it is convenient to define a few auxiliary functions of a, where x is taken to be a parameter. Holding x fixed, let $F^C(x|a)$ =conv$F(x|a)$ denote the convex envelope or convex hull of $F(x|a)$, i.e. the highest convex function such that $F^C(x|a) \leq F(x|a)$ for all a. See Rockafellar (1970) for details. Formally,

$$F^C(x|a) = \min_{\gamma, a_1, a_2} \{\gamma F(x|a_1) + (1-\gamma) F(x|a_2) | \gamma \in [0,1], \, a_1, a_2 \in [\underline{a}, \overline{a}], \, \gamma a_1 + (1-\gamma) a_2 = a\}. \tag{3}$$

Let

$$A_1^C = \{a \in (\underline{a}, \overline{a}) \, | F(x|a) = F^C(x|a) \text{ for all } x \in [\underline{x}, \overline{x}]\} \tag{4}$$

be the set of interior actions where $F(x|a)$ and $F^C(x|a)$ coincide for all x. Similarly,

[8] In case of third order stochastic dominance, a linear utility function can be used to establish the implication that $\int_{\underline{x}}^{\overline{x}} G(x)dx \leq \int_{\underline{x}}^{\overline{x}} H(x)dx$.

define

$$A_2^C = \left\{ a \in (\underline{a}, \overline{a}) \mid \int_{\underline{x}}^{x} F(y|a)dy = \text{conv} \int_{\underline{x}}^{x} F(y|a)dy \text{ for all } x \in [\underline{x}, \overline{x}] \right\}$$

and

$$\begin{aligned} A_3^C &= \left\{ a \in (\underline{a}, \overline{a}) \mid \int_{\underline{x}}^{x} \int_{\underline{x}}^{z} F(y|a)dydz = \text{conv} \int_{\underline{x}}^{x} \int_{\underline{x}}^{z} F(y|a)dydz \text{ for all } x \in [\underline{x}, \overline{x}] \right\} \\ &\cap \left\{ a \in (\underline{a}, \overline{a}) \mid \int_{\underline{x}}^{\overline{x}} F(y|a)dy = \text{conv} \int_{\underline{x}}^{\overline{x}} F(y|a)dy \right\}, \end{aligned}$$

where $A_1^C \subseteq A_2^C \subseteq A_3^C$.[9]

To preview the results, consider some fixed monotonic ($\mathcal{C}_1$) and L-IC$_{a^*}$ contract. Given the linear costs, it should be clear from (2) that a^* is a utility maximizing action for the agent if and only a^* is on the convex hull of the "aggregate" function $\int_{\underline{x}}^{\overline{x}} F(x|a)dv(w(x))$. However, this aggregate function depends on the specific contract, whereas the objective is to find conditions that apply to *any* monotonic and L-IC$_{a^*}$ contracts. It is perhaps not too surprising that it turns out to be necessary and sufficient to assume that $F(x|a^*)$ is on its convex envelope for all x, or $a^* \in A_1^C$. That is, $A_1^{L \Longrightarrow G} = A_1^C$. In similar fashion, repeat use of integration by parts explains why the sets A_2^C and A_3^C are interesting. The remainder of this section is devoted to an instructive proof of these results, where the method of proof has explicitly been designed to best bring out the relationships in Table 1 in the introduction and to explain how the generalization to third order stochastic dominance follows naturally from Rogerson's and Jewitt's condition.

The first step is to consider an auxiliary problem. Holding x fixed, let

$$F^L(x|a, a^*) = F(x|a^*) + (a - a^*)F_a(x|a^*) \tag{5}$$

be the tangent line to $F(x|a)$ at $a = a^*$. Figure 1 illustrates $F(x|a)$, $F^L(x|a, a^*)$, and $F^C(x|a)$. Now switch the roles of x and a. Holding a and a^* fixed, consider the function $F^L(x|a, a^*)$. Note that $F^L(x|a, a^*)$ is not necessarily monotonic in x, nor is it necessarily bounded between 0 and 1. Nevertheless, it is convenient to think of $F^L(x|a, a^*)$ as an (admittedly odd) distribution function and ask what the agent's "payoff" would be in an artificial problem where $F(x|a)$ is replaced with $F^L(x|a, a^*)$.

[9] To see that $A_1^C \subseteq A_2^C$, begin with $\int_{\underline{x}}^{x} F(y|a)dy \geq \text{conv}\int_{\underline{x}}^{x} F(y|a)dy \geq \int_{\underline{x}}^{x} \text{conv}F(y|a)dy$. If $a \in A_1^C$ then $\text{conv}F(y|a) = F(y|a)$ for all y, and so the terms on the far left and far right must coincide. Hence, if $a \in A_1^C$ then it must also be the case that $a \in A_2^C$.

Since F^L does have the key properties that $F^L(\underline{x}|a,a^*) = 0$ and $F^L(\overline{x}|a,a^*) = 1$, integration by parts can be used in the same manner as in (2). Thus, "expected utility" in this auxiliary problem is simply

$$EU^L(a|a^*) = v(w(\overline{x})) - \int_{\underline{x}}^{\overline{x}} F^L(x|a,a^*)dv(w(x)) - a, \tag{6}$$

or

$$EU^L(a|a^*) = EU(a^*) + (a-a^*)\left[-\int_{\underline{x}}^{\overline{x}} F_a(x|a^*)dv(w(x)) - 1\right], \tag{7}$$

which, by construction, has the property that $EU^L(a^*|a^*)$ coincides with $EU(a^*)$. This is because $F(x|a)$ and $F^L(x|a,a^*)$ coincide when $a = a^*$. Moreover, the term in brackets disappears if the contract satisfies L-IC$_{a^*}$. In other words, $EU^L(a|a^*) = EU(a^*)$ is independent of a. That is, the agent is indifferent between all actions in the auxiliary problem, or $EU^L(a|a^*) = EU(a^*)$ for all a. The reason is that $F(x|a)$ and $F^L(x|a,a^*)$ are tangent at a^*.

Lemma 1 *Fix $a^* \in (\underline{a},\overline{a})$. Then, for any L-IC$_{a^*}$ contract, $EU^L(a|a^*) = EU(a^*)$ for all $a \in [\underline{a},\overline{a}]$.*

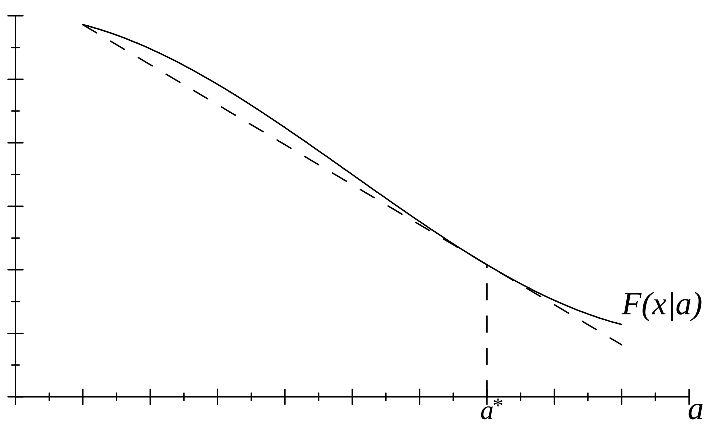

Figure 1: $F(x|a)$ and its tangent line, $F^L(x|a,a^*)$.

NOTE: In this example, $F^C(x|a) = F^L(x|a,a^*)$ when $a \leq a^*$ and $F^C(x|a) = F(x|a)$ when $a \geq a^*$.

Clearly, F^L and F^C are closely related. For instance, if $a^* \in A_1^C$ then the three functions F, F^L, and F^C are all tangent at a^*. The observation in the next lemma, illustrated in Figure 1, is particularly useful. Acknowledging that F^L is not a proper distribution function, $F^L(x|a, a^*)$ will nevertheless be said to first order stochastically dominate $F(x|a)$ if $F^L(x|a, a^*) \le F(x|a)$ for all $x \in [\underline{x}, \overline{x}]$. See Table 1 for definitions of second and third order stochastic dominance.

