SUBJECTIVE INDEPENDENCE AND CONCAVE EXPECTED UTILITY

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Abstract. We take a different approach to independence and assume that decision makers exhibit ambiguity neutrality when they find alternatives to be similar. Similarity is subjective and can be determined through preferences' property we refer to as *subjective codecomposable independence*. This axiom characterizes a large class of non-additive expected utility models resorting to the decision maker's betting behavior and a general integration scheme. The new approach allows us to: (a) introduce the *Concave Expected Utility* model of decision making, adhering to ambiguity aversion where uncertainty is captured through a non-additive belief; and (b) provide sufficient conditions, weaker than those employed by previous formulations hinging on the independence axiom, to subjective and Choquet expected utility models.

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1. Introduction

In the heart of decision theory underlies Savage's [16] and Anscombe and Aumann's [1] Subjective Expected Utility (*SEU*) theory. Among other standard axioms, Savage's theory hinges on the sure thing principle, while that of Anscombe–Aumann's on the independence axiom. *SEU* theory states that a decision maker entertains a subjective prior probability and uncertain alternatives (i.e., acts) are ranked according to their expected utility. Starting with Ellsberg [5], an abundance of thought and lab experiments suggest that in the presence of subjective uncertainty, termed ambiguity, individuals may be unable to entertain a prior belief and hence violate the sure thing principle and the independence axiom. The main observation is that individuals are not ambiguity neutral and exhibit preferences for hedging, a phenomenon that is termed *ambiguity aversion* (see Schmeidler [17] and Gilboa and Schmeidler [8]). Schmeidler [17] proposed to weaken the independence axiom and assumed that it applies only for *comonotonic* acts.[1] He argued that comonotonic acts are structurally similar to one another and hence there should not be a strict preference for hedging. By weakening Anscombe and Aumann's independence axiom to comonotonic-independence, Schmeidler presented Choquet Expected Utility (*CEU*) theory; ambiguity is captured through a subjective non-additive probability and alternatives are ranked according to their expected utility which is calculated by the Choquet integral.

We take a different approach to independence. The main idea is the following. Any act can be represented (or, decomposed) as a coin toss (a mixture) between betting on an event and a 'complementary' act. Typically, there are different possibilities to represent an act this way. The independence axiom we employ requires that any act can be decomposed in such a way that the decision maker exhibits ambiguity neutrality.[2] Note that according to this axiom, ambiguity neutrality is required to hold for

[1]Two functions over a state space are comonotonic if when ordering the states according to their associated outcomes, both functions induce the same ordering.

[2]More formally, any act can be decomposed in such a way that the decision maker exhibits ambiguity neutrality among acts represented by the same bet and complementary act. For brevity, we use the former wording throughout the Introduction.

a particular representation and not for all of them. Also, this particular decomposition depends on the decision maker and typically differs from one decision maker to another. We therefore refer to this axiom as *subjective codecomposable independence*.

Our approach, which hinges on codecomposable independence, gives rise to a large class of non-additive expected utility preferences. These preferences are represented by a subjective non-additive probability, capturing the decision maker's belief (or, betting behavior) over the state space, and a general integration scheme according to which expected utility is calculated. This class preferences contains *CEU*, but also other economically meaningful families of preferences. To show that we impose the classical ambiguity aversion axiom. It turns out that in this case, preferences can be represented by a concave integral (Lehrer [9]). These are referred to as *Concave Expected Utility* (*CavEU*) preferences.

How to motivate our codecomposable independence approach beyond the results it yields? First, codecomposable independence axioms can take different shapes, providing new ways of characterizing existing models. Thus, this approach sheds new light on known theories. For instance, given an act one could postulate that ambiguity neutrality applies to decompositions that involve bets that are comonotonic with the act. Such an axiom is sufficient to characterize the *CEU* model. Another version of codecomposable independence axiom yields the *SEU* model. This version requires that ambiguity neutrality would hold not only for a particular sub-class of decompositions, but to any decomposition to a bet and a complementary act.

This brings us to the second, and perhaps the more important, motivation for subjective codecomposable independence. Comonotonic independence (Schmeidler [17]) suggests that comonotonic acts are structurally 'similar', and therefore it is reasonable to assume that a decision maker will not have a strict preference for hedging over such acts. This argument obviously applies to the comonotonic version of our codecomposable independence axiom; a decision maker has no choice but to be ambiguity neutral when a bet and an act are comonotonic. However, we find this assumption rather strong; a decision maker might not find this particular structural similarity sufficient to imply ambiguity neutrality. Subjective codecomposable independence allows a decision maker the freedom to choose what similarity means and for which type of decompositions to exhibit ambiguity neutrality. Thus, ambiguity attitude towards different decompositions is completely subjective and may differ from one decision maker to another.

Among the class preferences we introduce, the ambiguity averse preferences, *CavEU*, are more flexible than *CEU* preferences in the sense that the acts among which ambiguity neutrality applies are subjectively determined and are not dictated by pre-specified structural similarity. Due to this flexibility, *CavEU* preferences are less vulnerable than *CEU* ones to 'paradoxes' such as those introduced by Machina [15].

To summarize, the contribution of the current paper is threefold. First, we introduce a subjective codecomposable independence axiom that allows us to characterize a general class of event-separable preferences. For such preferences, ambiguity is captured through a non-additive probability and expected utility is determined according to a general integration scheme (not necessarily the Choquet integral). Second, this approach allows us to introduce a model of decision making that always respects ambiguity aversion where uncertainty is captured through a non-additive probability. Lastly, it provides sufficient conditions, which are weaker than the previous formulations adhering to the independence axioms, to subjective and Choquet expected utility models.

