Remarks on the Utilitarian Approach to Optimal Taxation

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September 29, 2013

Abstract

We show using elementary arguments that the utilitarian approach to optimal taxation leads to results that are very sensitive to the underlying formulation of the problem. Furthermore, we present a legitimate optimal taxation problem that admits a solution of a different form than normally believed. Specifically, in this paper we provide a proof that the optimal marginal tax rates can be equal to zero for all but a measure zero set of income levels. In other words, we argue within the utilitarian framework that it is possible that the optimal tax function, if it exists, assumes the form of a step function. As a byproduct, we show that finding optimal tax schedules can occasionally be in fact much simpler than normally thought, but at the same time we question, given the sensitivity of solutions, the applicability of the approach as a basis for practical policy guidance.

Key Words: Optimal Taxation, Step Function, Zero Marginal Taxes

JEL: H21

1 Introduction

We present a series of novel and yet elementary arguments that cast doubt on the relevance of the utilitarian approach to optimal taxation as proposed by Mirrlees [7]. Specifically, we argue that the utilitarian approach is not well grounded in economic theory, and, consequently, leads to ambiguous and very sensitive results and, thus, cannot be used as a basis for policy recommendation. In particular, we identify

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and present optimal taxation problems that admit solutions of a different form than normally believed. Specifically, we show that the solutions can assume effectively the form of a step function implying zero marginal income tax rates for all except for a measure zero set of income levels.

We are not the first to criticize the approach of Mirrlees. Historically, researchers resented the approach on fundamental grounds that ranking of utilities and interpersonal comparisons of marginal utilities were meaningless. Subsequently, Ebert [3], and Lollivier and Rochet [5] noted technical shortcomings of the approach of Mirrlees and constructively showed that the first order conditions given by Mirrless are occasionally insufficient and fail to identify the true optimal tax function. More recently Mankiw and Weinzierl [6], and Weinzierl [17] argue that Mirrlees's methodology when taken seriously warrants policy prescriptions that stand at odds with numerous notions of fairness. Furthermore, in a new contribution Saez and Stantcheva [12] argue that the utilitarian approach is unnecessarily limiting and propose a modification that leaves discretion with regard to the social marginal weights. In this paper, we augment the existing criticisms with new and simple arguments and argue that the utilitarian approach despite its analytical elegance is characterized by internal inconsistencies and as such cannot be used as a basis for policy recommendations.

The utilitarian approach emerged as a response to the desire of economists and policy makers to be able to provide informed recommendations with regard to equity and efficiency tradeoffs. Specifically, it is routinely asserted that the framework allows to address the redistributive motive with regard to labor income and the efficiency considerations. Moreover, the framework appears natural to most formally trained economists. However, its popularity is not driven by its elegance or its formal grounding in economic theory, but, rather it is caused by the fact that the approach employes tools that are analytically manageable. In other words, the approach of Mirrlees, despite its appeal, is not unique. Furthermore, we argue that it may be necessary to reconsider the applicability of the approach as it may lead to results that are too sensitive to be of practical relevance.

It may appear that our findings stand as a stark contrast, in particular, with the results of Mirrlees [7], Diamond [1], and Saez [9]. However, this is not the case. We never question the validity of the efficiency conditions derived in the literature. Moreover, we consider them correct from a mathematical perspective. Nevertheless, we point out that their approach while mathematically sound, is based on an approach that utilizes a framework that is too broad to yield relevant policy prescriptions.

Many readers may find our assumptions to be too restrictive and, consequently, the results to be of little relevance as we study problems with a suitably defined utility function – related to that of Diamond [1]. Furthermore, numerous researchers may believe that our assumptions essentially allow us, especially given the simplicity of our approach, to trivialize a serious economic problem. We strongly disagree, as we study a legitimate optimal taxation problem, and we obtain our results in a framework that is frequently analyzed in the literature, e.g., Diamond [1]. Moreover, our results, at a technical level, do not stand in conflict with the literature on optimal taxation. We in fact consider our findings to be complementary to the results of Diamond [1], Saez [9], and Mirrlees [7], as we effectively enrich the class of solutions to optimal taxation problems. Nevertheless, it appears that it is possible to interpret the presence of our findings in a very drastic manner that puts into question the approach to optimal taxation taken in the literature. The optimal taxation problem is to address the equity and efficiency considerations and as such can be viewed in a purely economic context. Therefore, the solution, if it exists, should not be sensitive to the mathematical formulation of the problem, but as our findings reveal, it can be. Consequently, it may be necessary to reformulate the optimal taxation problem to allow for a richer class for solutions that are not sensitive to the mathematical representation of economic concepts.

This paper is divided into six sections. We outline the basic problem in the next section. In the following section we present the solution to the problem of dead weight burden minimization. In section four we extend our approach to a class of optimal taxation problems. The relevance of our approach is discussed in section five. Finally, section six contains conclusions.

2 Basic Problem

Following Saez [9] and effectively Diamond [1], let us consider an economy populated with economic agents whose preferences are represented with the following utility function

$$U(c, L) = h(c - \frac{1}{2}L^2), \tag{1}$$

where $h(\ )$ is an increasing, concave and differentiable function.

Furthermore, let us assume that $[a_L, a_H]$ represents the support of the relevant distribution of skills and let $f(\,)$ denote the corresponding *pdf* of the distribution of skills. Naturally, we assume that an agent whose productivity is equal to a delivers $y = aL$ units of output when she chooses to supply L units of labor.