Lemma 2 *Fix $a^* \in (\underline{a}, \overline{a})$. $F^L(\cdot|a, a^*)$ ith order stochastically dominates $F(\cdot|a)$ for all $a \in [\underline{a}, \overline{a}]$ if and only if $a^* \in A_i^C$, $i = 1, 2, 3$.*

Proof. For any fixed x, if $a^* \in A_1^C$ then F^C, F, and F^L coincide and are tangent at a^*. By definition, F is nowhere below F^C and since F^C is convex it lies everywhere above its tangent line. Hence, $F(x|a) \ge F^C(x|a) \ge F^L(x|a, a^*)$ for all x and all a. Conversely, if for some x, $F^C(x|a^*) < F(x|a^*) = F^L(x|a^*, a^*)$, then F^C must, like F^L, be linear between a_1 and a_2 (see (3)). Thus, $F^L(x|a, a^*)$ must exceed $F^C(x|a)$ at either a_1 or a_2 (or possibly both). Since F^C coincides with F at a_1 and a_2, it follows that $F^L(x|a, a^*) > F(x|a)$ for some a. This proves the lemma for $i = 1$. For $i = 2$, note first that the tangent line (as a function of a) to $\int_{\underline{x}}^x F(y|a)dy$ at $a = a^*$ is

$$\int_{\underline{x}}^x F(y|a^*)dy + (a - a^*) \int_{\underline{x}}^x F_a(y|a^*)dy = \int_{\underline{x}}^x F^L(y|a, a^*)dy.$$

The remainder of the proof proceeds at above, with the functions $F(x|a)$, $F^C(x|a)$, and $F^L(x|a, a^*)$ replaced by $\int_{\underline{x}}^x F(y|a)dy$, $\text{conv}\int_{\underline{x}}^x F(y|a)dy$, and $\int_{\underline{x}}^x F^L(y|a, a^*)dy$, respectively. The proof for $i = 3$ is analogous. ■

The implication of Lemma 1 is that *any* $\mathcal{C}_i$ and L-IC$_{a^*}$ contract is G-IC$_{a^*}$ if and only if for any such contract it holds that $EU^L(a|a^*) \ge EU(a)$ for all $a \in [\underline{a}, \overline{a}]$. It should now be obvious that this is the case if and only if $F^L(x|a, a^*)$ ith order stochastically dominates $F(x|a)$ for all $a \in [\underline{a}, \overline{a}]$.[10] The main result then follows immediately from Lemma 2.

Theorem 1 *$A_i^{L \Longrightarrow G} = A_i^C$, $i = 1, 2, 3$. Thus, fixing $a^* \in (\underline{a}, \overline{a})$, the following statements are equivalent:*

1. *Any $\mathcal{C}_i$ and L-IC$_{a^*}$ contract is G-IC$_{a^*}$ (or $a^* \in A_i^{L \Longrightarrow G}$).*

2. *$F^L(\cdot|a, a^*)$ ith order stochastically dominates $F(\cdot|a)$ for all a, (or $a^* \in A_i^C$).*

[10] To reiterate, it is irrelevant that F^L is not a proper distribution function. The stochastic dominance proofs rely only on $F^L(\underline{x}|a, a^*) = F(\underline{x}|a^*) = 0$, $F^L(\overline{x}|a, a^*) = F(\overline{x}|a^*) = 1$, and the relative magnitude (but not sign) of $F^L(x|a, a^*)$ and $F(x|a^*)$ and their antiderivatives.

Recall that Rogerson (1985) proved that monotonicity and L-IC$_{a^*}$ implies G-IC$_{a^*}$ regardless of a^* if $F_{aa}(x|a) \geq 0$ for all a and all x. Theorem 1 can be seen as a "local" version of Rogerson's "global" result. It is worth pointing out that monotonicity and L-IC$_{a^*}$ is not sufficient for G-IC$_{a^*}$ if it is merely assumed that $F_{aa}(x|a^*) \geq 0$ for all x. After all, such an assumption implies only that $EU(a)$ attains a local maximum at a^*. Instead, the proper "local" counterpart to Rogerson's sufficiency result is that a^* is on the convex envelope of $F(x|a)$ for all x, which is of course more restrictive than $F_{aa}(x|a^*) \geq 0$ for all x. Similar remarks apply to Jewitt's conditions. Kadan and Swinkels (2012, Proposition 2) prove that any action can be implemented if there is a set of outcomes, S, such that the probability that $x \in S$ is concave in the action (or equivalently that the complementary probability is convex). Specifically, a (increasing or decreasing) step contract can be used. However, the arguments leading to Theorem 1 reveals that it is sufficient that there exists a set of outcomes, $S(a^*)$, for each action possible action, such that a^* is on the convex envelope of $\Pr(x \in S(a^*)|a)$.

Moreover, Theorem 1 gives conditions that are not only sufficient but also necessary. However, this means only that L-IC$_{a^*}$ is *not always* sufficient if $F^C(x|a^*) = F(x|a^*)$ for only a proper subset of $x \in (\underline{x}, \overline{x})$. For instance, a L-IC$_{a^*}$ step contract with jump at $\widehat{x}$ is in fact G-IC$_{a^*}$ if $F^C(\widehat{x}|a^*) = F(\widehat{x}|a^*)$ for *some* $\widehat{x} \in (\underline{x}, \overline{x})$, even if $a^* \notin A_1^C$. Likewise, it is not necessarily the case that there is no monotonic G-IC$_{a^*}$ contract if $F^C(x|a^*) < F(x|a^*)$ for all $x \in (\underline{x}, \overline{x})$; it just means that L-IC$_{a^*}$ by itself is not sufficient.

4 Justifying the first-order approach

As mentioned in the introduction, Rogerson's and Jewitt's proofs can be decomposed into two parts. In one, sufficient conditions are given for L-IC$_a$ to imply G-IC$_a$ among $\mathcal{C}_1$ and $\mathcal{C}_2$ contracts, respectively, *for any a*. In the other, sufficient conditions are derived to ensure the FOA candidate solution is a $\mathcal{C}_1$ or $\mathcal{C}_2$ contract. This line of reasoning can obviously be extended to $\mathcal{C}_3$ contracts.

For the first part, the intention is to give conditions such that $A_i^{L \Longrightarrow G} = A_i^C = (\underline{a}, \overline{a})$, $i = 1, 2, 3$. For $i = 1$, for instance, the requirement is thus that $F(x|a)$ coincides with its convex hull (as a varies) for all a and all x. However, it is easy to see that a function coincides with its convex hull everywhere if and only if the function is itself convex.

Proposition 1 *The following conditions identify when $A_i^C = (\underline{a}, \overline{a})$, $i = 1, 2, 3$:*

1. *$A_1^C = (\underline{a}, \overline{a})$ if and only if $F_{aa}(x|a) \geq 0$ for all $x \in [\underline{x}, \overline{x}]$ and all $a \in [\underline{a}, \overline{a}]$.*

2. $A_2^C = (\underline{a}, \overline{a})$ *if and only if* $\int_{\underline{x}}^{x} F_{aa}(y|a)dy \geq 0$ *for all* $x \in [\underline{x}, \overline{x}]$ *and all* $a \in [\underline{a}, \overline{a}]$.

3. $A_3^C = (\underline{a}, \overline{a})$ *if and only if* $\int_{\underline{x}}^{x} \int_{\underline{x}}^{z} F(y|a)dydz \geq 0$ *for all* $x \in [\underline{x}, \overline{x}]$ *and all* $a \in [\underline{a}, \overline{a}]$ *and* $\int_{\underline{x}}^{\overline{x}} F_{aa}(y|a)dy \geq 0$ *for all* $a \in [\underline{a}, \overline{a}]$.