The rest of the paper is organized as follows. The next section provides an informal discussion regarding Choquet expected utility, concave expected utility and some of the differences between the two approaches. The formal framework of choice under uncertainty is presented and the basic axioms are formulated in Section 3.1. Subjective codecomposablity and the emergence of a capacity are presented in Sections 3.2 and 3.3. Ambiguity aversion and *CavEU* preferences are discussed in Section 4.1 where a short literature review appears in Section 4.2. Section 4.3 presents a recent paradox for *CEU* preferences raised by Machina [15] and shows how *CavEU* accommodates the paradox. Lastly, the relation of codecomposablity to *SEU* and *CEU* is presented in Section 5. All the proofs are in the appendix.

2. Choquet and Concave Expected Utility

This section provides an informal discussion and (partial) comparison between the classic Choquet integral and the alternative theories considered in this paper. The aim of this section is to present different decision theoretic concepts, relevant to the point the current paper is trying to make, to a reader who is not familiar with this literature. The formal study appears in the following sections.

Assume that the underlying domain of alternatives is the collection of (non-negative) utility acts, or random variables[3] given a state space $S = \{s_1, ..., s_n\}$. A *capacity* v over the state space is a function that assigns a number to each event in a monotonic fashion (with respect to inclusion). We interpret the capacity $v(E)$ of an event E as how likely E is with respect to v. A finite collection $(a_i, E_i)_i$, where a_i is a positive real number and E_i is an event,[4] is a *decomposition* of an act g if $\sum_i a_i \mathbb{1}_{E_i} = g$. That is, g can be decomposed to the collection of simpler functions of the form $a_i \mathbb{1}_{E_i}$. Similarly to the Lebesgue integral, the value of a decomposition $(a_i, E_i)_i$ with respect to a capacity v is simply $\sum_i a_i v(E_i)$.

For an act g, permute the state space by $\pi : \{1, ..., n\} \to \{1, ..., n\}$ such that $\pi(i) \geq \pi(j)$ if $g(s_i) \geq g(s_j)$. That is, $g(s_{\pi^{-1}(i)})$ is increasing with i. The Choquet integral of g with respect to a capacity v is

$$(1) \quad \int^C gdv = g(s_{\pi^{-1}(1)})v(\{s_{\pi^{-1}(1)}, ..., s_{\pi^{-1}(n)}\})+ \\ + \sum_{i=2}^{n} (g(s_{\pi^{-1}(i)}) - g(s_{\pi^{-1}(i-1)})v(\{s_{\pi^{-1}(i)}, ..., s_{\pi^{-1}(n)}\}).$$

From the right hand side of Eq. 1, the Choquet integral of g with respect to v is the value of a particular decomposition of g of the form $\left(g(s_{\pi^{-1}(1)}), \{s_{\pi^{-1}(1)}, ..., s_{\pi^{-1}(n)}\}\right)$, $\left(g(s_{\pi^{-1}(2)}) - g(s_{\pi^{-1}(1)}), \{s_{\pi^{-1}(2)}, ..., s_{\pi^{-1}(n)}\}\right)$, ..., $\left(g(s_{\pi^{-1}(n)}) - g(s_{\pi^{-1}(n-1)}), \{s_{\pi^{-1}(n)}\}\right)$. We refer to such a decomposition as the *Choquet decomposition*. Preferences $\succeq$ over the domain discussed are *CEU* if they can be represented by the Choquet integral with

[3]That is, we assume for the sake of simplicity that the vNM utility index was already identified.

[4]$\mathbb{1}_E$ is the indicator function of the event E.

respect to a capacity. That is there exists a capacity v such that $g \succeq h$ if and only if $\int^C gdv \geq \int^C hdv$.

Clearly, every alternative has more than one decomposition. There are numerous ways to evaluate an act, and resorting to the Choquet decomposition is only one of them. Consider the alternative in which an act is evaluated according to the maximum value over *all* of its decompositions. Such valuation mechanism is referred to as the *concave integral*, where the concave integral of g with respect to a capacity v will be denoted by $\int^{Cav} gdv$. We will refer to preferences that can be represented by the concave integral as *CavEU*.

To illustrate how *CavEU* may be different than *CEU*, consider the following example. Let the state space be $S = \{s_1, ..., s_4\}$ and define a capacity v over the state space as follows: $v(s) = \frac{1}{12}$ for every state s, $v(\{s_1, s_2\}) = v(\{s_1, s_3\}) = v(\{s_2, s_3\}) = v(\{s_1, s_4\}) = \frac{1}{6}$, $v(\{s_2, s_4\}) = v(\{s_3, s_4\}) = \frac{3}{12}$, $v(\{s_1, s_2, s_3\}) = v(\{s_1, s_3, s_4\}) = v(\{s_2, s_3, s_4\}) = \frac{1}{3}$, $v(\{s_1, s_2, s_4\}) = \frac{5}{6}$ and $v(S) = 1$. Note that the contribution of the state s_2 to any event that contains neither s_1 nor s_2 is greater than the contribution of s_1. Formally, for any event E that does not contain the states s_1, s_2, $v(E \cup \{s_1\}) \leq v(E \cup \{s_2\})$. Moreover, the inequality is strict when $E = \{s_4\}$. In this sense, under the belief v the state s_2 is more likely than s_1.