The problem of an economic agent whose productivity is equal to a and who faces tax function $\tau(y)$ can be summarized as

$$max_{\{y\}}U(c, L) = h(y - \tau(y) - \frac{1}{2}(\frac{y}{a})^2), \tag{2}$$

in which we treat consumption as the numeraire.

The relevant first order condition can be expressed as

$$h^{'}(y - \tau(y) - \frac{1}{2}(\frac{y}{a})^2)(1 - \tau'(y) - \frac{y}{a^2}) = 0, \tag{3}$$

which implicitly defines the optimal income earned, y_a.

Let us denote the optimal choice of an agent whose productivity is equal to a with y_a. Furthermore, let R_a denote the revenue collected from an agent whose productivity is equal to a, i.e., let $R_a = \tau(y_a)$. Finally, let $U_a = h(y_a - \tau(y_a) - \frac{1}{2}(\frac{y_a}{a})^2)$ denote the realized utility of an agent whose productivity is equal to a.

Under standard assumptions that individual productivities are not observable, we can express the optimal taxation problem as follows:

$$max_{\{\tau(\,)\}}W = \int_{a_L}^{a_H} G(U_a)f(a)da, \tag{4}$$

where

$$U_a = h(y_a - \tau(y_a) - \frac{1}{2}(\frac{y_a}{a})^2), \tag{5}$$

subject to

$$(i) \quad \int_{a_L}^{a_H} \tau(y_a)f(a)da = R \tag{6}$$

and

$$(ii) \quad \forall a \epsilon [a_L, a_H]: \; h^{'}(y_a - \tau(y_a) - \frac{1}{2}(\frac{y_a}{a})^2)(1 - \tau'(y_a) - \frac{y_a}{a^2}) = 0, \tag{7}$$

where R denotes the revenue requirement, $G(\,)$ is a strictly increasing and strictly concave function, in order to capture the redistributive motive of the government, and $h(\,)$ is normally assumed to be an increasing and concave function. Naturally, the series of constraints,[1] (7), captures the notion that the government recognizes that individual agents remain at their private optima when the government chooses tax function $\tau(\,)$. Let $\hat{\tau}(\,)$ denote the optimal tax function that solves the above optimal taxation problem.

In the main part of the paper, we assume[2] that $0 < a_L$, and that $a_H < \infty$. Moreover, we assume, to exclude the possibility of trivial solutions, that R is big enough, so that a uniform tax of $\frac{R}{\int_{a_L}^{a_H} f(a)da}$ on all agents is not feasible. Given these restrictions on the values of the underlying parameters we assume, in the main part of the paper, that agents of all types strictly prefer[3] to supply a strictly positive amount of labor when faced with the optimal tax function that solves the problem described with equations (4), (5), (6), and (7).

3 Dead Weight Burden

We begin our exposition from a related and fundamentally much simpler problem. First, we choose to abstract completely from any equity considerations and focus strictly on efficiency issues. Specifically, we are interested in identifying tax functions that minimize the dead weight burden. The problem of minimizing the dead weight burden has not been subject to extensive research with the exception of the work of Saez [10], who derives an explicit analytic expression that defines the dead weight burden minimization function when lump sum taxes are unavailable. It appears that the literature has mostly ignored the problem of dead weight burden minimization as the problem in its purest form is trivial, and the solution can be degenerate, and, most importantly, the problem has been superseded by a more general problem of optimal taxation. In this paper, we first embark on a simple problem of dead weight burden minimization for purely expositional purposes. In this section, we follow Dudek [2] and illustrate our basic - Dudek [2] provides detailed analysis - argument without obscuring it with unnecessary technicalities.

The concept of dead weight burden is routinely used in applied work. Nevertheless, there are some disagreements with regard to its precise meaning. In this paper, we follow Saez [10] and focus on the definition, which appears to be most commonly

[1] As pointed out by Ebert [3], and Lollivier and Rochet [5], there are cases when one must exercise additional care as the stated constraints are not sufficient.

[2] Dudek [2] shows that the main results remain true under less restrictive algebraic assumptions.

[3] Again, Dudek [2] shows that the results prevail when some agents choose to be idle.

accepted in the literature. Specifically, we choose to measure the dead weight burden relying on the notion of equivalent variation, i.e., we define the dead weight burden as the difference between taxes actually paid by an individual and the lump sum tax that would induce the same level of utility. Formally, we can express the dead weight burden as follows.

Let us consider the behavior of an agent whose productivity is equal to a and who solves the following maximization problem,

$$max_{\{c,L\}}U(c,L) = h(c - \frac{1}{2}L^2) \tag{8}$$

subject to

$$c = aL - \tau(aL), \tag{9}$$

where $\tau(\,)$ is the tax function faced by the agent.

The relevant efficiency condition is given by

$$h^{'}(aL - \tau(aL) - \frac{1}{2}L^2)(a - a\tau'(aL) - L) = 0, \tag{10}$$

which implicitly defines the actual choice of the number of hours worked, L_a, income earned, $y_a = aL_a$, and, in turn, the level of consumption $c_a = y_a - \tau(y_a)$, and, finally, the level of realized utility, U_a, given by

$$U_a = h(y_a - \tau(y_a) - \frac{1}{2}(\frac{y_a}{a})^2). \tag{11}$$

The tax function, $\tau()$, is of general form and presumably creates some distortions. Therefore, collecting the specific amount, $\tau(y_a)$, comes at a cost to the consumer that goes beyond the direct revenue cost. The concept of dead weight burden that we employ here is related to that additional cost. In fact, typically, the consumer could be willing to pay more in taxes to avoid this additional cost. Specifically, if lump sum taxes are available, then there are no distortions, assuming that a is large enough, and the choices of the consumer are given by

$$L_a^F = a, \text{ and } c_a^F = a^2, \tag{12}$$

which leads to the value of realized utility of

$$U_a^F = h(\frac{1}{2}a^2 - T_a), \tag{13}$$

where T_a denotes the lump sum tax actually paid by the agent.