Proof. If $F_{aa}(x|a) \geq 0$ for all $x \in [\underline{x}, \overline{x}]$ and all $a \in [\underline{a}, \overline{a}]$ then it is trivially true that F^C and F coincide for all x and all a. Equivalently, $A_1^C = (\underline{a}, \overline{a})$. For the other direction, assume $A_1^C = (\underline{a}, \overline{a})$. The proof is by contradiction. Assume $F_{aa}(x|a') < 0$ for some $x \in (\underline{x}, \overline{x})$ and some $a' \in (\underline{a}, \overline{a})$. Since F is concave in a at $a = a'$, F^C and F cannot coincide, which contradicts $A_1^C = (\underline{a}, \overline{a})$. If $F_{aa}(x|a') < 0$ for some $x \in (\underline{x}, \overline{x})$ and $a' \in \{\underline{a}, \overline{a}\}$, then, by continuity of F_{aa}, there is some $a \in (\underline{a}, \overline{a})$ for which $F_{aa}(x|a) < 0$, but this once again contradicts $A_1^C = (\underline{a}, \overline{a})$. This completes the proof of the first part of the proposition. The proof is analogous for the second and third part. ■

The conditions that $F_{aa}(x|a) \geq 0$ and $\int_{\underline{x}}^{x} F_{aa}(y|a)dy \geq 0$ for all x and all a coincide with Rogerson's and Jewitt's sufficient conditions.[11] Proposition 1 signifies that not only are these conditions sufficient, they are in fact the weakest conditions that can be imposed to ensure that L-IC implies G-IC for all a when the only characteristics of the contracts that are exploited are monotonicity or monotonicity and concavity.

Thus far, focus has been on interior actions, where L-IC is necessary for utility maximization. Consider now the corners, $\underline{a}$ and $\overline{a}$. At the corners, any function coincides with its convex envelope. However, unlike in the interior, the tangent line need not have the same slope as the convex hull. Thus, the characterization in Theorem 1 does not immediately extend to the corners; for instance, it is not true that any monotonic and L-IC$_{\overline{a}}$ contract is G-IC$_{\overline{a}}$, even though $\overline{a}$ is on the convex envelope. For instance, if $F_{aa} \leq 0$ for all x and a, then $EU(a)$ is convex for any monotonic contract. Thus, L-IC$_{\overline{a}}$ would imply that $\overline{a}$ is the worst possible action.

However, if Rogerson's condition is satisfied, then F and F^C coincide everywhere and so are of course always tangent. In this case, $EU'(\overline{a}) = 0$ (or L-IC$_{\overline{a}}$) is sufficient for G-IC$_{\overline{a}}$. Indeed, if $EU'(\overline{a}) \geq 0$, it follows from (7) that $EU^L(\overline{a}|\overline{a}) = EU(\overline{a}) \geq EU^L(a|\overline{a})$ for all $a \in [\underline{a}, \overline{a}]$. Since Rogerson's condition implies that $EU^L(a|\overline{a}) \geq EU(a)$ for any $\mathcal{C}_1$ contract, it must be the case that $EU(\overline{a}) \geq EU(a)$. Hence, at $\overline{a}$, any $\mathcal{C}_1$ contract that satisfies $EU'(\overline{a}) \geq 0$ is G-IC$_{\overline{a}}$. Similarly, any $\mathcal{C}_1$ contract that satisfies $EU'(\underline{a}) \leq 0$ is G-IC$_{\underline{a}}$ (a constant-wage contract is a special case). These arguments extend to $\mathcal{C}_2$ and $\mathcal{C}_3$ contracts.

[11] The condition $\int_{\underline{x}}^{x} F_{aa}(y|a)dy \geq 0$ is Jewitt's (1988) assumption (2.10a). He also imposes another assumption, (2.10b), but this assumption is redundant; see Conlon (2009a, 2009b). Assumptions (2.11) and (2.12) are used in the other step of his proof (see below).

If the principal can write only $\mathcal{C}_i$ contracts, and the corresponding condition from Proposition 1 applies, the principal's problem simplifies to selecting a wage schedule, $w(x)$, in $\mathcal{C}_i$ and a target action, $a^* \in [\underline{a}, \overline{a}]$, subject to

$$EU'(a^*) \begin{cases} \leq 0 \text{ if } a^* = \underline{a} \\ = 0 \text{ if } a^* \in (\underline{a}, \overline{a}) \\ \geq 0 \text{ if } a^* = \overline{a} \end{cases} \tag{8}$$

and whatever other constraints may be relevant, such as a participation constraint. While Innes (1990) argues that it is reasonable to restrict attention to $\mathcal{C}_1$ contracts in some environments, it may be preferable to impose no such restrictions ex ante. In fact, Rogerson and Jewitt, in the second step of their proofs, find conditions such that the candidate solution using only local incentive compatibility turns out to belong to $\mathcal{C}_1$ or $\mathcal{C}_2$ respectively. At this juncture, more assumptions must be imposed.

As in Jewitt (1988), assume the principal is risk neutral. Let $B(a)$ denote the expected gross benefit to the principal if the agent chooses action a. In many applications, $B(a)$ is simply the expected value of x. Assume $v(\cdot)$ is differentiable to the requisite degree, and that the agent is strictly risk averse, or $v''(\cdot) < 0$. Apart from incentive compatibility, the only other constraint is a participation constraint. It will be assumed the constraint-set is non-empty, i.e. that there exists a contract that satisfies both the participation constraint and L-IC for some a.

Assume the likelihood-ratio

$$l(x|a) = \frac{f_a(x|a)}{f(x|a)}$$

is bounded below. As in Rogerson and Jewitt, assume that the monotone likelihood ratio property (MLRP) is satisfied, or $l_x(x|a) \geq 0$. This assumption in fact implies $F_a(x|a) \leq 0$. Finally, assume that it is optimal to offer a wage $w(x)$ in state x that is in the interior of the domain of $v(\cdot)$. For a fixed utility function, this assumption is typically satisfied if the agent's reservation utility is high enough.[12] In this case, $w(x)$ is characterized by a first order condition which can be written

$$\frac{1}{v'(w(x))} = \lambda + \mu l(x|a^*), \tag{9}$$

where $\lambda > 0$ is the multiplier of the participation constraint and $\mu \geq 0$ the multiplier

[12] See e.g. Jewitt et al (2008), and in particular Gutiérrez (2012) for a detailed discussion. As can be seen from (9), below, this also explains why $l(x|a)$ must be bounded.

of the local incentive compatibility constraint. If $a^* = \underline{a}$, a flat wage is optimal ($\mu = 0$). However, if $a^* > \underline{a}$ then $\mu > 0$ and so, by the MLRP, the wage schedule belongs to $\mathcal{C}_1$ (it is monotonic).[13] These conclusions are due only to the assumptions that $v'(\cdot)$ is decreasing in w and $l(x|a^*)$ is increasing in x. Jewitt (1988) imposes more substantial joint conditions on the utility function and likelihood ratio. These conditions allow him to conclude that the optimal contract belongs to $\mathcal{C}_2$ (i.e. $v(w(x))$ is increasing and concave).

[TABLE 2 ABOUT HERE (SEE THE LAST PAGE)]

To aid the analysis, Jewitt defines the function

$$\omega(z) = v(v'^{-1}(1/z)), \; z > 0.^{14}$$

Note that $\omega'(z) > 0$ if and only if $v''(w) < 0$, which has already been assumed. Jewitt adds the assumption that $\omega''(z) \leq 0$ and $l_{xx}(x|a) \leq 0$. From (9),

$$v(w(x)) = \omega\left(\lambda + \mu l(x|a^*)\right).$$

Hence, Jewitt's assumptions imply that any contract that solves (9) belongs to $\mathcal{C}_2$.

It turns out that this pattern too can be continued. On top of the assumptions already accumulated, assume that $l_{xxx}(x|a) \geq 0$ and that $\omega'''(z) \geq 0$. Then, it can easily be shown that the resulting contract (from (9)) belongs to $\mathcal{C}_3$.