Now, consider the random variables $f = (0, 1, 2, 3)$ and $g = (1, 0, 2, 3)$. Note that f and g differ only in states s_1 and s_2. f assigns the lower outcome to the less likely state and the higher outcome to the more likely one. It is the opposite case for g; it assigns the higher outcome to the less likely state. It is plausible that preferences based on the capacity v would rank f over g. Nevertheless, the Choquet integral of both f and g is $\frac{8}{12}$: $\int^C fdv = v(\{s_2, s_3, s_4\}) + v(\{s_3, s_4\}) + v(\{s_4\}) = v(\{s_1, s_3, s_4\}) + v(\{s_3, s_4\}) + v(\{s_4\}) = \int^C gdv$. That is, *CEU* preferences represented by the capacity v rank f and g indifferent. However, *CavEU* preferences rank f strictly preferred to g: $\int^{Cav} fdv = v(\{s_2, s_4\}) + 2v(\{s_3, s_4\}) = \frac{9}{12} > \frac{8}{12} = v(\{s_1, s_3, s_4\}) + v(\{s_3, s_4\}) + v(\{s_4\}) = \int^{Cav} gdv$.

The capacity v above can be presented as $v(E) = \min_i p_i(E)$ for every event E, where $p_1 = (\frac{1}{12}, \frac{1}{12}, \frac{1}{6}, \frac{2}{3}), p_2 = (\frac{1}{12}, \frac{2}{3}, \frac{1}{12}, \frac{1}{6}), p_3 = (\frac{1}{12}, \frac{2}{3}, \frac{1}{6}, \frac{1}{12})$ and $p_4 = (\frac{2}{3}, \frac{1}{12}, \frac{1}{12}, \frac{1}{6})$. That is the capacity, as a modeling tool of perception of ambiguity, displays pessimism. In this case it is natural to assume that the decision maker will exhibit ambiguity aversion.

Nevertheless, the Choquet integral with respect to this capacity does not exhibit such aversion.[5] On the other hand, the concave integral does. As will be formally shown, the example above is a generic one in the sense that the capacity representing any *CavEU* preferences can always be constructed as the minimum of measures over the state space. In contrast, the Choquet integral with respect to such capacities typically does not exhibit ambiguity aversion.

3. Subjective Decomposability

3.1. **Environment.** Consider a decision making framework in which an object of choice is an act from the state space to utility outcomes. More formally, let S be a finite non–empty set of *states of nature*. An act[6] is a function from S to $\mathbb{R}_+$. The collection of acts is denoted by $\mathcal{F}$ with typical elements being f, g, h. Abusing notation, for an act $f \in \mathcal{F}$ and a state $s \in S$, we denote by $f(s)$ the constant act that assigns the utility $f(s)$ to every state of nature. Utils (and constant acts) will be typically denoted by a, b, c. Mixtures (convex combinations) of acts are performed pointwise. That is, if $f, g \in \mathcal{F}$ and $\delta \in [0,1]$, then $\delta f + (1-\delta)g$ is the act in $\mathcal{F}$ that yields $\delta f(s) + (1-\delta)g(s)$ utility for every $s \in S$. Mixtures coefficients will be denoted by δ, α, etc.

In our framework, a decision maker is associated with a binary relation $\succeq$ over $\mathcal{F}$ representing his ranking. $\succ$ is the asymmetric part of the relation. That is $f \succ g$ if $f \succeq g$ but it is not true that $g \succeq f$. $\sim$ is the symmetric part, that is $f \sim g$ if $f \succeq g$ and $g \succeq f$.

We interpret $f(s)$ as the payoff induced by act $f \in \mathcal{F}$ in state $s \in S$ and assume it is the utility exerted by the decision maker if f is chosen and s is the realized state. That is, we assume that the vNM utility function of the decision maker has already been identified.[7]

[5]The capacity v as defined in the example is not a convex one. According to Schmeidler [17] the Choquet integral with respect to v does not adhere to ambiguity aversion.

[6]See Remark 1 for a discussion on the unboundedness assumption.

[7]One can also consider the restatement by Fishburn [7] of the classical Anscombe-Aumann [1] set-up. In that case, standard axioms imply that the vNM utility index can be identified and that the formulation of alternatives as utility acts, as we do here, is well defined. Such results have been established in many papers and we here rely on these results for convenience and brevity.

A binary relation $\succeq$ is *reflexive* if $f \sim f$ for every act f. $\succeq$ is *complete* if for every $f, g \in \mathcal{F}$, either $f \succeq g$ or $g \succeq f$. It is *transitive* if for $f, g, h \in \mathcal{F}$, $f \succeq g$ and $g \succeq h$ imply $f \succeq h$. The following is a list of assumptions (axioms) regarding a binary relation $\succeq$ over acts. We will postulate these assumption throughout.

Preference. $\succeq$ is complete and transitive.

Monotonicity. For every $f, g \in \mathcal{F}$, $f(s) \geq g(s)$ for all $s \in S$ implies $f \succeq g$.

Continuity. For every $f \in \mathcal{F}$ the sets $\{g \in \mathcal{F} : g \succeq f\}$ and $\{g \in \mathcal{F} : g \preceq f\}$ are closed.

3.2. **Codecomposable Independence.** A *bet* is an act that yields some utility $b \in \mathbb{R}_+$ over an event $E \subseteq S$ and the utility 0 over the complement event. Such a bet will be denoted by b_E. An act which is not a bet can always be represented as a convex combination, or a decomposition, of some bet and another act. That is, for $f \in \mathcal{F}$ we can find a bet b_E, an act f', and $\delta \in [0,1]$ such that $f = \delta b_E + (1-\delta)f'$. Clearly, there are many decompositions of this sort for an act.

