Limit Theorems for the Empirical Distribution Function of Scaled Increments of Itô Semimartingales at high frequencies

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Motivation

The standard model

$$dX_t = \alpha_t dt + \sigma_t dW_t + dY_t,$$

where

- α_t is the drift,
- W_t is Brownian motion,
- σ_t is stochastic volatility,
- Y_t is the jump component.

Motivation

High-frequency data makes possible recovering functions of realized volatility path from discrete observations of X:

- integrated volatility $\int_0^T \sigma_s^2 ds$,
- integrated functions of volatility $\int_0^t f(\sigma_s) ds$ for some smooth $f(\cdot)$,
- spot volatility σ_t^2,
- volatility occupation times $\int_0^T 1_{\{\sigma s \in A\}} ds$.

Motivation

The volatility high-frequency estimators are based on the "local Gaussianity" in X:

$$\frac{1}{\sqrt{h}}(X_{t+sh} - X_t) \xrightarrow{\mathcal{L}} \sigma_t \times (B_{t+s} - B_s), \quad \text{as } h \to 0 \text{ and } s \in [0,1],$$

where B_t is a Brownian motion and the above convergence is for the Skorokhod topology.

Local Gaussianity has two important features:

- the scaling factor of the increments is $1/\sqrt{h}$,
- the limiting distribution of the increments is Gaussian.

Main Results

- The local Gaussianity critical for the statistical/econometric work.
- The goal of the paper is to make it testable.
- We estimate locally volatility.
- We scale the high-frequency increments by the local volatility estimates.
- We derive the limiting behavior of the empirical cdf of the scaled increments when X is jump-diffusion or when it is pure-jump.
- We apply the limit theory to propose Kolmogorov-Smirnov type tests for the jump-diffusion Itô semimartingale class of models.

Outline

- Construction of the Empirical Distribution of Scaled Increments of Itô Semimartingales
- Convergence in Probability
- Feasible CLT and testing Local Gaussianity
- Monte Carlo
- Empirical Illustration

Empirical CDF of “Devolatilized” Increments

Setting:

- we observe X on the discrete grid $0, \frac{1}{n}, \frac{2}{n}, \ldots .1$ with $n \to \infty$,
- we split high-frequency observations into blocks containing k_n observations with $k_n \to \infty$ such that $k_n/n \to 0$.

To “devolatilize” increments we use a local jump-robust estimator of volatility:

$$\widehat{V}_j^n = \frac{\pi}{2}\frac{n}{k_n}\sum_{i=(j-1)k_n+2}^{jk_n} |\Delta_{i-1}^n X||\Delta_i^n X|, \quad j = 1, \ldots, \lfloor n/k_n \rfloor,$$

which is local Bipower Variation.

Note: for the behavior of our statistic in the pure-jump case it is important to use Bipower Variation.

Empirical CDF of “Devolatilized” Increments

To form the statistic we need to “filter” the “big” jumps. The total number of the remaining high-frequency observations is

$$N^n(\alpha, \varpi) = \sum_{j=1}^{\lfloor n/k_n \rfloor} \sum_{i=(j-1)k_n+1}^{(j-1)k_n+m_n} 1\left(|\Delta_i^n X| \le \alpha \sqrt{\widehat{V}_j^n} n^{-\varpi} \right),$$

where $\alpha > 0$ an $\varpi \in (0, 1/2)$ and $0 < m_n < k_n$.

Note: the truncation depends on the local volatility estimator.

Empirical CDF of "Devolatilized" Increments

The empirical CDF of the "devolatilized" and truncated increments is

$$\widehat{F}^n(\tau) = \frac{1}{N^n(\alpha,\varpi)} \sum_{j=1}^{\lfloor n/k_n \rfloor} \sum_{i=(j-1)k_n+1}^{(j-1)k_n+m_n} 1\left\{ \frac{\sqrt{n}\Delta_i^n X}{\sqrt{\widehat{V}_j^n}} 1_{\left\{|\Delta_i^n X| \le \alpha \sqrt{\widehat{V}_j^n} n^{-\varpi}\right\}} \le \tau \right\}.$$

Limit Behavior when X is Jump-Diffusion

We have under some regularity conditions:

$$\widehat{F}_n(\tau) \xrightarrow{\mathbb{P}} \Phi(\tau), \quad \text{as } n \to \infty,$$

where the above convergence is uniform in τ over compact subsets of $(-\infty, 0) \cup (0, +\infty)$ and $\Phi(\tau)$ is the cdf of a standard normal variable.

Limit Behavior when X is Pure-Jump

A more general setting for X is the following model

$$dX_t = \alpha_t dt + \sigma_{t-} dS_t + dY_t,$$

where α_t, σ_t and Y_t are processes with càdlàg paths adapted to the filtration and Y_t is of pure-jump type. S_t is a stable process with a characteristic function given by

$$\log\left[\mathbb{E}(e^{iuS_t})\right] = -t|cu|^{\beta}\left(1 - i\gamma \mathrm{sign}(u)\Phi\right), \quad \Phi = \begin{cases} \tan(\pi\beta/2) & \text{if } \beta \neq 1, \\ -\frac{2}{\pi}\log|u|, & \text{if } \beta = 1, \end{cases}$$

where $\beta \in (0, 2]$ and $\gamma \in [-1, 1]$.