Table 2 summarizes the main conclusions thus far. The first row identifies sufficient conditions for L-IC$_a$ (or rather (8)) to imply G-IC$_a$ among $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$ contracts, respectively, for any a. The second row identifies sufficient conditions for the FOA candidate solution in (9) to be a $\mathcal{C}_1$, $\mathcal{C}_2$, or $\mathcal{C}_3$ contract, respectively, for any a. The validity of the FOA follows by imposing both sets of assumptions. To make the following statement more succinct, assume the second best action is in $(\underline{a}, \overline{a})$.[15]

[13] One of the contributions in Rogerson (1985) and Jewitt (1988) is to establish that $\mu > 0$. In fact, Jewitt's (1988) paper appears to be cited more often for this result (and its very elegant proof) than for his conditions justifying the FOA. As in Conlon (2009a), I omit the proof here. Rogerson (1985) allows the principal to be risk averse. It is considerable harder to allow a risk averse principal in Jewitt's framework; see Conlon (2009a).

[14] To clarify, $v'^{-1}(\cdot)$ refers to the inverse of $v'(\cdot)$.

[15] Rogerson (1985) make a similar assumption (see his Assumption A.10).

Proposition 2 *Assume the second best action is in* $(\underline{a}, \overline{a})$.[16] *Assume the joint conditions in one of the columns of Table 2 are satisfied. Then, the optimal contract solves the problem*

$$\max_{w,a} B(a) - \int_{\underline{x}}^{\overline{x}} w(x) f(x|a) dx$$

$$st. \quad \int_{\underline{x}}^{\overline{x}} v(w(x)) f(x|a) dx - a \geq \overline{u}$$

$$\int_{\underline{x}}^{\overline{x}} v(w(x)) f_a(x|a) dx - 1 = 0,$$

where $\overline{u}$ *is the agent's reservation utility. The optimal contract takes the form in (9).*

Evidently, the assumptions in the first row of Table 1 become weaker as one moves rightward from one column to the next. As for the second row, consider the following possible utility functions:

$$v_1(w) = \ln w,\ v_2(w) = 1 - e^{-\alpha w},\ v_3(w) = w^\beta,$$

where $\alpha, \beta > 0$. The domains of the functions are $(0, \infty)$, $(-\infty, \infty)$, and $[0, \infty)$, respectively (or convex subsets thereof). For these functions, $\omega(z)$ can be shown to be

$$\omega_1(z) = \ln z,\ \omega_2(z) = 1 - \frac{1}{\alpha z},\ \text{and } \omega_3(z) = (\beta z)^{\frac{\beta}{1-\beta}}$$

respectively. Thus, the first two functions satisfy $\omega_i'(z) > 0$, $\omega_i''(z) < 0$, $\omega_i'''(z) > 0$, $i = 1, 2$. The third function satisfies $\omega_3'(z) > 0$, $\omega_3''(z) < 0$ if and only if $\beta \in \left(0, \frac{1}{2}\right)$, i.e. if the agent is sufficiently risk averse. However, for β in this range it is also the case that $\omega_3'''(z) > 0$. Thus, in these examples, the assumptions on $\omega(z)$ in the third column of Table 1 are not any stronger than those in the second column. Thus, the main strengthening from Jewitt's conditions to the new conditions in the third column is in the added requirement that $l_{xxx}(x|a) \geq 0$. Incidentally, all Jewitt's (1988, page 1183) examples have this feature.

[16] It is easy to check this assumption. Just compare the solution to the problem stated in the proposition with the payoffs the principal would get from optimally inducing $\underline{a}$ or $\overline{a}$. To find these payoffs, use the constraints in (8). Since the principal is risk neutral, however, the optimal contract that induces $\underline{a}$ is a fixed-wage contract.

5 A modified first-order approach

As Mirrlees (1999) pointed out early on, the FOA is not always valid. The purpose of this section is to propose a modified FOA that works in some specific, but important, models in which the FOA is not generally valid. I will also demonstrate that the modified FOA simplifies the analysis of some examples in the literature.

More concretely, consider distribution functions of the form

$$F(x|a) = p(a)G(x) + (1 - p(a))\, H(x), \tag{10}$$

where $p(a) \in [0, 1]$ for all $a \in [\underline{a}, \overline{a}]$ and G and H are non-identical distribution functions with support $[\underline{x}, \overline{x}]$, and strictly positive densities $g(x)$ and $h(x)$, respectively. While this model is certainly too specialized to capture all principal-agent relationships, it should be stressed that it does have a compelling interpretation. For instance, $p(a)$ could be the proportion of time the parts-supplier (the agent) spends using the new and advanced technology G rather than the less reliable but more user-friendly old technology, H. Given such interpretations of the model, the most meaningful economic assumption is that $p(a)$ is monotonic. Thus, as is common in the literature, assume that $p'(a) > 0$ for all $a \in (\underline{a}, \overline{a}]$. The case where $p(a)$ is non-monotonic is not that much more difficult. It is discussed briefly later.

Distributions of this form have been studied extensively. Grossman and Hart (1983) say that the *spanning condition* is satisfied if $F(x|a)$ can be written as in (10). Since (10) is linear in p, Hart and Holmström (1987) refer to (10) as the Linear Distribution Function Condition (LDFC). The significance of the model and its place in the literature is discussed in detail after the formal analysis.

Typically, additional assumptions are imposed on the curvature of $p(a)$ as well as on the relationship between G and H. For instance, Sinclair-Desgagné (1994) points out that the FOA is valid if $p(a)$ is concave and $\frac{g(x)}{h(x)}$ is nondecreasing. The latter assumption implies the MLRP, while the former ensures concavity of the agent's objective function when he faces a monotonic contract. The second assumption also implies that G first order stochastically dominates H. Without assumptions on $p(a)$, Grossman and Hart (1983) prove that if $\frac{g(x)}{h(x)}$ is nondecreasing then any optimal contract must feature monotonic wages.[17] Ke (2011, Proposition 7) shows that the FOA is valid if $p(a)$ is concave, even without the MLRP.

Here, I impose no such conditions on (10). For instance, $p(a)$ may be concave

[17] In their discrete model, Grossman and Hart (1983) allow multiple incentive compatibility constraints to bind. I will show, in the continuous model, that if a can be implemented then all but the local incentive compatibility constraint are redundant.

only locally, or not at all, and G and H may cross, as would be the case if H is a mean-preserving spread over G.[18] Contrary to the first part of the paper, I also allow contracts to be non-monotonic. However, as before, the crucial first step is to explore the link between L-IC and G-IC.

First, on a technical note, I restrict attention to contracts that give the agent bounded utility. In principle, any action could be implemented by specifying a contract that provides unbounded utility to the agent.[19,20] However, restricting the analysis to contracts with bounded utility is of little consequence since the cost of providing unbounded utility must, by Jensen's inequality, itself be unbounded, and thus cannot be optimal. Hence, in the following, contracts are assumed to yield bounded utility and to be integrable.

Given the spanning condition, for any $a^* \in (\underline{a}, \overline{a})$, L-IC$_{a^*}$ is

$$p'(a^*) \int v(w(x)) \left(g(x) - h(x)\right) dx - 1 = 0. \tag{11}$$

Since $p'(a^*) > 0$, the integral must take the strictly positive value $\frac{1}{p'(a^*)}$ in order to satisfy (11). The agent's expected utility can be written

$$EU(a) = \frac{p(a)}{p'(a^*)} - a + \int v(w(x))h(x)dx,$$

should he take action a. It follows that

$$EU(a^*) - EU(a) = \frac{(-p(a)) - [-p(a^*) + (a - a^*)(-p'(a^*))]}{p'(a^*)} \tag{12}$$

for all $a \in [\underline{a}, \overline{a}]$. The term in the square brackets is the tangent line to $-p(a^*)$. Let

[18] Note that if $G(x) = H(x)$ then $F_a(x|a) = 0$, which was ruled out, by assumption, in the previous sections. Indeed, $F_a(x|a) > 0$ if it should happen that $G(x) > H(x)$. Though it need not be imposed, it is natural to assume that $G(x) < H(x)$ for some x. Otherwise, higher (more costly) actions lead to stochastically worse outcomes.