Pick one decomposition of f, say, to the bet b_E and the complementary act f'. In particular, $f \in [b_E, f'] = \{\alpha b_E + (1-\alpha)f' : \alpha \in [0,1]\}$. Now, every act $g \in [b_E, f']$ can be decomposed, similarly to f, to the bet b_E and the act f'. In that sense one may say that the acts f and g, and any other act in $[b_E, f']$, have a "similar" structure. Thus, it is reasonable to assume, as in Schmeidler [17], that a decision maker will not have a strict preference for hedging (or, ambiguity neutrality) between such acts. However, different decision makers may perceive "similarity" in different ways; while one decision maker may show ambiguity neutrality when mixing between f and g due to their similarity, another decision maker will not find f and g similar and exhibit other ambiguity attitudes for such mixtures. Our main axiom, *Subjective Codecomposable Independence*, postulates that a "similarity" structure exists but is subjective: every act can be decomposed to some bet and a complementary act such that, the decision maker exhibits ambiguity neutrality across all acts that can be decomposed in a similar way. Formally:

Subjective Codecomposable Independence. For every non-bet act f, there exist a bet b_E and f' such that $f \in [b_E, f']$ and $\succeq$ satisfies independence[8] over $[b_E, f']$.

The next axiom can be considered as complementary to the main axiom presented above, which does not have any bite when the act under consideration is a bet. In this case we assume that different bets on the same event are considered as similar.

Bet Independence. $\succeq$ satisfies independence over $[0, b_E]$ for every bet b_E.

3.3. **A Capacity Emerges.** To explore the implications of *subjective codecomposable independence* we need to present some notations and definitions. A *capacity* v over S is a function $v : 2^S \to [0,1]$ satisfying: (i) $v(\phi) = 0$ and $v(S) = 1$; and (ii) $K \subseteq T \subseteq S$ implies $v(K) \leq v(T)$.

Definition 1. *We say that a binary relation $\succeq$ over all acts $\mathcal{F}$ admits a* decomposition representation *if there exist a functional $V : \mathcal{F} \to \mathbb{R}$ and a capacity $v : 2^S \to [0,1]$ such that:*

1. V represents $\succeq$, that is $V(f) \geq V(g) \iff f \succeq g$ for every $f, g \in \mathcal{F}$;

2. $V(b_E) = b \cdot v(E)$ for every bet b_E; and

3. for every act $f \in \mathcal{F}$,

$$V(f) = \sum a_E v(E) \text{ for some } \sum a_E \mathbb{1}_E = f,$$

where $a_E \geq 0$.

A collection $\{(a_E, E) : E \subseteq S, a_E \geq 0\}$ is a *decomposition* of f if $\sum a_E \mathbb{1}_E = f$. Given a capacity v over events, the value of such a decomposition is $\sum a_E v(E)$. Thus, a binary relation admits a decomposition representation if an act is ranked according to the value, with respect to the capacity, of one of its decompositions into bets.

Theorem 1. *Let $\succeq$ be a binary relation over $\mathcal{F}$ satisfying* preferences, monotonicity, continuity, bet independence *and* subjective codecomposable independence*. Then, $\succeq$ admits a decomposition representation. Moreover, if both (V, v) and (V', v') represent $\succeq$, then $V = V'$ and $v = v'$.*

[8]We say that a binary relation $\succeq$ satisfies independence over a collection of acts $\mathcal{F}' \subseteq \mathcal{F}$ if for every $f, g, h \in \mathcal{F}'$ and every $\delta \in (0,1)$, $f \succeq g$ if and only if $\delta f + (1-\delta)h \succeq \delta g + (1-\delta)h$.

Theorem 1 states that given standard assumptions, *bet independence* and *subjective codecomposable independence*, a binary relation admits a decomposition representation. The axioms are sufficient to identify a unique non-additive belief and to state that alternatives are ranked according to the value of one of their decompositions.[9]

Another way to think about the representation (or, the axioms) is the following. In the eyes of the decision maker, every act is equivalent to a lottery over different bets on different events. A bet is being evaluated according to the (non-additive) belief; similar to *SEU* theory, the act itself is being evaluated as the expected value, with respect to the lottery, of all such bets. Unlike *SEU* and *CEU*, different decision makers may consider different decompositions (as lotteries over bets) for the same alternative.

Note that *subjective codecomposable independence* is a weak assumption; it is not possible to determine exactly what is the decomposition according to which an alternative is ranked. A question is whether making stronger behavioral assumptions can help identify the belief (capacity) and the integration mechanisms, and whether such integration mechanisms are natural and interesting while being different than Choquet? We investigate this direction in the following section.

Remark 1. *We assume that the vN–M utility is unbounded from above.* Subjective codecomposable independence *is stronger when the utility is bounded. It might be impossible to obtain some decompositions for an act since the complementary act to a particular bet may require levels of utility that are not specified (or identified) by the decision maker's preferences. That implies more structure on the capacity or the integration mechanism. To exemplify, assume that the vN-M utility is bounded by* $[0, 1]$ *and consider the utility act* $f = (0.5, 1)$ *over the state space* $\{s_1, s_2\}$*. Considering the bet* $1\!\!1_{s_1}$ *then* f *can be decomposed as* $f = 0.5 \cdot 1\!\!1_{\{s_1\}} + 0.5 \cdot 2 \cdot 1\!\!1_{\{s_2\}}$*, however the complementary act, in this case* $2 \cdot 1\!\!1_{\{s_2\}}$*, is not in the realm of the vN-M utility.This discussion is continued in Remark 2 below when ambiguity aversion is introduced.*

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[9]Note that, unlike the customary representation result, Theorem 1 provides only sufficient conditions in order to obtain a decomposition representation. However, assuming that the representing functional V satisfies standard properties as continuity and monotonicity, and in addition a "subjective additivity" property in the line of our main axiom, it is possible to show that the axioms we present are also necessary. We believe that the interesting part of the result is the one presented.