Limit Behavior when X is Pure-Jump

- When $\beta = 2$, above model is the standard jump-diffusion.
- When $\beta < 2$, the above model is of pure-jump type with "locally stable" jumps.

Local Gaussianity generalizes to local Stability:

$$h^{-1/\beta}(X_{t+sh} - X_t) \xrightarrow{\mathcal{L}} \sigma_t \times (S'_{t+s} - S'_t), \quad \text{as } h \to 0 \text{ and } s \in [0, 1],$$

for every t and where S'_t is a Lévy process identically distributed to S_t.

Note:

- the different scaling factor,
- and the different limiting distribution.

Limit Behavior when X is Pure-Jump

What happens with $\widehat{F}_n(\tau)$ in the pure-jump setting?

Recall the scaled "devolatilized" increments are

$$\frac{\sqrt{n}\Delta_i^n X}{\sqrt{\widehat{V}_j^n}} = \frac{n^{1/\beta}\Delta_i^n X}{\sqrt{n^{2/\beta-1}\widehat{V}_j^n}},$$

and $\sqrt{n^{2/\beta-1}\widehat{V}_j^n}$ is a consistent estimator for σ_t in the pure-jump setting.

Limit Behavior when X is Pure-Jump

Under some regularity conditions we have if $\beta \in (1, 2]$

$$\widehat{F}_n(\tau) \xrightarrow{\mathbb{P}} F_\beta(\tau), \quad \text{as } n \to \infty,$$

where the above convergence is uniform in τ over compact subsets of $(-\infty, 0) \cup (0, +\infty)$; $F_\beta(\tau)$ is the cdf of $\sqrt{\frac{2}{\pi}\frac{S_1}{\mathbb{E}|S_1|}}$ (S_1 is the value of the β-stable process S_t at time 1) and $F_2(\tau)$ equals the cdf of a standard normal variable $\Phi(\tau)$.

Note:

- $F_\beta(\tau)$ corresponds to the cdf of a random variable Z with $\mathbb{E}|Z| = \sqrt{\frac{2}{\pi}}$,
- $\Longrightarrow$ the difference between $\beta < 2$ and $\beta = 2$ will be in the relative probability assigned to "big" versus "small" values of τ.

Limit Behavior when X is Itô semimartingale + Noise

What happens if X (either jump-diffusion or pure-jump) is contaminated with noise:

$$X^*_{\frac{i}{n}} = X_{\frac{i}{n}} + \epsilon_{\frac{i}{n}},$$

where $\left\{\epsilon_{\frac{i}{n}}\right\}_{i=1,\dots,n}$ are i.i.d. random variables defined on a product extension of the original probability space and independent from $\mathcal{F}$ and we further assume $\mathbb{E}|\epsilon_{\frac{i}{n}}|^{1+\iota} < \infty$ for some $\iota > 0$.

Limit Behavior when X is Itô semimartingale + Noise

What happens with $\widehat{F}_n(\tau)$ in the noisy setting?

Recall the scaled "devolatilized" increments are

$$\frac{\sqrt{n}\Delta_i^n X^*}{\sqrt{\widehat{V}_j^n}} = \frac{\Delta_i^n X^*}{\sqrt{n^{-1}\widehat{V}_j^n}},$$

and n^{-1} is the correct scaling factor that ensures $\widehat{V}_j^n$ converges to a non-degenerate limit.

Limit Behavior when X is Itô semimartingale + Noise

Under certain regularity conditions we have

$$\widehat{F}_n(\tau) \xrightarrow{\mathbb{P}} F_\epsilon(\tau), \quad \text{as } n \to \infty,$$

where the above convergence is uniform in τ over compact subsets of $(-\infty, 0) \cup (0, +\infty)$ and

- we denote $\mu = \sqrt{\frac{\pi}{2}} \sqrt{\mathbb{E}\left(|\epsilon_{\frac{i}{n}} - \epsilon_{\frac{i-1}{n}}| |\epsilon_{\frac{i-1}{n}} - \epsilon_{\frac{i-2}{n}}| \right)}$,
- $F_\epsilon(\tau)$ is the cdf of $\frac{1}{\mu}\left(\epsilon_{\frac{i}{n}} - \epsilon_{\frac{i-1}{n}}\right)$.

Limit Behavior when X is Itô semimartingale + Noise

If $\epsilon_{\frac{i}{n}}$ is normally distributed then

$$\frac{\sqrt{n}\Delta_i^n X^*}{\sqrt{\widehat{V}_j^n}} \approx N(0, \sigma^2),$$

where

$$\sigma^2 = \frac{2}{\frac{\pi}{2}\mathbb{E}\left(|\xi_1 + \xi_2||\xi_2 + \xi_3|\right)},$$

with ξ_1, ξ_2 and ξ_3 independent standard normals.