[19] Fix some action a^* to be implemented. Assuming $v(\cdot)$ is unbounded above, offer a contract such that $v(w(x)) = [(\overline{x} - x) f(x|a)]^{-1}$ if $x < \overline{x}$ and $v(w(\overline{x})) = 0$, say. This non-monotonic and unbounded contract gives unbounded utility for a set of actions including a^*, and so the agent would be willing to choose a^*.

[20] In the previous sections, any monotonic contract must be bounded because the support $[\underline{x}, \overline{x}]$ is compact. Hermalin and Katz (1991) use tools from convex analysis to characterize the set of implementable actions in a model with a finite set of actions and a finite set of outcomes. Since the set of outcomes is finite, any contract must be bounded. Note that their analysis does not reveal when L-IC implies G-IC, or when the optimal contract is monotonic.

A_p^C denote the set of actions in $(\underline{a}, \overline{a})$ for which $-p(a)$ coincides with its convex hull. As in Lemma 2, $a^* \in A_p^C$ if and only if (12) is non-negative for any a.

Proposition 3 *With the LDFC, there exists a G-IC$_{a^*}$ contract (that yields bounded utility) if and only if $a^* \in A_p^C \cup \{\underline{a}, \overline{a}\}$.*

Proof. Assume $a^* \notin A_p^C$. If there is a G-IC$_{a^*}$ contract, then that contract must necessarily be L-IC$_{a^*}$, and so (12) should apply. However, since $a^* \notin A_p^C$, there is some $a \in (\underline{a}, \overline{a})$ for which (12) is strictly negative, which contradicts G-IC$_{a^*}$.

For the other direction, assume $a^* \in A_p^C$. Since G and H are distinct, there is some $x \in (\underline{x}, \overline{x})$ for which $G(x) \neq H(x)$, or $F_a(x|a^*) \neq 0$. Now, construct a step contract that delivers utility v_0 if the outcome is worse than x, and v_1 otherwise. Pick v_0 and v_1 in such a manner that L-IC$_{a^*}$ is satisfied. Regardless of the sign of $F_a(x|a^*)$, such a step contract exists given the assumption that utility is unbounded either above or below. Since $a^* \in A_p^C$, (12) is everywhere non-negative. Hence, the contract is G-IC$_{a^*}$.

Now assume $a^* \in \{\underline{a}, \overline{a}\}$. By modifying the steps that led to (12), it is easy to see that a step contract that makes $EU'(\underline{a})$ sufficiently small or $EU'(\overline{a})$ sufficiently large is G-IC$_{\underline{a}}$ or G-IC$_{\overline{a}}$, respectively. ■

There are two differences between Proposition 3 and Theorem 1. First, the latter considers only certain classes of contracts, all of them monotonic. Second, it considers only contracts for which L-IC$_{a^*}$ implies G-IC$_{a^*}$. In the present model, $A_p^C \cup \{\underline{a}, \overline{a}\}$ identifies the feasible set of implementable actions without imposing monotonicity, and without insisting that L-IC is sufficient (for interior actions). Nevertheless, as recorded next, it should be clear from the proof of Proposition 3 that L-IC$_{a^*}$ is in fact *necessary and sufficient* for G-IC$_{a^*}$, for any $a^* \in A_p^C$.

Proposition 4 *If $a^* \in A_p^C$ then L-IC$_{a^*}$ is necessary and sufficient for G-IC$_{a^*}$.*

Proof. Necessity is obvious. As in the proof of Proposition 3, sufficiency follows from the fact that (12) is everywhere positive if $a^* \in A_p^C$. ■

As a consequence of Propositions 3 and 4, a modified FOA suggests itself. In the first step, the feasible set is identified, $A_p^C \cup \{\underline{a}, \overline{a}\}$. The feasible set is closed (but not necessarily convex). In the second step, the FOA is applied to this set (i.e. with the constraint that $a \in A_p^C \cup \{\underline{a}, \overline{a}\}$). In a third step, the solution is compared to the payoff from optimally implementing $\underline{a}$ and $\overline{a}$. Whichever contract is superior is then chosen.

To find the optimal contract that implements $\underline{a}$ or $\overline{a}$, it turns out that the continuum of incentive compatibility constraints can again be summarized by one lone condition. For instance, consider implementing $\underline{a}$. Let $\underline{a}^c = \inf A_p^C$ if A_p^C is non-empty and let $\underline{a}^c = \overline{a}$ otherwise. If $\underline{a} = \underline{a}^c$, then $EU'(\underline{a}) \leq 0$ is sufficient for G-IC$_{\underline{a}}$. On the other hand, if $\underline{a} < \underline{a}^c$ then it can be shown that any contract that leaves no incentive for the agent to pick $\underline{a}^c$ over $\underline{a}$ is G-IC$_{\underline{a}}$. To implement $\overline{a}$, the relevant counterpart to $\underline{a}^c$ is $\overline{a}^c = \sup A_p^C$ when A_p^C is non-empty and $\overline{a}^c = \underline{a}$ otherwise.

Proposition 5 *Implementing boundary actions:*

1. *If* $\underline{a}^c = \underline{a}$ *then* $EU'(\underline{a}) \leq 0$ *is necessary and sufficient for G-IC*$_{\underline{a}}$*. If* $\underline{a}^c > \underline{a}$ *then* $EU(\underline{a}) \geq EU(\underline{a}^c)$ *is necessary and sufficient for G-IC*$_{\underline{a}}$*.*

2. *If* $\overline{a}^c = \overline{a}$ *then* $EU'(\overline{a}) \geq 0$ *is necessary and sufficient for G-IC*$_{\overline{a}}$*. If* $\overline{a}^c < \overline{a}$ *then* $EU(\overline{a}) \geq EU(\overline{a}^c)$ *is necessary and sufficient for G-IC*$_{\underline{a}}$*.*

Proof. Necessity is obvious. For sufficiency in the first part of the proposition, consider first the "no-gap" case, $\underline{a}^c = \underline{a}$. Here, the slope of $-p(a)$ coincides with the slope of its convex hull at $\underline{a}$. As in the proof of Proposition 3, a modification of (12) then establishes that $EU'(\underline{a}) \leq 0$ is sufficient for G-IC$_{\underline{a}}$. However, this is not necessarily true in the "gap" case, where $\underline{a}^c > \underline{a}$. Note that

$$EU(\underline{a}) - EU(a) = (a - \underline{a}) \left[\frac{-p(a) - (-p(\underline{a}))}{a - \underline{a}} \int v(w(x)) \, (g(x) - h(x)) \, dx + 1 \right],$$

and so $EU(\underline{a}) \geq EU(\underline{a}^c)$ implies that the term in brackets must be non-negative when $a = \underline{a}^c$. If the integral is negative, then the term in brackets is positive for all a, or $EU(\underline{a}) \geq EU(a)$ for all a. That is, the contract is G-IC$_{\underline{a}}$. If the integral is positive, then the term in brackets is *minimized* at $a = \underline{a}^c$. This follows by definition of the convex hull, since the line from $(\underline{a}, -p(\underline{a}))$ to $(\underline{a}^c, -p(\underline{a}^c))$ is steeper than the line from $(\underline{a}, -p(\underline{a}))$ to any other point on $-p(\cdot)$. Hence, if $EU(\underline{a}) \geq EU(\underline{a}^c)$ then $EU(\underline{a}) \geq EU(a)$ for all $a \in [\underline{a}, \overline{a}]$, thus implying G-IC$_{\underline{a}}$. The proof of the second part of the proposition is analogous. ■

The assumption that $p(a)$ is monotonic seems justified on economic grounds. However, it is possible to allow $p(a)$ to be non-monotonic. First, note that the argument following (12) remains valid if $p'(a^*) > 0$ even if $p'(a^{**}) < 0$ for some $a^{**} \neq a^*$. That is, a^* can be implemented, and L-IC$_{a^*}$ is sufficient, if and only if a^* is on the convex hull of $-p(a^*)$. By similar reasoning, $a^{**} \in (\underline{a}, \overline{a})$ can be implemented, and L-IC$_{a^{**}}$ is sufficient, if and only if a^{**} is on the convex hull of $p(a^{**})$.[21] Thus, the set of implementable interior actions can be obtained by piecing together the

[21] This is easily seen by multiplying both numerator and denominator in (12) by -1.

sets of implementable actions with $p'(\cdot) > 0$ and $p'(\cdot) < 0$, respectively. Of course, if $a^* \in (\underline{a}, \overline{a})$ and $p'(a^*) = 0$ then no L-IC$_{a^*}$ contract exist (with bounded utility), as can be seen from (11). Similarly, if $p(a^*) = p(a')$, then a^* cannot be implemented if $a' < a^*$ because it would be cheaper for the agent to pick a' rather than a^*.[22] Note that such actions cannot be on the convex hull of either $-p(a)$ or $p(a)$ when $p'(a^*) \neq 0$.