4. Ambiguity Aversion

4.1. **Concave Expected Utility.** Since Schmeidler [17] and Gilboa and Schmeidler [8] ambiguity aversion has been one of the most studied phenomenon in the theory of decision making. Unlike Schmeidler [17] who focused on comonotonic-independence, we here wish to impose ambiguity aversion while assuming only *subjective codecomposable independence.*

Ambiguity Aversion. For every $f, g \in \mathcal{F}$, if $f \sim g$ then $\delta f + (1-\delta)g \succeq g$ for every $\delta \in [0,1]$.

In addition we impose more structure by strengthening the *bet-independence* axiom to allow for independence across all mixtures with the worst outcome.[10]

Worst-Outcome Independence. $\succeq$ satisfies independence over $[0, f]$ for every act f.

Lehrer [10] presented an integration scheme for capacities based on concavity: the *concave integral* of an act $f : S \to \mathbb{R}_+$ with respect to a capacity v is defined by

$$\int^{Cav} f dv = \max\left\{\sum a_E v(E) : \sum a_E \mathbb{1}_E = f, a_E > 0\right\}. \tag{2}$$

The integral evaluates an act according to the decomposition with the maximal value (with respect to the capacity) among all the act's decompositions. We refer to preferences $\succeq$ over all acts $\mathcal{F}$ as *CavEU* if there exist a capacity $v : 2^S \to [0,1]$, such that for all $f, g \in \mathcal{F}$

$$f \succeq g \iff \int^{Cav} f dv \geq \int^{Cav} g dv.$$

The following result states that along with the standard assumptions, if preferences satisfy *worst-outcome independence*, *subjective codecomposable independence* and *ambiguity aversion*, then it must be that these preferences are *CavEU*.

Theorem 2. *Let $\succeq$ be a binary relation over $\mathcal{F}$. Then the following are equivalent:*
1. $\succeq$ are preferences *that satisfy* monotonicity, continuity, worst-outcome independence, subjective codecomposable independence *and* ambiguity aversion*; and*
2. $\succeq$ is CavEU.

[10] Similar versions of this axiom appeared in Chateauneuf and Faro [4] and Cerreia et al. [3].

Note that the concave integral is a particular instance of a decomposition representation presented in Definition 1. To see this, we need to make sure that condition 2 in the definition is satisfied. Clearly, the concave integral is well defined given any capacity v , but it may be the case that there exists an event $E \subseteq S$ such that $\int^{Cav} \mathbb{1}_E dv > v(E)$. Nevertheless, letting $\hat{v}$ be defined by $\hat{v}(E) = \int^{Cav} \mathbb{1}_E dv$ for every event E solves the problem. This is due to the fact that (see Lehrer and Teper [12]) $\int^{Cav} f dv = \int^{Cav} f d\hat{v}$ for every act f. In other words, $\hat{v}$ represents the same preferences as v, where unlike v, the capacity $\hat{v}$ satisfies condition 2 in Definition 1.

We know, due to Theorem 1, that the belief (or, capacity) representing the preferences is unique. However, the Theorem allows for very general preferences and there is not much more that can be said about such beliefs. Since now we have restricted attention to preferences that adhere to *ambiguity aversion*, it is possible to identify a particular structure for the decision maker's beliefs.

Proposition 1. *v can represent a* CavEU *preference relation (in the sense of Definition 1) if and only if v can be written as a minimum of finitely many measures over S (that is, $v = \min_i \mu_i$).*

The proposition states that *CavEU* preferences can be represented by a belief which, as a modeling tool of perception of ambiguity, displays pessimism. It is reasonable to assume that a decision maker entertaining such beliefs will exhibit ambiguity aversion. Nevertheless, other integration mechanisms, such as the Choquet integral, with respect to pessimism exhibiting capacities do not display such aversion.[11]

We conclude this section by a complementary comment to Remark 1 above.

Remark 2. *Consider the least monotonic capacity v over the state space $S = \{s_1, s_2, s_3, s_4, s_5, s_6\}$ that satisfies $v(s_1, s_4, s_5) = v(s_2, s_5, s_6) = v(s_3, s_4, s_6) = \frac{2}{3}$ and $v(S) = 1$. Note that $v(E) \leq \frac{2}{3}$ whenever $E \neq S$. It is easy to verify that v is totally balanced. Consider the act $f = (0.5, 0.5, 0.5, 1, 1, 1)$. It can be decomposed as $f = \frac{1}{2}\mathbb{1}_{\{s_1,s_4,s_5\}} + \frac{1}{2}\mathbb{1}_{\{s_2,s_5,s_6\}} + \frac{1}{2}\mathbb{1}_{\{s_3,s_4,s_6\}}$, which implies that $\int^{Cav} f dv \geq 3 \cdot \frac{1}{2} \cdot \frac{2}{3} = 1$. Let the utility be bounded in $[0, 1]$ and assume, adhering to* subjective codecomposeable

[11]For *CEU* preferneces, for example, it holds as such capacities are typically not convex; see Schmeidler [17].