Note: $\sigma^2 < 1$.

CLT when X is jump-diffusion

Theorem 1. *Let X_t be jump-diffusion satisfying some regularity conditions. Further, let the block size grow at the rate*

$$\frac{m_n}{k_n} \to 0, \quad k_n \sim n^q, \quad \text{for some } q \in (0, 1/2), \text{ when } n \to \infty.$$

We then have locally uniformly in subsets of $(-\infty, 0) \cup (0, +\infty)$

$$\widehat{F}_n(\tau) - \Phi(\tau) = \widehat{Z}_1^n(\tau) + \widehat{Z}_2^n(\tau) + \frac{1}{k_n}\frac{\tau^2\Phi''(\tau) - \tau\Phi'(\tau)}{8}\left(\left(\frac{\pi}{2}\right)^2 + \pi - 3\right) + o_p\left(\frac{1}{k_n}\right),$$

CLT when X is jump-diffusion

with the pair $(\widehat{Z}_1^n(\tau),\ \widehat{Z}_2^n(\tau))$ *having the following limit behavior*

$$\left(\sqrt{\lfloor n/k_n \rfloor m_n}\widehat{Z}_1^n(\tau) \quad \sqrt{\lfloor n/k_n \rfloor k_n}\widehat{Z}_2^n(\tau)\right) \quad \stackrel{\mathcal{L}}{\longrightarrow} \quad (Z_1(\tau) \quad Z_2(\tau))\,,$$

where $\Phi(\tau)$ *is the cdf of a standard normal variable and* $Z_1(\tau)$ *and* $Z_2(\tau)$ *are two independent Gaussian processes with covariance functions*

$$\mathit{Cov}\,(Z_1(\tau_1), Z_1(\tau_2)) = \Phi(\tau_1 \wedge \tau_2) - \Phi(\tau_1)\Phi(\tau_2),$$

$$\mathit{Cov}\,(Z_2(\tau_1), Z_2(\tau_2)) = \left[\frac{\tau_1\Phi'(\tau_1)}{2}\frac{\tau_2\Phi'(\tau_2)}{2}\right]\left(\left(\frac{\pi}{2}\right)^2 + \pi - 3\right), \quad \tau_1, \tau_2 \in \mathbb{R} \setminus 0.$$

CLT when X is jump-diffusion

Comments:

- $Z_1(\tau)$ is the standard Brownian bridge appearing in the Donsker theorem for empirical processes
- $Z_2(\tau)$ is due to the estimation of the local scale σ_t via $\widehat{V}_j^n$
- the third component in $\widehat{F}_n(\tau) - \Phi(\tau)$ is asymptotic bias
- picking the rate of growth of m_n and k_n arbitrary close to $\sqrt{n}$, we can make the rate of convergence of $\widehat{F}^n(\tau)$ arbitrary close to $\sqrt{n}$
- asymptotic bias and variances are constant $\Longrightarrow$ feasible inference is straightforward
- $\sqrt{n}$ rate is in general not possible because of the presence of the drift term in X

Kolmogorov Smirnov test

The critical region of our proposed test is given by

$$C_n = \left\{ \sup_{\tau \in \mathcal{A}} \sqrt{N^n(\alpha, \varpi)} |\widehat{F}^n(\tau) - \Phi(\tau)| > q_n(\alpha, \mathcal{A}) \right\}$$

where recall $\Phi(\tau)$ denotes the cdf of a standard normal random variable, $\alpha \in (0,1)$, $\mathcal{A} \in \mathbb{R} \setminus 0$ is a finite union of compact sets with positive Lebesgue measure, and $q_n(\alpha, \mathcal{A})$ is the $(1-\alpha)$-quantile of

$$\sup_{\tau \in \mathcal{A}} \left| Z_1(\tau) + \sqrt{\frac{m_n}{k_n}} Z_2(\tau) + \sqrt{\frac{m_n}{k_n}} \frac{\sqrt{n}}{k_n} \frac{\tau^2 \Phi''(\tau) - \tau \Phi'(\tau)}{8} \left(\left(\frac{\pi}{2} \right)^2 + \pi - 3 \right) \right|,$$

with $Z_1(\tau)$ and $Z_2(\tau)$ being the Gaussian processes defined in the Theorem.

Kolmogorov Smirnov test

We have

$$\lim_n \mathbb{P}(C_n) = \alpha, \quad \text{if } \beta = 2 \ \text{ and } \ \liminf_n \mathbb{P}(C_n) = 1, \quad \text{if } \beta \in (1, 2).$$

Monte Carlo

We test performance on the following models:

- Jump-Diffusion Model

$$dX_t = \sqrt{V_t}dW_t + \int_{\mathbb{R}} x\mu(ds, dx), \quad dV_t = 0.03(1.0 - V_t)dt + 0.1\sqrt{V_t}dB_t,$$

  where (W_t, B_t) is a vector of Brownian motions with $\mathrm{corr}(W_t, B_t) = -0.5$ and μ is a homogenous Poisson measure with compensator $\nu(dt, dx) = dt \otimes \frac{0.25e^{-|x|/0.4472}}{0.4472}dx$ which corresponds to double exponential jump process with intensity of 0.5.