We can now find the value of T_a, which makes the consumer just indifferent to facing the original tax function, $\tau(\,)$, and paying T_a by equating U_a and U_a^F, which yields

$$h(y_a - \tau(y_a) - \frac{1}{2}(\frac{y_a}{a})^2) = h(\frac{1}{2}a^2 - T_a) \tag{14}$$

and further reduces to

$$T_a = \frac{1}{2}(\frac{y_a}{a} - a)^2 + \tau(y_a). \tag{15}$$

Observe that originally the agent actually paid $\tau(y_a)$ in taxes and, at the same time, could be willing to pay up to T_a, given by (15), if lump sum taxes were available. Alternatively, we can say that the discrepancy between T_a and $\tau(y_a)$ reflects the additional cost born by the society due to the presence of the distortions induced by $\tau(\,)$. We choose to identify this *unnecessary* revenue loss stemming from existing distortions with the dead weight burden. Consequently, we have

$$DWB_a = T_a - \tau(y_a) = \frac{1}{2}(\frac{y_a}{a} - a)^2. \tag{16}$$

Now, we are in a position to define the problem of the government. First of all, the government is interested in minimizing the economy wide dead weight burden. Therefore, we can state the objective function as

$$min_{\{\tau(\,)\}} DWB = \int_{a_L}^{a_H} DWB_a f(a) da. \tag{17}$$

Note that individual dead weight burdens are expressed in common units. Therefore, we do not encounter standard comparability issues by choosing to express the objective function with equation (17).

Naturally, the government must meet its specific revenue needs, R, and must take into account the fact that economic agents rationally respond to the tax schedule.

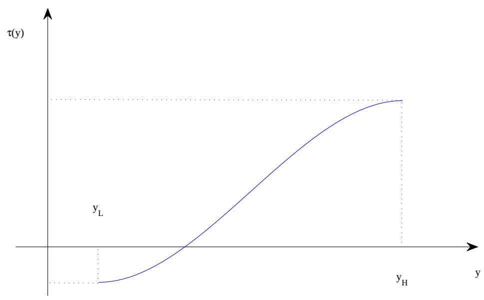

Figure 1: The Generic Form of an Optimal Tax Function.

Formally we state the problem as,

$$min_{\{\tau(\,)\}} DWB = \int_{a_L}^{a_H} \frac{1}{2}(\frac{y_a}{a} - a)^2 f(a)da \qquad (18)$$

subject to

$$R = \int_{a_L}^{a_H} \tau(y_a) f(a)da \qquad (19)$$

and to

$$\forall a \epsilon [a_L, a_H] : \; h^{'}(y_a - \tau(y_a) - \frac{1}{2}(\frac{y_a}{a})^2)(1 - \tau'(y_a) - \frac{y_a}{a^2}) = 0. \qquad (20)$$

The above optimization problem, expressed with equations (18), (19), and (20), can be dealt with using the proper mathematical techniques. Let $\tilde{\tau}(\,)$ be the solution, possibly obtained with the approach of Saez [10] applied to bounded distributions, of the above problem. There is some evidence, e.g., Sadka [8], Seade [13, 14], that optimal tax function can be of the form presented in figure (1); please note zero marginal taxes at the lowest, y_L, and the highest, y_H, incomes earned. We assume, here, for expositional purposes, that $\tilde{\tau}(\,)$ is of the same form. Moreover, let L_a^*, c_a^*, and $y_a^* = aL_a^*$ denote the choices of a consumer whose productivity is equal to a and who faces the tax schedule, $\tilde{\tau}(\,)$.

Observe that the revenue brought by an agent whose productivity is equal to a is given by $R_a^* = \tilde{\tau}(y_a^*)$ and that the total amount of revenue collected, naturally, must

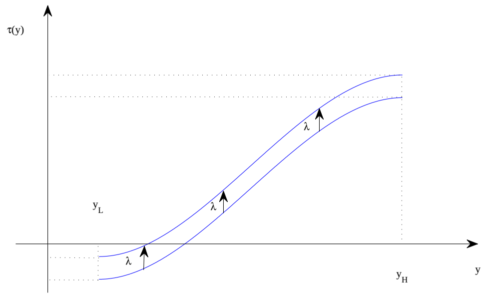

Figure 2: The form of $\tilde{\tau}_\lambda(\,)$ obtained from $\tilde{\tau}(\,)$ with an upward shift by λ.

be equal to R, i.e., we have

$$R = \int_{a_L}^{a_H} \tau(y_a^*) f(a) da. \tag{21}$$

Furthermore, the realized value of the objective functional is given by

$$DWB^* = \int_{a_L}^{a_H} \frac{1}{2}(\frac{y_a^*}{a} - a)^2 f(a) da, \tag{22}$$

and the realized utility of an individual whose productivity is equal to a can be written as

$$U_a^* = h(y_a^* - \tilde{\tau}(y_a^*) - \frac{1}{2}(\frac{y_a^*}{a})^2). \tag{23}$$

Recall that we have already assumed that all agents prefer to supply a strictly positive amount of labor, the alternate case is considered in Dudek [2], when they face $\tilde{\tau}(\,)$.