independence*, that* $\int^{Cav} f dv = \sum a_E v(E)$ *for some decomposition* $f = \sum a_E \mathbb{1}_E$*, where* $\sum a_E = 1$*. Thus,* $\sum a_E v(E) = \sum_{E \neq S} a_E v(E) + a_S v(S) \leq \frac{2}{3} \sum_{E \neq S} a_E + a_S = \frac{2}{3}(1 - a_S) + a_S \leq \frac{2}{3} \cdot \frac{1}{2} + \frac{1}{2} = \frac{5}{6} < \int^{Cav} f dv$ *(the second inequality is due to that* a_S *cannot exceed* $\frac{1}{2}$ *because* $f = \sum a_E \mathbb{1}_E$*). This is a contradiction. Hence, it cannot be that the capacity* v *represents preferences that adhere to* subjective codecomposeable independence *and* ambiguity aversion *when the utility is bounded. This illustrates how* subjective codecomposeable independence *entails more structure on preferences when utility is bounded relative to unbounded utility. Note that the capacity specified above is non-convex. Indeed, when the concave integral is taken with respect to a convex capacity, the resulting preferences must satisfy* subjective codecomposeable independence *and* ambiguity aversion. *It appears, however, that convexity is not necessary. In a separate note we show (see Lehrer and Teper* [13]*) that a weaker property than convexity, the Sandwich property, is necessary and sufficient for the representation of* CavEU *preferences when utility is bounded.*

4.2. **How Does It Fit in the Lit?** There are numerous models of choice under uncertainty. The most related ones are *confidence preferences* presented by Chateauneuf and Faro [4], maxmin expected utility (*MEU*) that were axiomatized by Gilboa and Schmeidler [8] and, of course, *CEU* preferences.

CavEU is clearly a particular case of confidence preferences, but require more structure since not every confidence preferences satisfy the decomposability property. [12] To see that, consider *MEU* preferences, which are a particular case of confidence preferences. Not every *MEU* preference relation can be represented as a (concave) integral; *MEU* satisfies translation covariance (due to the c-independence axiom) while it is clear from *subjective codecomposable independence* that it does not have to be satisfied by *CavEU*. The subclass of *CavEU* preferences that do admit an *MEU* representation are those that can be represented with a capacity having a *large core* (see, Lehrer [10]).[13] This brings us to *CEU* preferences. Schmeidler [17] shows that the Choquet

[12]In particular, *CavEU* are those confidence preferences with finitely many extreme points in the support of the confidence function.

[13]The definition of large core is due to Sharkey [18].

integral is a concave one if and only if the capacity is *convex*. Hence we have that when the capacity is not convex *CavEU* and *CEU* differ. In addition, due to Lehrer [10] and Teper and Lehrer [12], *CEU* and *CavEU* coincide if and only if the capacity representing the preferences is *convex* (and in this case it is also *MEU*).

The latter point emphasizes that given ambiguity aversion, the class of *CavEU* preferences is more general than that of *CEU*. Below we illustrate this point by discussing an example by Machina [15] and showing that *CavEU* can explain behavior that may lead to a "paradox" for *CEU* (and also for *MEU* as shown by Baillon, L'Hardion and Placido [14]).

4.3. **On an Example by Machina.** Machina [15] in a recent paper "exploits" structural independence (in particular, tail separability, as he refers to it) exhibited by *CEU* preferneces and constructs several examples, in the spirit of Ellsberg, in which such preferences can not accommodate choices that may be considered natural. This has been reinforced by Baillon, L'Hardion and Placido [14] who showed that a large number of subjects exhibit such choices. As discussed in the Introduction, *CavEU* are more flexible than *CEU* preferences in the sense that event-separability is subjective and is not pre-specified structurally. This is why *CavEU* preferences are less vulnerable than *CEU* ones to such 'paradoxes'.

One of Machina's examples is the following. Consider an urn containing 100 balls, each marked with a number from 1 through 4. All you know is that there are 50 balls that are marked either 1 or 2, and 50 balls that are marked either 3 or 4. You are being offered a pair of bets f and g, as described in Table 1, that depend on a draw of one ball from the urn.[14] In addition you are being offered another pair of bets h and k that depend as well on a draw of one ball from the urn.

Machina notices that by tail separability, as he refers to it, *CEU* maximizer prefers f to g if and only if she prefers h to k. From the tables above it is clear that acts h and k are obtained from f and g by a pair of common-outcome tail shifts; *CEU* preferences

[14]Even though the analysis would go through if entries are monetary, we consider utiles for brevity and simplicity.

Table 1. The Reflection Example

Bet	s_1	s_2	s_3	s_4
f	0	200	100	100
g	0	100	200	100
h	100	200	100	0
k	100	100	200	0

cannot explain a "reversal" such as the preference of f over g and at the same time the preference of k over h.

CavEU preferences can accommodate the reversal of preferences indicated by Machina.[15] If $v(s_2) > v(s_3) = 0$, $v(s_2, s_3, s_4) > v(s_2, s_3) + v(s_3, s_4)$ and $v(s_2, s_4) = v(s_2)$ then $\int^{Cav} f dv = v(s_2, s_3, s_4) + v(s_2)$, $\int^{Cav} g dv = v(s_2, s_3, s_4) + v(s_3)$ and $\int^{Cav} f dv > \int^{Cav} g dv$. On the other hand, if in addition $v(s_1, s_2, s_3) - v(s_2, s_3) > v(s_1, s_2) - v(s_2)$ and $v(s_1, s_2, s_3) < v(s_1, s_3) + v(s_2, s_3)$ then $\int^{Cav} h dv = v(s_1, s_2, s_3) + v(s_2)$, $\int^{Cav} k dv = v(s_1, s_3) + v(s_2, s_3)$ and $\int^{Cav} k dv > \int^{Cav} h dv$.

5. Codecomposable Independence and Expected Utility Models

It is interesting to see the links between the codecomposable independence approach to existing models. Clearly, both *SEU* and *CEU* are particular classes of preferences admitting a decomposition representation. In what follows, we provide stronger versions of our independence axiom that will yield exactly *SEU* and *CEU*.

Fix an act f. Recall that *subjective codecomposable independence* states that independence holds over at least one interval $[b_E, f']$ that contains f. The following axiom postulates that for every such decomposition to b_E and f', the preference relation satisfies independence over $[b_E, f']$.