- Pure-Jump Model

$$X_t = S_{T_t}, \quad \text{with} \quad T_t = \int_0^t V_s ds,$$

  where S_t is a symmetric tempered stable martingale with Lévy measure $\frac{0.1089e^{-|x|}}{|x|^{1+1.8}}$ and V_t is the square-root diffusion given above.

Monte Carlo

Tuning parameters:

- time span: 252 days
- frequency: $n = 100$ and $n = 200$ corresponding to 5-min and 2-min sampling
- $\lfloor n/k_n \rfloor$ in the range $1 - 3$ blocks per day
- $\lfloor m_n/k_n \rfloor = 0.75$ for $n = 100$ and $\lfloor m_n/k_n \rfloor = 0.70$ for $n = 200$
- $\alpha = 3$ and $\varpi = 0.49$ for jump cutoff

Monte Carlo

Table 1: Monte Carlo Results for Jump-Diffusion Model

Nominal Size	**Rejection Rate**		
	Sampling Frequency $n = 100$		
	$k_n = 33$	$k_n = 50$	$k_n = 100$
$\alpha = 1\%$	4.1	0.4	2.2
$\alpha = 5\%$	15.4	3.6	10.6
	Sampling Frequency $n = 200$		
	$k_n = 67$	$k_n = 100$	$k_n = 200$
$\alpha = 1\%$	1.2	2.0	7.8
$\alpha = 5\%$	4.4	7.1	26.2

Note: For the cases with $n = 100$ we set $\lfloor m_n/k_n \rfloor = 0.75$ and for the cases with $n = 200$ we set $\lfloor m_n/k_n \rfloor = 0.70$.

Monte Carlo

Table 2: Monte Carlo Results for Pure-Jump Model

Nominal Size	Rejection Rate		
	Sampling Frequency $n = 100$		
	$k_n = 33$	$k_n = 50$	$k_n = 100$
$\alpha = 1\%$	27.5	87.4	99.9
$\alpha = 5\%$	66.6	97.5	100.0
	Sampling Frequency $n = 200$		
	$k_n = 67$	$k_n = 100$	$k_n = 200$
$\alpha = 1\%$	100.0	100.0	100.0
$\alpha = 5\%$	100.0	100.0	100.0

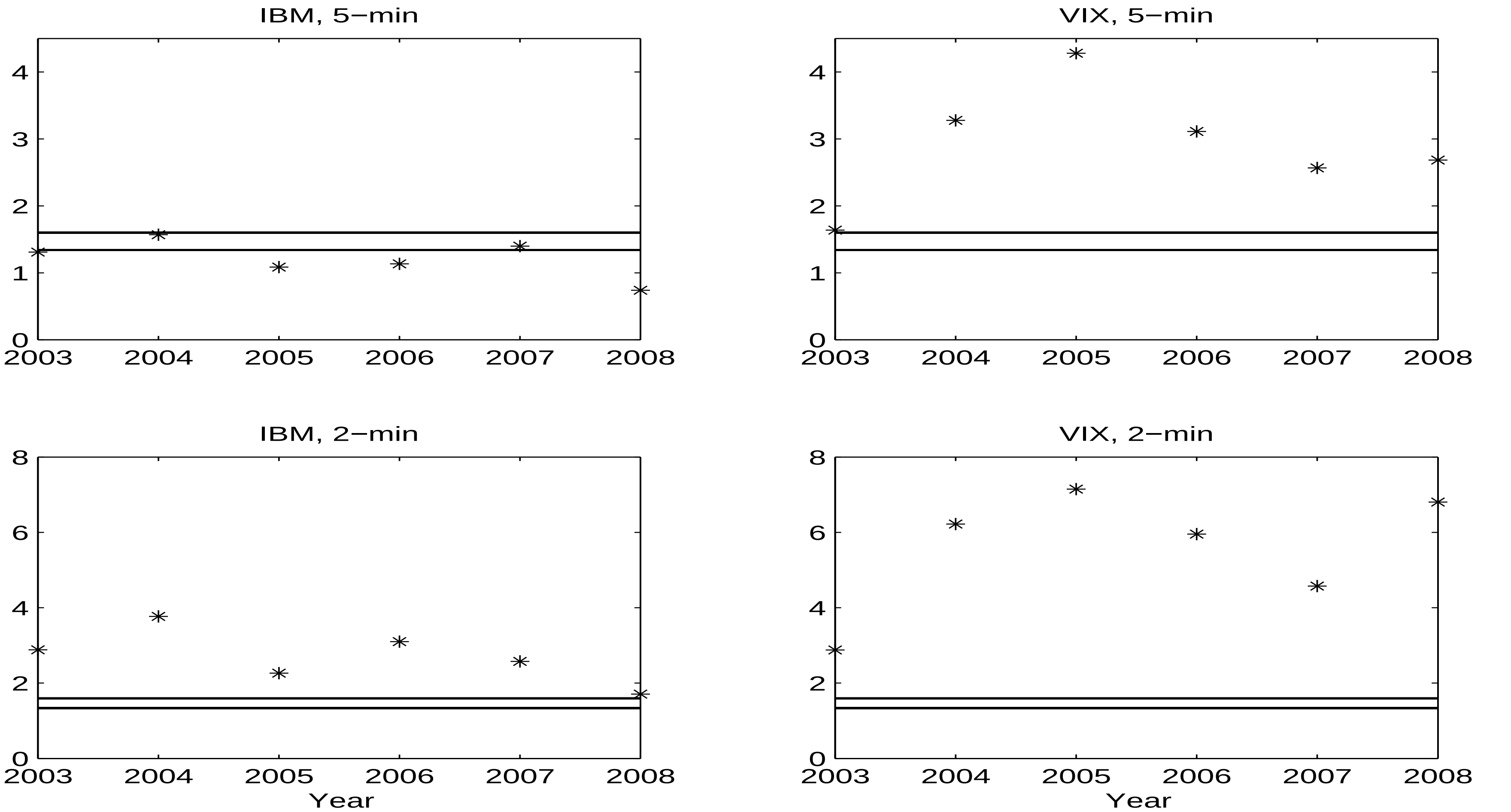
Note: For the cases with $n = 100$ we set $\lfloor m_n/k_n \rfloor = 0.75$ and for the cases with $n = 200$ we set $\lfloor m_n/k_n \rfloor = 0.70$.