Let us now consider a simple variation[4] of $\tilde{\tau}(y)$. Specifically, let us consider the following tax function

$$\tilde{\tau}_\lambda(y) = \tilde{\tau}(y) + \lambda, \tag{24}$$

where λ is a constant.

Figure (2) presents the new tax function, $\tilde{\tau}_\lambda(\,)$. Note that the new tax function given by (24) is simply equal to the previous one increased by a lump sum tax of λ.

[4] Again, it is argued in Dudek [2] that such a variation remains feasible.

Observe that given our assumptions of positive labor supply on the part of all agents it must be the case, given the form of the utility function, that the labor supply choice of agents will remain unchanged when tax liabilities are dictated with $\tilde{\tau}_\lambda(\,)$ rather than by $\tilde{\tau}(\,)$ for λ sufficiently small - detailed arguments are presented in Dudek [2]. Furthermore, given the form of the utility function and that λ is just a lump sum transfer, there should be no further efficiency losses as compared to the case when tax liabilities are determined with $\tilde{\tau}(\,)$. However, the values of individual utilities are affected, and they now become

$$U_a^*(\lambda) = h(y_a^* - \tilde{\tau}(y_a^*) - \lambda - \frac{1}{2}(\frac{y_a^*}{a})^2). \tag{25}$$

Similarly, we can, now, express the revenue collected from a single individual as

$$R_a^*(\lambda) = \tilde{\tau}(y_a^*) + \lambda = R_a^* + \lambda, \tag{26}$$

which, in particular, implies that the total amount of revenue collected by the government is higher and given by

$$R(\lambda) = \int_{a_L}^{a_H} R_a(\lambda) f(a) da = R + \lambda \int_{a_L}^{a_H} f(a) da, \tag{27}$$

i.e., the government collects more revenue than it needs.

Let us now consider the following class of tax functions

$$\tilde{\tau}_{\lambda,\bar{y}}(y) = \begin{cases} \tilde{\tau}_\lambda(y) & \text{for } y \leq \bar{y} \\ \tilde{\tau}_\lambda(\bar{y}) & \text{for } \bar{y} < y. \end{cases} \tag{28}$$

Naturally, $\tilde{\tau}_{\lambda,\bar{y}}(\,)$ looks just like $\tilde{\tau}_\lambda(\,)$, with its right hand end flattened starting from $\bar{y}$, figure (3).

Recall that y_L denotes the level of income earned by the lowest type when she faces $\tilde{\tau}(\,)$ and y_H denotes the level of income earned by the highest type when taxes are paid according to $\tilde{\tau}(\,)$. Imagine, now, that tax liabilities are to be dictated by $\tilde{\tau}_{\lambda,\bar{y}}(\,)$ rather than by $\tilde{\tau}(\,)$. Observe that when we choose $\bar{y} = y_L$ then the revenue collected from all agents must be, for λ small enough, necessarily smaller than the one that is collected from all agents when taxes are paid in line with $\tilde{\tau}(\,)$. Furthermore, when $\bar{y} = y_H$ then $\tilde{\tau}_{\lambda,\bar{y}}(\,) = \tilde{\tau}_\lambda(\,)$ and the revenue collected from all agents is necessarily

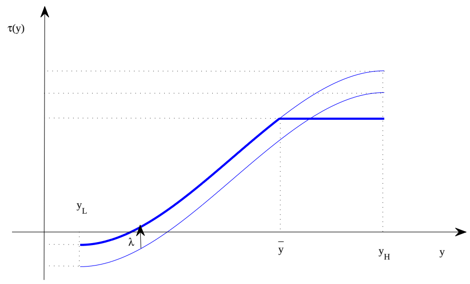

Figure 3: The Form of $\tilde{\tau}_{\lambda,\bar{y}}(\ )$.

equal to the one that is obtained when taxes are paid according to $\tilde{\tau}_{\lambda}(\ )$ and larger than the revenue collected from all agents when tax liabilities are dictated with the optimal tax function, $\tilde{\tau}(\ )$. Therefore, we can expect, relying on an intuitive notion of the mean value theorem[5], that there exists a value of $\bar{y}$ for which he revenue collected from all agents, who face $\tilde{\tau}_{\lambda,\bar{y}}()$, is the same as the revenue collected from agents when agents face the optimal tax function, $\tilde{\tau}(\ )$. Let us denote such a value of $\bar{y}$ with $\bar{y}_R$.

Let us now assume that agents' tax liabilities are determined by $\tilde{\tau}_{\lambda,\bar{y}_R}(\ )$. Furthermore, let the labor supply choice, given tax function $\tilde{\tau}_{\lambda,\bar{y}_R}(\ )$, of an agent whose productivity is equal to a be denoted with L_a^{**} and the corresponding income earned with $y_a^{**} = aL_a^{**}$, and let the level of revenue collected be equal to

$$R_a^{**}(\lambda, \bar{y}_R) = \tilde{\tau}_{\lambda,\bar{y}_R}(y_a^{**}). \tag{29}$$

Note that given the choice of $\bar{y}_R$ the revenue collected with $\tilde{\tau}_{\lambda,\bar{y}_R}(\ )$ must be the same as the revenue collected with the original optimal tax function, $\tilde{\tau}(\ )$, i.e., we must have

$$\int_{a_L}^{a_H} R_a^{**}(\lambda, \bar{y}_R) f(a)da = \int_{a_L}^{a_H} \tilde{\tau}_{\lambda,\bar{y}_R}(y_a^{**}) f(a)da = \int_{a_L}^{a_H} R_a^{*} f(a)da = R. \tag{30}$$

Furthermore, the realized utility in this case is given by

$$U_a^{**} = h(y_a^{**} - R_a^{**}(\lambda, \bar{y}_R) - \frac{1}{2}(\frac{y_a^{**}}{a})^2). \tag{31}$$

[5]We provide a rigorous argument in Appendix A.