The remainder of this section is devoted to demonstrating the significance of the spanning condition as well as illustrating some uses of the preceding characterization.

First, it is useful to recognize that the textbook case in which there are two outcomes (but a continuum of actions) is in fact a special case of (10). Specifically, this model corresponds to assuming that G and H are degenerate distributions, with all mass concentrated at opposite ends of the support. Hence, Propositions 3 – 5 makes it possible to reexamine some important examples in the literature.

Example 1 (Araujo and Moreira (2001)): Araujo and Moreira (2001) propose a general Lagrangian approach to the moral hazard problem that applies when the FOA is not valid. Their leading example is the following. There are two states, where state 1 is the bad state and state 2 is the good state. The agent picks an effort level, e, from $[\underline{e}, \overline{e}] \subseteq [0, 1]$. With effort e, the probability of the good state is $q(e) = e^3$. The cost of effort is $c(e) = e^2$. To reparameterize the model, let $a \equiv c(e) = e^2$ and $p(a) = q(c^{-1}(a)) = a^{\frac{3}{2}}$, $a \in [\underline{a}, \overline{a}] = [\underline{e}^2, \overline{e}^2]$. Note that $p(a)$ is increasing and convex. Thus, $-p(a)$ is concave and so A_p^C is empty. In other words, no interior action can be implemented. Moreover, the boundary actions can be implemented, and the only relevant incentive compatibility constraint is that the desired action be preferable to the action on the opposite end of the support. Consequently, this example essentially reduces to the textbook example with two outcomes and two actions, $\underline{a}$ and $\overline{a}$, and is therefore trivial to solve once a participation constraint is added. In contrast, to use their general approach to solve the example, Araujo and Moreira (2001) (having added assumptions on $v(w)$ and on the principal's payoff) construct an algorithm in Mathematica and use this to solve 20 non-linear systems of equations. As expected, they find the optimal action is at a corner. While their method is obviously powerful, using it on their leading example is overkill (not to mention labor intensive) and obscures the intuition. ▲

Mirrlees (1999) offers a famous example to illustrate how the FOA may fail. In their textbook, Bolton and Dewatripont (2005, p. 148) remark that: "This example

[22] Consequently, once $p(a)$ is allowed to be non-monotonic, it is no longer necessarily the case that $\overline{a}$ can be implemented.

is admittedly abstract, but this is the only one to our knowledge that addresses the technical issue." Next, I will show that Mirrlees' (1999) example can be analyzed using the techniques presented earlier in this section. In particular, the modified FOA correctly solves the problem. Thereafter, I will provide a more straightforward example of how the FOA may fail. Again, the modified FOA allows the correct solution to be obtained.

EXAMPLE 2 (MIRRLEES (1999)): Consider an agent with payoff function $U(w,z) = we^{-(z+1)^2} - \left(-e^{-(z-1)^2}\right)$. It may be helpful to think of this example as a special environment with two outcomes, where, for some reason, the wage in one state is exogenously fixed at 0. The principal controls the "bonus" w (which may be positive or negative) if the other state materializes. The agent's action is $z \in \mathbb{R}$. Think of $e^{-(z+1)^2}$ roughly as the probability of the state in which a bonus is paid out, and think of $-e^{-(z-1)^2}$ as the cost function. This example fits rather well with the model in (10). In particular, with the spanning condition and only two states, the agent's expected utility is separable in the action and the difference between utility in the two states (the bonus). Next, let $a = -e^{-(z-1)^2}$, and note that $a \in [-1, 0)$. Think of the agent as having a two-dimensional problem. First, he has to decide which cost level, a, to incur, and, second, whether to incur this cost with a z that is above or below 1 (since $z_- = 1 - \sqrt{-\ln(-a)}$ and $z_+ = 1 + \sqrt{-\ln(-a)}$ both yield the same a). Depending on whether $z < 1$ or $z > 1$, expected utility can be written as $V^-(w,a) = wp^-(a) - a$ or $V^+(w,a) = wp^+(a) - a$, respectively, where

$$p^-(a) = e^{-\left(2-\sqrt{-\ln(-a)}\right)^2}, \text{ and } p^+(a) = e^{-\left(2+\sqrt{-\ln(-a)}\right)^2}, \ a \in [-1, 0).$$

Clearly, $p^-(a) > p^+(a)$. Hence, $V^-(w,a) > V^+(w,a)$ if and only if w is strictly positive. It is now possible to split the problem into two entirely conventional problems. In one, the principal is constrained to $w \leq 0$ and the agent's payoff function is effectively $V^+(w,a)$. In the other, the constraint is $w \geq 0$ and the agent's payoff function is $V^-(w,a)$.

For the first problem, it can be shown that $p^+(a)$ is decreasing. Hence, negative wages are indeed necessary for L-IC. Moreover, $p^+(a)$ is convex and so coincides with its convex envelope. It follows from Propositions 3 and 4 that any interior action can be implemented and that the FOA is valid (see the remarks following Proposition 4).

The second problem is more interesting. Here, $p^-(a)$ is increasing on $[-1, -e^{-4})$, and decreasing on $(-e^{-4}, 0)$. Since only non-negative wages can be used, there is no permissible contract that satisfies L-IC for any $a \geq -e^{-4} \approx -0.0183$. On the remaining support, $-p^{-}(a)$ coincides with its convex envelope if and only if $a \in [-1, -0.9982] \cup [-0.0217, -e^{-4})$. By Propositions 3 and 4, the modified FOA is valid on this set (and actions in $(-0.9982, -0.0217)$ cannot be implemented).

Next, Mirrlees specifies an objective function for the principal. There is no participation constraint. The principal seeks to maximize $-(z-1)^2 - (w-2)^2$ or, equivalently, $\ln(-a) - (w-2)^2$. The agent's first order condition yields $w = 1/p^{-\prime}(a)$ and $w = 1/p^{+\prime}(a)$, respectively. Substituting this into the principal's objective function and plotting the resulting functions reveals that positive bonuses are superior to negative bonuses and that the solution is at a corner of the feasible set, specifically at $w = 1$ and $a = -0.9982$ (or $z_{-} = 0.957$). This of course coincides with the solution Mirrlees found, but not with the solution one would obtain from the standard FOA (which yields $a = -0.988\,97$ or $z_{-} = 0.895$, as demonstrated by Mirrlees). ▲

EXAMPLE 3 (SIMPLIFIED COUNTEREXAMPLE): There are two outcomes. Let v_1 be the agent's utility (from wages) if the outcome is bad and v_2 be his utility if the outcome is good. The outcomes are worth x_1 and x_2 to the principal, respectively. The probability of the good outcome is $p(a)$, with $p'(a) > 0$. The participation constraint and L-IC constraint yield the system

$$\begin{aligned} v_1 + p(a)(v_2 - v_1) - a &= \overline{u} \\ p'(a)(v_2 - v_1) - 1 &= 0 \end{aligned}$$

with solution

$$v_1 = \overline{u} + a - \frac{p(a)}{p'(a)},\ v_2 = \overline{u} + a + \frac{1 - p(a)}{p'(a)}.$$

If L-IC is sufficient, the risk-neutral principal's expected payoff is

$$\pi(a) = (1 - p(a))(x_1 - v^{-1}(v_1)) - p(a)(x_2 - v^{-1}(v_2)).$$

Assume $p(a) = a + \frac{1}{2}(a^2 - a^3)$, $a \in [0, \overline{a}]$, $\overline{a} \in (\frac{1}{2}, 1]$. Note that if $a = 0$ then $v_1 = v_2 = \overline{u} + a$, so in this special case, with $p(0) = 0$, $\pi(0)$ also describes the optimal way of implementing the lowest action.[23] Here, $p(a)$ is convex when $a < \frac{1}{3}$ and concave when $a > \frac{1}{3}$. However, the relevant set is A_p^C, which is $A_p^C = [\frac{1}{2}, \overline{a})$. Thus, the set of implementable actions is $\{0\} \cup [\frac{1}{2}, \overline{a}]$, with $\underline{a}^c = \frac{1}{2} > \underline{a}$ and $\overline{a}^c = \overline{a}$. Assume $\overline{u} = 2$, $v(w) = \sqrt{w}$ and $x_1 = 5$, $x_2 = 9.4$. Figure 2 plots $\pi(a)$ when $\overline{a} = \frac{2}{3}$.