[15]In an unpublished manuscript, Lehrer [11] shows that the concave integral can accommodate the other reversal, that is, the preference of g over f and that of h over k. L'Hardion and Placido [14] find that the more common reversal is the one discussed in the discussion above. Baillon, L'Haridon, and Placido [2] rely on the particular example in Lehrer's note and claim that *CavEU* preferences cannot accommodate the common reversal. The example above shows that this is in fact inaccurate.

Codecomposable Independence. For every bet b_E and act f', $\succeq$ satisfies independence over $[b_E, f']$.

Assuming *codecomposable independence* along with the axioms specified above allows us to formulate the following result.

Proposition 2. *The following two statements are equivalent:*
1. $\succeq$ *satisfies* preference, continuity, monotonicity *and* codecomposable independence*;*
2. $\succeq$ *admits an* SEU *representation.*

Proposition 2 states that given the standard axioms, *codecomposable independence* allows us to identify a subjective probability with respect to which the decision maker calculates the expected utility of the different alternatives and ranks them accordingly. Note that *bet independence* is no longer needed as it is implied by *codecomposable independence.*

For an act f and a utility level $a \in \mathbb{R}_+$, let $E_a^f = \{s \in S : f(s) \geq a\}$ be the event in which f performs better that a. We refer to such an event as a *cumulative* event for f. When considering a cumulative event for an act f, we may ignore the utility level at times and write E^f. A weaker codecomposable independence axiom can be formulated taking into account only decomposition of acts to bets over (respectively) cumulative events.

Cumulative Codecomposable Independence. For every act f, bet b_{E^f} and f' such that $f \in [b_{E^f}, f']$, $\succeq$ satisfies independence over $[b_{E^f}, f']$.

The axiom postulates that if $f, g, h \in \mathcal{F}$ can all be decomposed to a bet b_{E^f} and an f', then independence involving f, g, h holds. Note that, in this case, f, g and h are comonotonic. Resulting from such weakening of codecomposable independence is the following proposition. Again, *bet independence* is implied by *cumulative codecomposable independence.*

Proposition 3. *The following two statements are equivalent:*
1. $\succeq$ *satisfies* preference, continuity, monotonicity *and* cumulative codecomposable independence*;*
2. $\succeq$ *admits a* CEU *representation.*

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Appendix: Proofs

Proof of Theorem 1. preferences and *continuity* imply that $\succeq$ admits a (continuous) representation $V : \mathbb{R}^S_+ \to \mathbb{R}$. That is, for every $f, g \in \mathcal{F}$, $f \succeq g$ if and only if $V(f) \geq V(g)$.

Now, pick a non-bet act $f \in \mathcal{F}$. *Subjective codecomposable independence* implies that there exist an event $E_1 \subseteq S$ and an f_1 such that $f = \delta b_{E_1} + (1-\delta) f_1$, V is affine on the interval spanned by b_{E_1} and f_1, and in particular $V(f) = \delta V(b_{E_1}) + (1-\delta)V(f_1)$. Let δ^*_1 and b^{*1} such that their product is maximized across all pairs of δ and b that satisfy the latter equalities. Such a maximum exists due to *continuity*. Let f^*_1 be the act in the decomposition of f corresponding to such δ^*_1 and b^{*1}. If f^*_1 is not a bet, the process above can be repeated and f^*_1 can be represented as well by $f^*_1 = \delta^*_2 b^{*2}_{E_2} + (1-\delta^*_2) f^*_2$, where V is affine on the interval spanned by $b^{*2}_{E_2}$ and f^*_2, and in particular $V(f^*_1) = \delta^*_2 V(b^{*2}_{E_2}) + (1-\delta^*_2)V(f^*_2)$. This process can be repeated and in the nth step, if f^*_{n-1} is not a bet, we get that there exist an E_n, δ^*_n, f^*_n such that $f^*_{n-1} = \delta^*_n b^{*n}_{E_n} + (1-\delta^*_n) f^*_n$ where V is affine over the interval spanned by $b^{*n}_{E_n}$ and f^*_n. Now, due to maximality of the $\delta^*_j b^{*j}$'s and the fact that V is affine along the path of decompositions, it cannot be the case that there are $k \neq j$ such that $E_k = E_j$. Hence, since the state space is finite, the procedure above must be of finite m iterations.[16]

We obtained that $f = \delta^*_1 b^{*1}_{E_1} + (1-\delta^*_1)\delta^*_{E_2} b^{*2}_{E_2} + \cdots + (1-\delta^*_1)\cdots(1-\delta^*_{m-1})\delta^*_m b^{*m}_{E_m} + (1-\delta^*_1)\cdots(1-\delta^*_m) b^* m + 1_{E_{m+1}}$ and $V(f) = \delta^*_1 V(b^{*1}_{E_1}) + (1-\delta^*_1)\delta^*_{E_2} V(b^{*2}_{E_2}) + \cdots + (1-\delta^*_1)\cdots(1-\delta^*_{m-1})\delta^*_m V(b^{*m}_{E_m}) + (1-\delta^*_1)\cdots(1-\delta^*_m) V(b^{*m+1}_{E_{m+1}}) = \delta^*_1 b^{*1} V(\mathbb{1}_{E_1}) + (1-\delta^*_1)\delta^*_{E_2} b^{*2} V(\mathbb{1}_{E_2}) + \cdots + (1-\delta^*_1)\cdots(1-\delta^*_{m-1})\delta^*_m b^{*m} V(\mathbb{1}_{E_m}) + (1-\delta^*_1)\cdots(1-\delta^*_m) b^{*m+1} V(\mathbb{1}_{E_{m+1}})$, where the last equality is due to homogeneity of V which is a result of *betcome independence*. Defining a set function $v : 2^S \to [0,1]$ by $v(E) = V(\mathbb{1}_E)$, we have that $V(f) = \sum a_E v(E)$ for some $\sum a_E \mathbb{1}_E = f$. Now, v is a capacity. Indeed, due to homogeneity of V we have that $v(\emptyset) = 0$, $v(S)$ can be normalized to 1 without loss of generality, and v is monotonic since V is monotone.