Finally, the realized value of the objective functional, the dead weight burden, can be expressed as

$$DWB^{**} = \int_{a_L}^{a_H} \frac{1}{2}(\frac{y_a^{**}}{a} - a)^2 f(a)da. \tag{32}$$

Observe that agents can be now split into two categories. Those who remain at their original choices when $\tilde{\tau}_\lambda(\,)$ is replaced with $\tilde{\tau}_{\lambda,\bar{y}_R}(\,)$, and those who alter their behavior. Specifically, when the behavior is not changed, then by definition, we have $y_a^{**} = y_a^*$, and consequently it must be

$$y_a^{**} - \frac{1}{2}(\frac{y_a^{**}}{a})^2 = y_a^* - \frac{1}{2}(\frac{y_a^*}{a})^2. \tag{33}$$

It is necessary to consider two separate cases when the behavior is affected. First note that agents who initially chose incomes below $\bar{y}_R$ and now choose incomes above $\bar{y}_R$ pay more in taxes, i.e., we must have

$$R_a^{**}(\lambda, \bar{y}_R) = \tilde{\tau}_{\lambda,\bar{y}_R}(y_a^{**}) > \tilde{\tau}_\lambda(y_a^*) = R_a^* + \lambda. \tag{34}$$

Furthermore, their previous choices remain available, so invoking the revealed preference argument we can write

$$h(y_a^{**} - R_a^{**}(\lambda, \bar{y}_R) - \frac{1}{2}(\frac{y_a^{**}}{a})^2) \geq h(y_a^* - R_a^* - \lambda - \frac{1}{2}(\frac{y_a^*}{a})^2), \tag{35}$$

which can be simplified to

$$y_a^{**} - R_a^{**}(\lambda, \bar{y}_R) - \frac{1}{2}(\frac{y_a^{**}}{a})^2 \geq y_a^* - R_a^* - \lambda - \frac{1}{2}(\frac{y_a^*}{a})^2. \tag{36}$$

In turn, by adding up (34) and (36), we can establish the following

$$y_a^{**} - \frac{1}{2}(\frac{y_a^{**}}{a})^2 > y_a^* - \frac{1}{2}(\frac{y_a^*}{a})^2. \tag{37}$$

The situation is slightly more complicated in the case of agents who initially chose incomes above $\bar{y}_R$. First of all, note that such agents would never switch to levels of income below $\bar{y}_R$, as those choices were available to them originally, and they chose incomes above $\bar{y}_R$. Moreover, now, they can actually pay less in taxes if they remain above $\bar{y}_R$, so switching to income levels below $\bar{y}_R$ is surely suboptimal. Furthermore, remaining above $\bar{y}_R$ entails, first of all, a lower amount of tax liabilities, and secondly, it implies that agents face a marginal tax of zero. Consequently, we can conclude that agents who were initially above $\bar{y}_R$ remain in the range of incomes above $\bar{y}_R$ and, in fact, do change their behavior in response to zero marginal taxes, with the exception of agents with the highest skill level. In other words, the new labor supply choices coincide with the values of labor supply that would be chosen if taxes were lump sum. Accordingly, we must have

$$h(y_a^{**} - R_a^* - \lambda - \frac{1}{2}(\frac{y_a^{**}}{a})^2) > h(y_a^* - R_a^* - \lambda - \frac{1}{2}(\frac{y_a^*}{a})^2), \tag{38}$$

as supplying a given level of revenue, $R_a^* + \lambda$, is more efficient when marginal taxes are zero rather than positive. Equation (38) naturally reduces to

$$y_a^{**} - R_a^* - \lambda - \frac{1}{2}(\frac{y_a^{**}}{a})^2 > y_a^* - R_a^* - \lambda - \frac{1}{2}(\frac{y_a^*}{a})^2. \tag{39}$$

Combining (33), (37), and (39), we can be sure that the following must be true

$$\forall a \epsilon [a_L, a_H] | \; y_a^{**} - \frac{1}{2}(\frac{y_a^{**}}{a})^2 \geq y_a^* - \frac{1}{2}(\frac{y_a^*}{a})^2, \tag{40}$$

which after subtracting $\frac{1}{2}a^2$ from both sides and multiplying by -1 becomes

$$\forall a \epsilon [a_L, a_H] | \; \frac{1}{2}(\frac{y_a^*}{a} - a)^2 \geq \frac{1}{2}(\frac{y_a^{**}}{a} - a)^2 \tag{41}$$

with a strict inequality on a non-degenerate set. Furthermore, the collection of inequalities (41) implies that

$$\int_{a_L}^{a_H} \frac{1}{2}(\frac{y_a^*}{a} - a)^2 f(a) da > \int_{a_L}^{a_H} \frac{1}{2}(\frac{y_a^{**}}{a} - a)^2 f(a) da, \tag{42}$$