[23] In general, the cost of implementing a given action is discontinuous at $\underline{a}$. The highest action, $a = \frac{2}{3}$, can be implemented with any contract for which $EU'(\frac{2}{3}) \geq 0$. However, it is easy to see that any contract with $EU'(\frac{2}{3}) > 0$ cannot be optimal. The reason is that such a contract unnecessarily imposes more risk on the agent ($v_2 - v_1$ is larger).

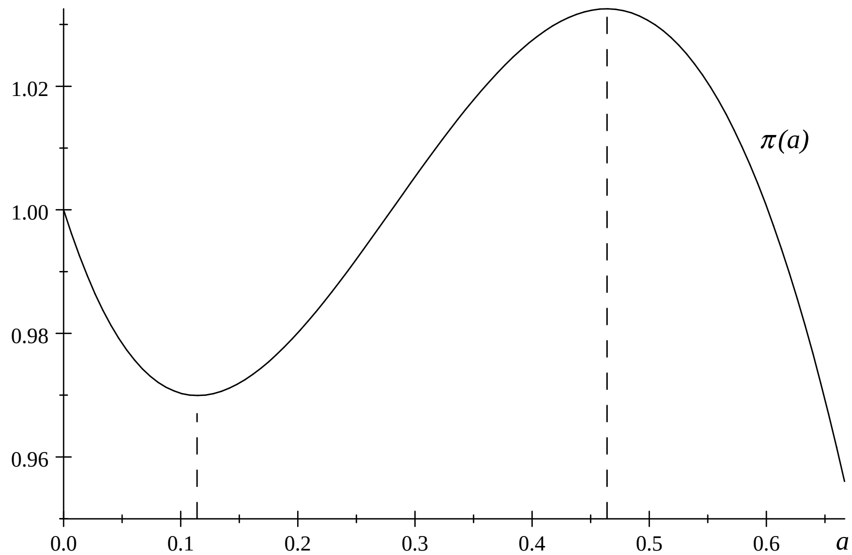

Figure 2: Simplified counterexample.

There are two stationary points. The first, at $a^* = 0.114$, minimizes the principal's payoff and is not even implementable because the agent's payoff is convex whenever $a < \frac{1}{3}$. The second stationary point, at $a^{**} = 0.464$, is the global maximum of $\pi(a)$. However, a^{**} is not implementable either. Though the agent's payoff is locally concave at a^{**}, it is profitable for the agent to deviate to $\underline{a} = 0$. One way to see this is that $v_2 > v_1 > 2 = \overline{u}$ whenever $a \in \left(0, \frac{1}{2}\right)$. Given the feasible set is $\{0\} \cup [\frac{1}{2}, \overline{a}]$, it is clear from the figure that the optimal action to induce is $a = \frac{1}{2}$ (with $v_1 = 2$, $v_2 = \frac{26}{9}$). ▲

The spanning condition has often been implicitly imposed in papers with a continuum of actions. Perhaps the most significant example of this is in LiCalzi and Spaeter (2003) who provide two classes of distributions for which Rogerson's conditions are satisfied. The first family of distributions is

$$F(x|a) = x + \beta(x)\gamma(a), \ x \in [0,1]. \tag{13}$$

Obviously, conditions must be imposed on $\beta(\cdot)$ and $\gamma(\cdot)$ to ensure that $F(x|a)$ is a proper distribution function. LiCalzi and Spaeter (2003) identify additional assumptions on both $\beta(\cdot)$ and $\gamma(\cdot)$ which ensure $F_{aa}(x|a) \geq 0$ and the MLRP. Note, however, that these distribution functions are separable in x and a. Thus, although it seems to not have been observed before, it should be clear that $F(x|a)$ could be stated as in (10). Thus, the modified FOA is always valid in this family of distributions, even without LiCalzi and Spaeter's (2003) additional assumptions.

Example 1 in Jewitt et al (2008) uses the Farlie-Gumbel-Morgenstern copula ($f(x|a) = 1 + \frac{1}{2}(1-2x)(1-2a)$, $x, a \in [0,1]$). This distribution is also separable in x and a and thus can be written as in (10). Example 1 in Kadan and Swinkels (2012) can be written as

$$F(x|a) = p(a)x + (1-p(a))\left(x + x^2 - x^3\right),$$

where $p(a) = 2a^2 - a^3$ and $x \in [0,1]$ and they assume $a \in [\frac{2}{3}, 1]$. Ke (2011, Proposition 10) prove that LiCalzi and Spaeter's (2003) additional assumptions on $\beta(x)$ are not necessary for the validity of the FOA. Note the overlap between his Propositions 7 and 10.

6 Multi-signal principal-agent problems

Rogerson's (1985) condition is equivalent to assuming that $1 - F(x|a)$ is concave in a or $-(1 - F(x|a))$ is convex in a. Sinclair-Desgagné (1994) asked how Rogerson's condition can be generalized to the case where there are multiple signals. Let $F(x_1, x_2, ..., x_k|a)$ denote the joint distribution of k signals. Alternatively, singling out the i'th signal, write the joint distribution as $F(x_i, x_{-i}|a)$ or the joint density as $f(x_i, x_{-i}|a)$, where x_{-i} is the vector of all signals except for x_i. Define

$$F^i(x_i, x_{-i}|a) \equiv \int_{\underline{x}_i}^{x_i} f(z, x_{-i}|a)dz.$$

Assume that F satisfies (the multi-signal) MLRP. With the additional assumption that utility is non-negative, Sinclair-Desgagné (1994) proved that the FOA is valid if there is *some* i such that

$$Q^i(x_i, x_{-i}|a) \equiv \left(F^i(\overline{x}_i, x_{-i}|a) - F^i(x_i, x_{-i}|a)\right) \tag{14}$$

is concave in a *for all* (x_i, x_{-i}, a), or $-Q^i$ is convex.[24] The results in Section 3 can be used to strengthen Sinclair-Desgagné (1994) result. To do so, it is instructive to

[24]Note that if utility is bounded below, an affine transformation can always be used to ensure utility is non-negative.

start once again by examining the agent's expected utility from some action a,

$$EU(a) = \int_{\underline{x}_{-i}}^{\overline{x}_{-i}} \left(\int_{\underline{x}_i}^{\overline{x}_i} v(w(x_i, x_{-i})) f(x_i, x_{-i}|a) dx_i \right) dx_{-i} - a.$$

Assuming $v(w(x_i, x_{-i}))$ is differentiable, the term in the parenthesis can be written

$$\begin{aligned} T^i(a) &= \left(F^i(\overline{x}_i, x_{-i}|a) - F^1(\underline{x}_i, x_{-i}|a)\right) v(w(\underline{x}_i, x_{-i})) \\ &\quad + \int_{\underline{x}_i}^{\overline{x}_i} \left(F^i(\overline{x}_i, x_{-i}|a) - F^i(\underline{x}_i, x_{-i}|a)\right) \frac{\partial v(w(x_i, x_{-i}))}{\partial x_i} dx_i \end{aligned}$$

since $F(\underline{x}_i, x_{-i}|a) = 0$, or

$$T^i(a) = Q^i(\underline{x}_i, x_{-i}|a) v(w(\underline{x}_i, x_{-i})) + \int_{\underline{x}_i}^{\overline{x}_i} Q^i(x_i, x_{-i}|a) \frac{\partial v(w(x_i, x_{-i}))}{\partial x_i} dx_i \qquad (15)$$

By assumption, v and its derivative are both positive. Hence, $EU(a)$ is concave in a if Q^i satisfies Sinclair-Desgagné's (1994) assumption.