Empirical Application-I

- We use two data sets: IBM stock price and the VIX volatility index, sample period 2003-2008,
- test is performed for each of the years in the sample,
- we perform test at 5-minute and 2-minute frequencies,
- $\lfloor n/k_n \rfloor = 2$ for the five-minute sampling frequency and $\lfloor n/k_n \rfloor = 3$ for the two-minute frequency,
- the range for the KS test is

$$\mathcal{A} = [Q(0.01) : Q(0.40)] \ \cup \ [Q(0.60) : Q(0.99)],$$

where $Q(\alpha)$ is the α-quantile of standard normal.


IBM, 5−min
VIX, 5−min
IBM, 2−min
VIX, 2−min
0
1
2
3
4
6
8
2003
2004
2005
2006
2007
2008
Year

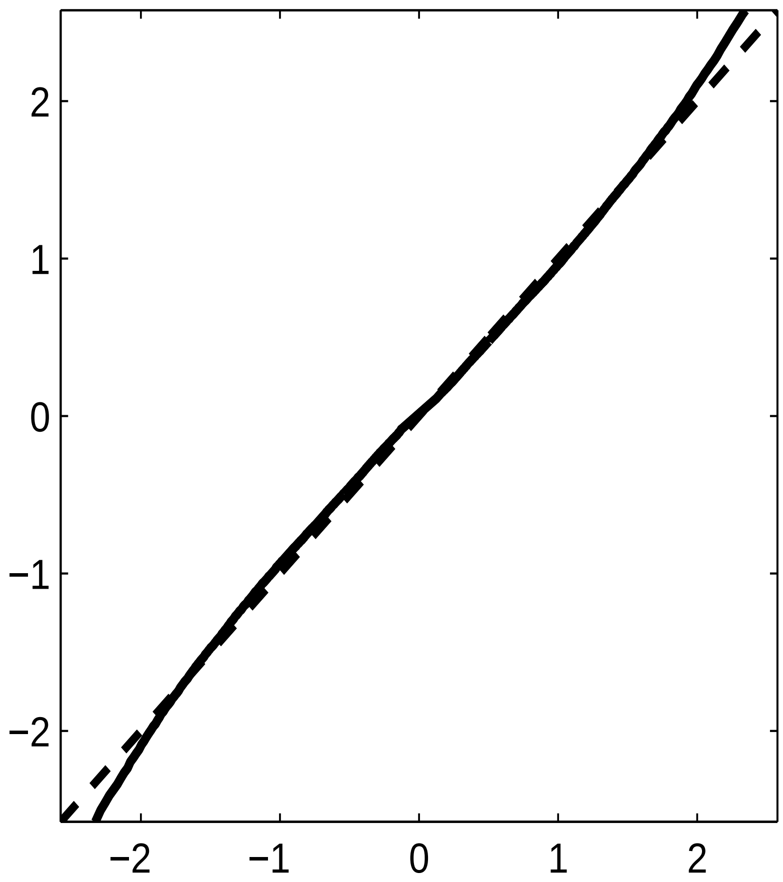
Empirical Application-II

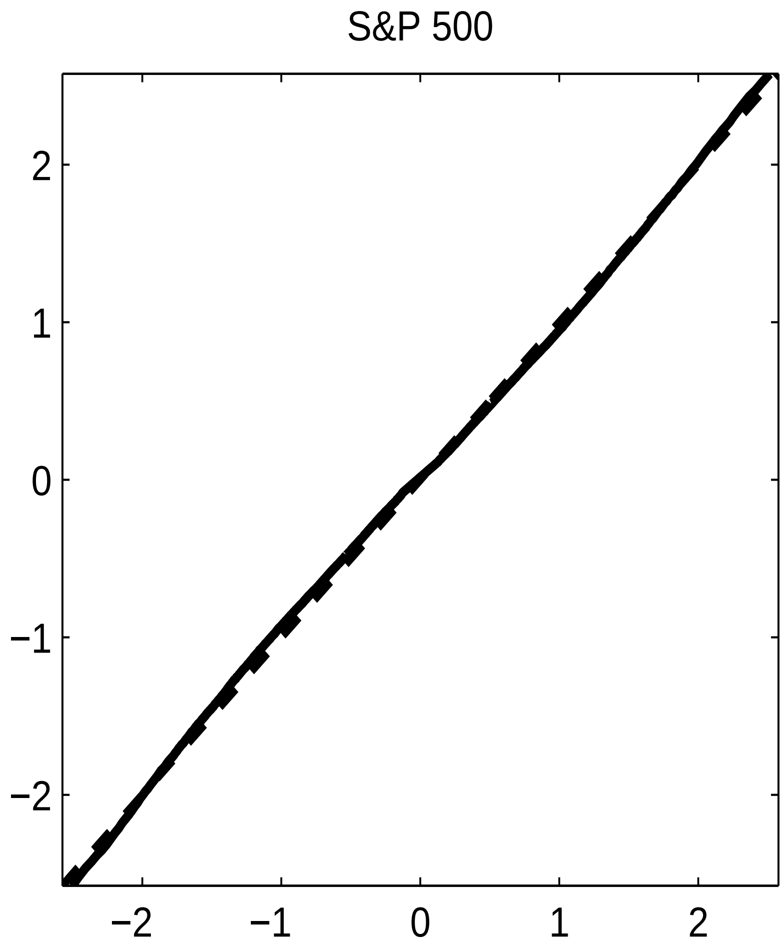
- S&P index, 5-min 2007–2012
- VIX futures prices also 2007–2012

Examine Q-Q plots before and after truncating large jumps.