i.e., noting (22) and (32), it leads to

$$DWB^* > DWB^{**}. \tag{43}$$

Observe that by replacing the optimal tax function $\tilde{\tau}(\,)$ with $\tilde{\tau}_\lambda(\,) = \tilde{\tau}(\,) + \lambda$, we increase the revenue collected, and we do not affect individual choices, i.e., we do not affect efficiency. Furthermore, by replacing $\tilde{\tau}_\lambda(\,) = \tilde{\tau}(\,) + \lambda$ with $\tilde{\tau}_{\lambda,\bar{y}_R}(\,)$, we enhance efficiency and reduce the revenue collected to the level obtained with $\tilde{\tau}(\,)$. Clearly, as we replace $\tilde{\tau}(\ )$ with $\tilde{\tau}_{\lambda,\bar{y}_R}(\ )$ we improve efficiency without compromising revenue. In other words, it is possible to collect, in more efficient way, the same amount of revenue as collected with $\tilde{\tau}(\ )$.

Therefore, we must conclude that the original function, $\tilde{\tau}(\ )$, cannot be optimal, as by replacing $\tilde{\tau}(\ )$ with $\tilde{\tau}_{\lambda,\bar{y}_R}(\ )$, we not only preserve revenue, but also enhance efficiency since we lower the value of the objective functional expressed with (18). In other words, a given tax function cannot be optimal unless it is flat in a neighborhood of the highest income earned. Consequently, we can state that a given tax function cannot be optimal unless it is of the shape of $\tilde{\tau}_{\lambda,\bar{y}_R}(\ )$ to start with. However, by originating at y_H and moving to the left, we can repeat the above argument starting from any point y that is actually chosen and at which the optimal tax function stops being flat. Thus, we can argue - see Dudek [2] for details - that any optimal tax function must be a step function at least over an non-degenerate set of income levels.

To reiterate we can state that it turns out that the class of functions implicitly considered plausible by Saez [10] can be too limited. Specifically, we can apply Saez's approach even when the distribution of skills is bounded. Let $\tilde{\tau}_S()$ be the optimal tax function obtained with Saez's method. Naturally, we can always apply our procedure to $\tilde{\tau}_S()$. First we shift it up by λ, small enough, and then we flatten its right hand tail to ensure revenue neutrality. Naturally, by doing so, we obviously enhance efficiency, i.e., we make the value of the total dead weight burden smaller. In other words, it is possible to improve[6] upon Saez's solution if one is willing to extend the class of admissible tax functions and, in particular, consider step functions as admissible.

Furthermore, Saez [10] argues that his dead weight burden minimization problem is equivalent to an optimal taxation problem with properly defined social welfare weights. However, this signals a possibility that solutions to some of the optimal taxation problems can be improved upon by extending the class of admissible functions.

[6]Note that we initially assumed that the revenue requirement, R, was sufficiently high to exclude the feasibility of a trivial solution in the form of a uniform tax of $\frac{R}{\int_{a_L}^{a_H} f(a)da}$. Consequently, our improvement is applicable in cases, considered of interest by Saez [10], when some agents are not able to pay a uniform poll tax.

4 Optimal Taxation Problem

In this section we employ the reasoning from the above section and show that certain, suitably defined, optimal taxation problems admit solutions of the form of a step function. Specifically, let us consider the following optimal taxation problem

$$max_{\{\tau(\,)\}} W = \int_{a_L}^{a_H} G(U_a) f(a) da, \tag{44}$$

where

$$U_a = h(y_a - \tau(y_a) - \frac{1}{2}(\frac{y_a}{a})^2), \tag{45}$$

subject to

$$(i) \quad \forall a \in [a_L, a_H] | \; \frac{dU_a}{dy_a} = 0 \tag{46}$$

and

$$(ii) \quad \int_{a_L}^{a_H} \tau(y_a) f(a) da = R. \tag{47}$$

Naturally, we assume that $G(\,)$ is an increasing and *concave* function and that $h(\,)$ is an increasing and *convex* function. Observe that here we make a nonstandard assumption as normally $h(\,)$ is assumed to be concave - see section 2. We address this issue - convexity of $h(\,)$ - in detail in the subsequent section. Obviously, our assumptions imply that the inverse of $h(\,)$ exists and is increasing and *concave*.

Assume that $\hat{\tau}()$ solves the above problem. Moreover, let $y_a^* = aL_a^*$, $c_a^* = y_a^* - \hat{\tau}(y_a^*)$ denote the corresponding choices of an agent whose productivity is equal to a when she faces the optimal tax function, $\hat{\tau}(\,)$. The level of individual realized utility in this case is given by

$$U_a^* = h(y_a^* - \hat{\tau}(y_a^*) - \frac{1}{2}(\frac{y_a^*}{a})^2), \tag{48}$$

the value of the social welfare function takes the form

$$W^* = \int_{a_L}^{a_H} G(h(y_a^* - \hat{\tau}(y_a^*) - \frac{1}{2}(\frac{y_a^*}{a})^2)) f(a) da. \tag{49}$$