Extending the arguments in Section 3, replace Q^i with its tangent line (with respect to a), Q^{iL}. As before, if the contract satisfies L-IC$_{a^*}$ then $EU^L(a|a^*) = EU(a^*)$, independently of a. The difference $EU^L(a|a^*) - EU(a)$ is determined by

$$\begin{aligned} T^{iL}(a|a^*) - T^i(a) &= \left[-Q^i(\underline{x}_i, x_{-i}|a) - \left(-Q^{iL}(\underline{x}_i, x_{-i}|a, a^*)\right)\right] v(w(\underline{x}_i, x_{-i})) \qquad (16) \\ &\quad + \int_{\underline{x}_i}^{\overline{x}_i} \left[-Q^i(x_i, x_{-i}|a) - \left(-Q^{iL}(x_i, x_{-i}|a, a^*)\right)\right] \frac{\partial v(w(x_i, x_{-i}))}{\partial x_i} dx_i. \end{aligned}$$

Sinclair-Desgagné's (1994) assumption is of course equivalent to assuming $-Q^i$ is convex in a. However, imagine instead that a^* is merely on the convex envelope of $-Q^i$ for all (x_i, x_{-i}). Then $T^{iL}(a|a^*) - T^i(a) \geq 0$ and thus $EU(a^*) - EU(a) \geq 0$. That is, L-IC$_{a^*}$ implies G-IC$_{a^*}$.

It is important to note that Sinclair-Desgagné's (1994) justification of the FOA relies on the existence of some i for which $-Q^i$ is convex in a *for all* a. However, the argument following (16) relies only on the existence of some i for which a^* – but not necessarily other actions – are on the convex envelope of $-Q^i$. In other words, different signals, or different i, can be used for different values of a^*. This is not the case in Sinclair-Desgagné (1994).

Proposition 6 *Assume utility is non-negative and that MLRP is satisfied. Assume the second best action is in* $(\underline{a}, \overline{a})$*. Then, the FOA is valid if for every* a *there is* *some i (not necessarily the same for all a) for which a is on the convex envelope of $-Q^i(x_i, x_{-i}|\cdot)$ for all (x_i, x_{-i}).*

As mentioned, the assumptions in Proposition 6 are weaker than those in Sinclair-Desgagné (1994). Conlon (2009a) suggests other assumptions that are also weaker than Sinclair-Desgagné's. However, the assumptions in Proposition 6 are different from Conlon's assumptions as well. In fact, the assumptions in Rogerson (1985), Jewitt (1988), Sinclair-Desgagné (1994), and Conlon (2009a) all imply that the agent's payoff is concave in his action. However, the assumptions in Proposition 6 does not have any such implication.

REMARK: The next version of the paper will explore the multi-signal problem in more detail, including multi-signal extensions of Jewitt's one-signal conditions.

7 Conclusion

In this paper, a new approach to the moral hazard problem has been suggested. The approach is based on reformulating the problem in terms familiar to any economist. In particular, standard results from the theory of choice under uncertainty can be invoked to prove new and old results.

Restricting attention to certain classes of contracts, the subset of actions for which L-IC implies G-IC was characterized. In special cases, this set contains all actions. Thus, a unified proof of Rogerson's (1985) and Jewitt's (1988) justifications of the FOA was provided. Indeed, the insights gained from reformulated the problem permitted a third set of sufficient conditions to be derived.

In the second part of the paper, a more specific model was considered. Though the spanning condition looks simple and dates back to Grossman and Hart (1983), the first full characterization of its solution is given here. Mirrlees' (1999) famous counterexample can also be solved used the techniques presented here.

Evidently, the FOA or the modified FOA are not always valid. Future research may uncover other environments where the binding incentive compatibility constraints can be determined and further economic insights obtained.

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Rogerson	Jewitt	Third set of conditions
$F_{aa}(x\|a) \geq 0$, $\forall x, a$ $\Downarrow$ Any nondecreasing and L-IC contract is G-IC	$\int_{\underline{x}}^{x} F_{aa}(y\|a)dy \geq 0$, $\forall x, a$ $\Downarrow$ Any nondecreasing, concave, and L-IC contract is G-IC	$\int_{\underline{x}}^{x} \int_{\underline{x}}^{z} F_{aa}(y\|a)dydz \geq 0$, $\forall x, a$ and $\int_{\underline{x}}^{\overline{x}} F_{aa}(y\|a)dy \geq 0$, $\forall a$ $\Downarrow$ Any nondecreasing, concave, positively skewed, and L-IC contract is G-IC
FOSD	SOSD	TOSD
$G(x) \leq H(x)$, $\forall x$ $\Updownarrow$ $E_G[u(x)] \geq E_H[u(x)]$ for any nondecreasing $u(x)$	$\int_{\underline{x}}^{x} G(y)dy \leq \int_{\underline{x}}^{x} H(y)dy$, $\forall x$ $\Updownarrow$ $E_G[u(x)] \geq E_H[u(x)]$ for any nondecreasing and concave $u(x)$	$\int_{\underline{x}}^{x} \int_{\underline{x}}^{z} G(y)dydz \leq \int_{\underline{x}}^{x} \int_{\underline{x}}^{z} H(y)dydz$, $\forall x$ and $\int_{\underline{x}}^{\overline{x}} G(y)dy \leq \int_{\underline{x}}^{x} H(y)dy$ $\Updownarrow$ $E_G[u(x)] \geq E_H[u(x)]$ for any nondecreasing, concave, and positively skewed $u(x)$.

Table 1: Rogerson, Jewitt and stochastic dominance.

Note: In the first row, $F(\cdot|a)$ is the distribution over outcomes given action a. Regardless of a, the support is $[\underline{x}, \overline{x}]$. It is implicitly assumed that $F_a(x|a) \leq 0$, $\forall x, a$ (whenever it is defined). In the third column, the second condition is, more formally, that $u'(x)$ is non-negative, decreasing, and convex (where $u(x)$ is utility if the outcome is x).

Rogerson	Jewitt	Third set of conditions
$F_{aa}(x\|a) \geq 0$, $\forall x, a$.	$\int_{\underline{x}}^{x} F_{aa}(y\|a)dy \geq 0$, $\forall x, a$.	$\int_{\underline{x}}^{x} \int_{\underline{x}}^{z} F_{aa}(y\|a)dydz \geq 0$, $\forall x, a$ and $\int_{\underline{x}}^{\overline{x}} F_{aa}(y\|a)dy \geq 0$, $\forall a$.
$\omega'(z) > 0$ $l_x(x\|a) \geq 0$	$\omega'(z) > 0$, $\omega''(z) \leq 0$ $l_x(x\|a) \geq 0$, $l_{xx}(x\|a) \leq 0$	$\omega'(z) > 0$, $\omega''(z) \leq 0$, $\omega'''(z) \geq 0$ $l_x(x\|a) \geq 0$, $l_{xx}(x\|a) \leq 0$, $l_{xxx}(x\|a) \geq 0$

Table 2: Justifying the first order approach.

Note: The assumptions in the first row ensure that any locally incentive compatible and $\mathcal{C}_1$, $\mathcal{C}_2$ or $\mathcal{C}_3$ contract, respectively, is globally incentive compatible. The assumptions in the second row ensure the candidate solution is a $\mathcal{C}_1$, $\mathcal{C}_2$ or $\mathcal{C}_3$ contract, respectively.