Examine Q-Q plots for stable-like prices

QQ-Plot S&P 500, raw and truncated for large jumps

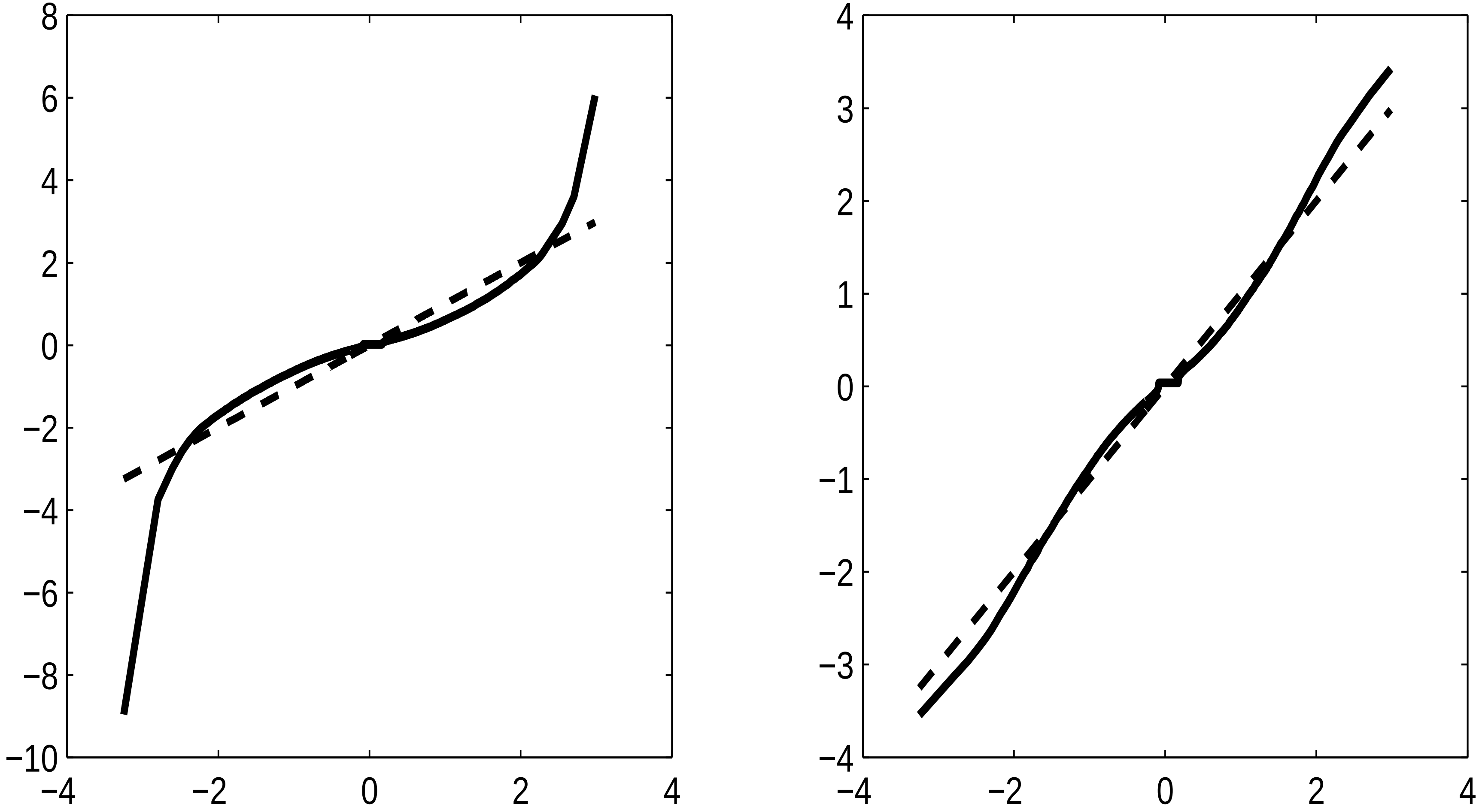




VIX Futures

Things are not nearly as clear-cut with these data.

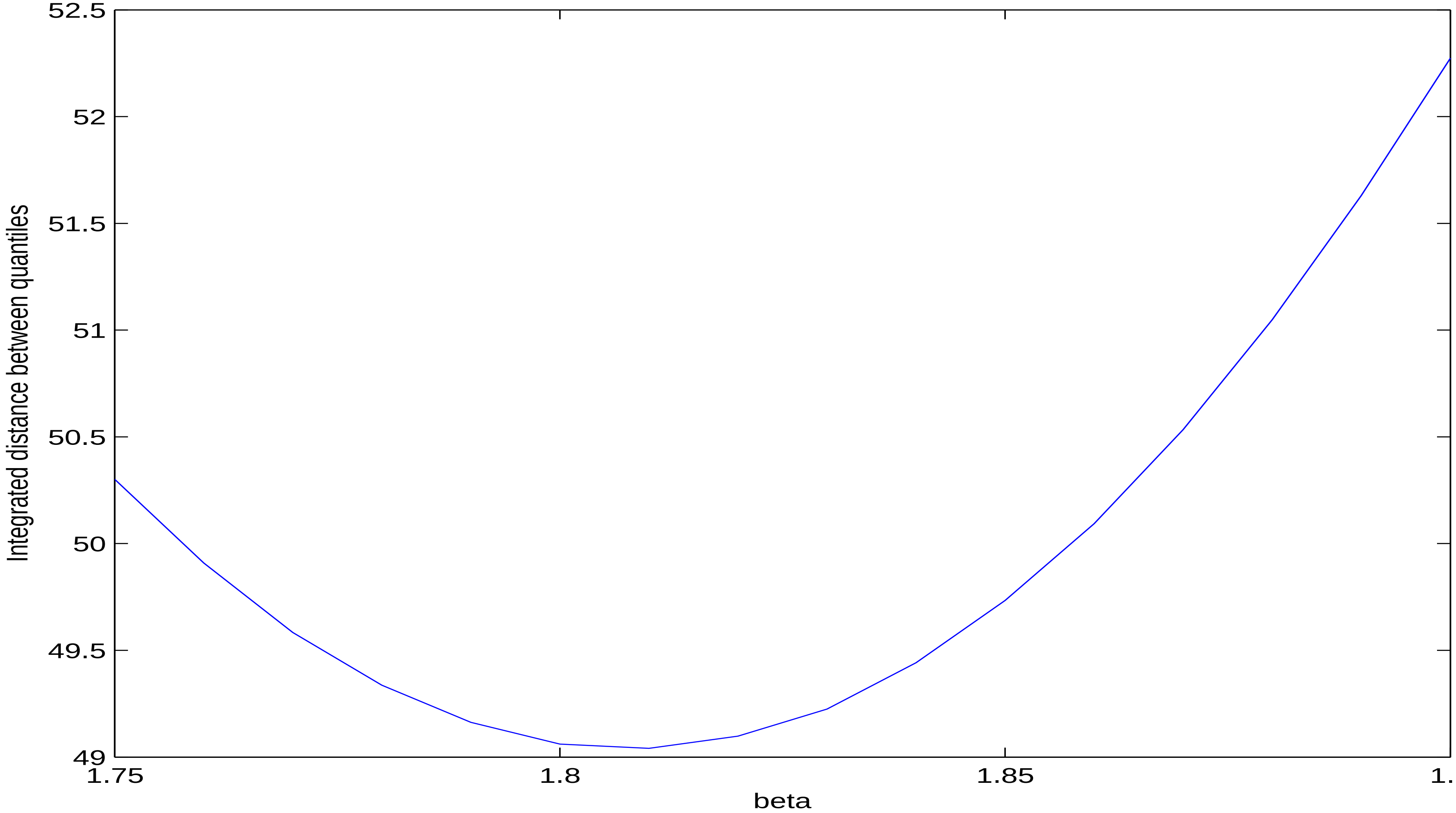
QQ-Plot VIX futures, raw and truncated for large jumps



VIX futures: Determination of the index β

We need to know the activity index β in order to get a reference stable distribution. We minimize the mean squared difference of OBS-PRED for Q-Q plot data over a grid of β.