Furthermore, the government budget constraint is satisfied and given by

$$\int_{a_L}^{a_H} \hat{\tau}(y_a^*) f(a) da = R. \tag{50}$$

Again, resorting to the findings of Sadka [8] and Seade [13, 14] we can expect that the shape of $\hat{\tau}(\,)$ is of the form depicted in figure (1). Note that the specified form assumes that the marginal taxes at both ends are equal to zero. Let us now engage in the process of distortions described in detail in the previous section. First let us assume, again, that all agents supply a positive amount of labor given $\hat{\tau}(\,)$. Now let us consider a new tax function, $\hat{\tau}_\lambda(\,)$, obtained from $\hat{\tau}(\,)$ with an upward shift by λ, figure (2). Naturally, we have

$$\hat{\tau}_\lambda(y) = \hat{\tau}(y) + \lambda. \tag{51}$$

Obviously , see Dudek [2] for details, economic agents do not change their behavior when $\hat{\tau}(\,)$ is replaced with $\hat{\tau}_\lambda(\,)$ for λ small enough. However, the amount of revenue collected by the government in this case is higher and equal to $R + \lambda \int_{a_L}^{a_H} f(a)da$.

As in the previous section, let us dispose of the additional revenue collected by the government by considering modifications of $\hat{\tau}_\lambda(\,)$ of the form, figure (3),

$$\hat{\tau}_{\lambda,\bar{y}}(y) = \begin{cases} \hat{\tau}_\lambda(y) & \text{for } y \leq \bar{y} \\ \hat{\tau}_\lambda(\bar{y}) & \text{for } \bar{y} < y \end{cases}. \tag{52}$$

Naturally, by replacing $\hat{\tau}_\lambda(\,)$ with $\hat{\tau}_{\lambda,\bar{y}}(\,)$, we reduce the amount of revenue collected. Moreover, given that the amount of revenue collected when $\bar{y} = y_H$ is definitely greater than the amount of revenue collected with $\hat{\tau}(\,)$ and the amount of revenue collected when $\bar{y} = y_L$ is necessarily smaller than the amount of revenue collected with $\hat{\tau}(\,)$ we expect − see Appendix A for a formal argument − that there is a value of $\bar{y} \in (y_L, y_H)$ such that the amount of revenue collected with $\hat{\tau}_{\lambda,\bar{y}}(\,)$ is just equal to the amount of revenue collected with $\hat{\tau}(\,)$. Let us denote this special value of $\bar{y}$ with $\bar{y}_R$.

Let us now assume that agents' tax liabilities are determined with $\hat{\tau}_{\lambda,\bar{y}_R}(\,)$ and let L_a^{**}, $y_a^{**} = aL_a^{**}$, and $c_a^{**} = y_a^{**} - \tau(y_a^{**})$ be the corresponding choices of an agent whose productivity is equal to a. Naturally, the amount of revenue collected from an agent whose productivity is equal to a is given by

$$R_a^{**} = \hat{\tau}_{\lambda,\bar{y}_R}(y_a^{**}). \tag{53}$$

Furthermore, by construction it must be the case that

$$\int_{a_L}^{a_H} \hat{\tau}_{\lambda,\bar{y}_R}(y_a^{**})f(a)da = \int_{a_L}^{a_H} \hat{\tau}(y_a^*)f(a)da, \tag{54}$$

which implies that $\hat{\tau}_{\lambda,\bar{y}_R}(\,)$ is feasible for the original optimal taxation problem described with (44), (45), (46), and (47). Consequently, given the assumed optimality of $\hat{\tau}(\,)$ we can write the following

$$\int_{a_L}^{a_H} G(h(y_a^{**} - \hat{\tau}_{\lambda,\bar{y}_R}(y_a^{**}) - \frac{1}{2}(\frac{y_a^{**}}{a})^2))f(a)da \leq \int_{a_L}^{a_H} G(h(y_a^* - \hat{\tau}(y_a^*) - \frac{1}{2}(\frac{y_a^*}{a})^2))f(a)da. \tag{55}$$

As before, observe that we can now divide economic agents into two categories: those who do not change their behavior when $\hat{\tau}_\lambda(\,)$ is replaced with $\hat{\tau}_{\lambda,\bar{y}_R}(\,)$ and those who actually do. Naturally, if the behavior is not changed, it must be the case that $y_a^{**} = y_a^*$ and consequently that

$$y_a^{**} - \frac{1}{2}(\frac{y_a^{**}}{a})^2 = y_a^* - \frac{1}{2}(\frac{y_a^*}{a})^2. \tag{56}$$

In the case of a behavior change we must consider two separate cases. First note that agents who initially chose incomes below $\bar{y}_R$ and now choose incomes above $\bar{y}_R$ pay more in taxes, i.e., we must have

$$\hat{\tau}_{\lambda,\bar{y}_R}(y_a^{**}) > \hat{\tau}_\lambda(y_a^*) = \hat{\tau}(y_a^*) + \lambda. \tag{57}$$

Furthermore, their previous choices remain available, so by revealed preference we must have

$$h(y_a^{**} - \hat{\tau}_{\lambda,\bar{y}_R}(y_a^{**}) - \frac{1}{2}(\frac{y_a^{**}}{a})^2) > h(y_a^* - \hat{\tau}(y_a^*) - \lambda - \frac{1}{2}(\frac{y_a^*}{a})^2), \tag{58}$$

which given the assumed monotonicity of $h(\,)$ can be simplified to

$$y_a^{**} - \hat{\tau}_{\lambda,\bar{y}_R}(y_a^{**}) - \frac{1}{2}(\frac{y_a^{**}}{a})^2 > y_a^* - \hat{\tau}(y_a^*) - \lambda - \frac{1}{2}(\frac{y_a^*}{a})^2, \tag{59}$$

which when combined with (57) reduces to

$$y_a^{**} - \frac{1}{2}(\frac{y_a^{**}}{a})^2 > y_a^* - \frac{1}{2}(\frac{y_a^*}{a})^2. \tag{60}$$