[16]An alternative proof would be to look at a sequence of decompositions to bets as long the complementary act is non 0. At the end of a transfinite decomposition process we do obtain the act 0. Since the state space is finite and the image of an act is bounded, the coefficients of a particular event can be summed up. This summation yields the event's coefficient in the decomposition of the particular act.

As for the uniqueness of the representation assume that there are two $V, V'\mathcal{F} \to \mathbb{R}$ that represent $\succeq$. Without loss of generality assume that $V(f) < V'(f)$ for some $f \in \mathcal{F}$. Let $b \in \mathbb{R}_+$ such that $V(E) < V(b_S) = b = V'(b_S) < V'(E)$. This contradicts the assumption that both V and V' represent $\succeq$. Thus $V = V'$ and by definition $v = v'$.

Proof of Theorem 2. The concave integral satisfies *subjective codecomposable independence* due to Proposition 5 in Even and Lehrer [6] (and it is immediate that the rest of the axioms are implied by integral).

Following Proposition 1, *worst-outcome independence* and *continuity* we have that $\succeq$ is represented by a homogeneous and continuous V such that $V(\mathbb{1}_E) \geq v(E)$. *Ambiguity aversion* implies that V is a concave functional. By Lemma 1 in Lehrer [10] we have that $V(\cdot) \geq \int^{Cav}(\cdot)dv$. However, for every $f \in \mathcal{F}$ concavity of V implies that $V(u(f)) \leq \sum \alpha_E V(\mathbb{1}_E)$ for all decompositions of f, implying that $V(u(f)) \leq \int^{Cav} u(f)dv$. Therefore $V(\cdot) = \int^{Cav}(\cdot)dv$.

Proof of Proposition 1. Due to Lemma 1 in Lehrer and Teper [12], without loss of generality we can assume that v is totally balanced. Now, let $\tilde{v}$ be a different totally balanced capacity and assume that $\tilde{v}$ represents $\succeq$. Since v and $\tilde{v}$ are different there exist $E \subseteq S$ such that without loss of generality $v(E) < \tilde{v}(E)$. Let $b \in \mathbb{R}_+$ such that $v(E) < b < v'(E)$. Then we have that $\int^{Cav} \mathbb{1}_E dv < \int^{Cav} b dv$ and $\int^{Cav} \mathbb{1}_E d\tilde{v} > \int^{Cav} b d\tilde{v}$, which contradicts the assumption that $\tilde{v}$ represents $\succeq$. Thus, 1 is proven. 2 is due to 1 and Theorem 1 in Kalai and Zemel [9].

Proof of Proposition 3. It is clear that the axioms are satisfied by the *CEU* preferences. As for the other implication, all that is needed to show is that given *cumulative codecomposable independence* the decomposition of any act obtained in the proof of Theorem 1 is the Choquet one.

To see that pick an act $f \in \mathcal{F}$ and, let $a_1 = \max\{f(s) : s \in S\}$ and $E_1 = \{s \in S : f(s) = a_1\}$. Also denote $a_2 = \max\{f(s) : s \in E_1^c\}$. Let f' be the act defined by $f'(s) = f(s)$ whenever $s \in E_1^c$ and a_2 otherwise (that is, f' coincides with f over the complement of E_1, and over E_1 it is defined as the second highest value f attains). Now, $f = f' + (a_1 - a_2)\mathbb{1}_{E_1} = \frac{a_2}{a_1}(\frac{a_1}{a_2}f') + \frac{a_1 - a_2}{a_1}(a_{1E_1})$. Note that E_1

is cumulative to f, hence by *cumulative codecomposable independence* we have that $V(f) = \frac{a_2}{a_1}V\left(\frac{a_1}{a_2}f'\right) + \frac{a_1-a_2}{a_1}V\left(a_{1E_1}\right) = V(f') + (a_1-a_2)V(\mathbb{1}_{E_1}) = V(f') + (a_1-a_2)v(E_1)$. Repeating the same procedure to f' we get that the desired result.

Proof of Proposition 2. It is clear that the axioms are satisfied by the EU preferences. As for the other implication, all that is needed to show is that given *codecomposable independence* the capacity obtained in the proof of Theorem 1 is additive, hence a probability.

Pick any event $E \subset S$ and state $s \in S \setminus E$ and consider an act of the form $f = 2\mathbb{1}_{\{s\}} + \mathbb{1}_E$. On one hand, from the proof of Proposition 3 we know that $V(f) = v(E \cup \{s\}) + v(\{s\})$. On the other hand, we can write $f = \frac{1}{2}(4\mathbb{1}_{\{s\}}) + \frac{1}{2}(2\mathbb{1}_E)$ and due to *codecomposable independence* we have that $V(f) = \frac{1}{2}(4\mathbb{1}_{\{s\}}) + \frac{1}{2}(2\mathbb{1}_E) = 2v(\{s\}) + v(E)$. Thus, $v(\{s\} + v(E \cup \{s\}) = 2v(\{s\}) + v(E)$, implying that $v(\{s\}) + v(E) = v(E \cup \{s\})$. Since E is an arbitrary event, we get that $v(F) = \sum_{s \in F} v(s)$ for any event $F \subset S$, implying that v is a probability over S.