Objective Function

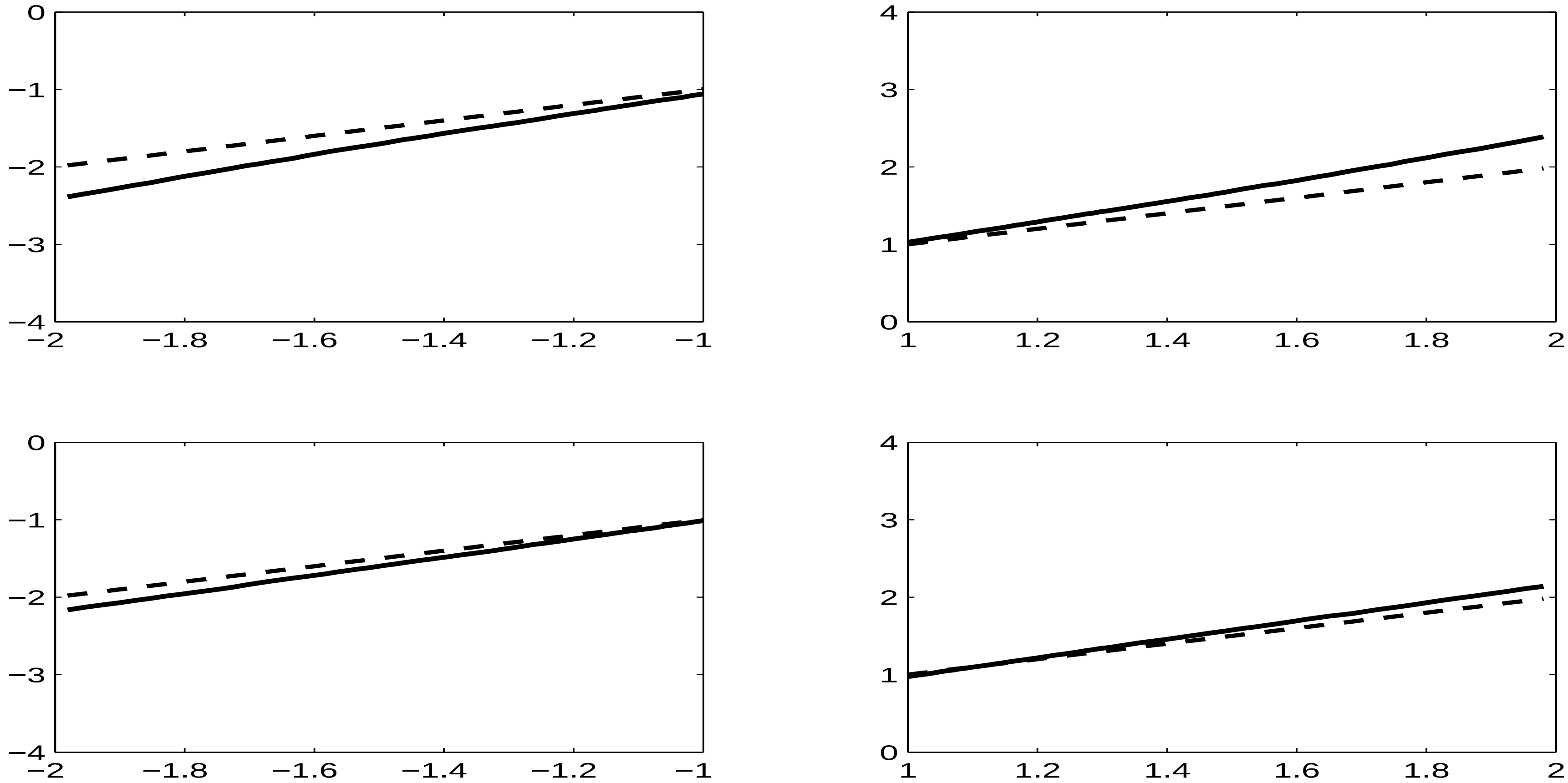


Final Q-Q Plots

We look at Q-Q plots using the Gaussian distribution as the reference and then using the stable($\hat{\beta} = 1.82\ldots$) as the reference distribution.

We now look at the left and right tails of Q-Q plots:

Left and right sides of QQ-Plots of unscaled and scaled VIX futures vs stable$(\hat{\beta})$



Conclude

- Can test the core distributional assumption of financial modeling
- Useful for examining risk premiums of jumps of different size.
- Potentially very relevant to regulators who monitor markets to identify unusual trading patterns.
- Other multivariate applications in progress,