In addition, agents who initially chose incomes above $\bar{y}_R$ and decided to change their behavior remain[7] in the range above $\bar{y}_R$ and face a marginal tax of zero whereas previously they faced positive marginal tax rates. In other words, their new labor supply choices coincide with the values of labor supply when taxes are lump sum. Accordingly, we must have

$$h(y_a^{**} - \hat{\tau}(y_a^*) - \lambda - \frac{1}{2}(\frac{y_a^{**}}{a})^2) > h(y_a^* - \hat{\tau}(y_a^*) - \lambda - \frac{1}{2}(\frac{y_a^*}{a})^2), \tag{61}$$

since it is more efficient to supply a given level of revenue, $\hat{\tau}(y_a^*) + \lambda$, when marginal taxes are zero rather than positive.

Equation (61) naturally can be simplified to

$$y_a^{**} - \frac{1}{2}(\frac{y_a^{**}}{a})^2 > y_a^* - \frac{1}{2}(\frac{y_a^*}{a})^2. \tag{62}$$

Combining (56), (60), and (62) we can establish that the following must be true

$$\forall a \in [a_L, a_H] | \; y_a^{**} - \frac{1}{2}(\frac{y_a^{**}}{a})^2 \geq y_a^* - \frac{1}{2}(\frac{y_a^*}{a})^2 \tag{63}$$

with a strict inequality on a non-degenerate set.

Observe that by replacing the optimal tax function, $\hat{\tau}(\,)$, with $\hat{\tau}_\lambda(\,)$ we increase revenue collected, and we do not affect individual choices, i.e., in particular, we do not affect efficiency. Furthermore, by replacing $\hat{\tau}_\lambda()$ with $\hat{\tau}_{\lambda,\bar{y}_R}()$ we enhance efficiency and reduce the revenue collected to the level obtained with $\hat{\tau}()$. Clearly, as we replace $\hat{\tau}()$ with $\hat{\tau}_{\lambda,\bar{y}_R}(\,)$ we improve efficiency without compromising revenue. In other words, it is possible to collect in a more efficient way the same amount of revenue as is collected with $\hat{\tau}(\,)$. Presumably, this can be expected because in the process of derivation of $\hat{\tau}(\,)$, equity consideration, in addition to efficiency issues, played a role as well.

Recall that $\hat{\tau}(\,)$ solves the optimal taxation problem described with (44), (45), (46), and (47). Imagine, now that our optimal tax function $\hat{\tau}(\,)$ was found when function $G(\,)$, which captures equity considerations, was of the form,

$$G(U) = h^{-1}(U). \tag{64}$$

[7] The choices of income below $\bar{y}_R$ were available to them initially and they chose incomes above $\bar{y}_R$ and were willing to pay higher taxes. Naturally, their tax liabilities above $\bar{y}_R$ are, if anything, smaller than before. Consequently, choosing income levels below $\bar{y}_R$ must be suboptimal.

Given the specific form of $G()$, note that $G()$ is increasing and concave, expressed by equation (64), the value of the objective functional at the optimum, formula (49), takes the form

$$W^* = \int_{a_L}^{a_H} h^{-1}(h(y_a^* - \hat{\tau}(y_a^*) - \frac{1}{2}(\frac{y_a^*}{a})^2))f(a)da, \tag{65}$$

which reduces to

$$W^* = \int_{a_L}^{a_H} (y_a^* - \hat{\tau}(y_a^*) - \frac{1}{2}(\frac{y_a^*}{a})^2)f(a)da, \tag{66}$$

and further to

$$W^* = \int_{a_L}^{a_H} (y_a^* - \frac{1}{2}(\frac{y_a^*}{a})^2)f(a)da - \int_{a_L}^{a_H} \hat{\tau}(y_a^*)f(a)da, \tag{67}$$

and finally to

$$W^* = \int_{a_L}^{a_H} (y_a^* - \frac{1}{2}(\frac{y_a^*}{a})^2)f(a)da - R. \tag{68}$$

The value of the objective functional when tax liabilities are determined according to $\hat{\tau}_{\lambda,\bar{y}_R}(\,)$ is given by

$$W^{**} = \int_{a_L}^{a_H} h^{-1}(h(y_a^{**} - \hat{\tau}_{\lambda,\bar{y}_R}(y_a^{**}) - \frac{1}{2}(\frac{y_a^{**}}{a})^2))f(a)da, \tag{69}$$

which reduces to

$$W^{**} = \int_{a_L}^{a_H} (y_a^{**} - \frac{1}{2}(\frac{y_a^{**}}{a})^2)f(a)da - \int_{a_L}^{a_H} \hat{\tau}_{\lambda,\bar{y}_R}(y_a^{**})f(a)da, \tag{70}$$

and further to

$$W^{**} = \int_{a_L}^{a_H} (y_a^{**} - \frac{1}{2}(\frac{y_a^{**}}{a})^2)f(a)da - R, \tag{71}$$

as $\bar{y}_R$ was originally chosen to ensure revenue neutrality.

Furthermore, given that (63) holds as a strict inequality on a non-degenerate set, we can establish that

$$W^{**} > W^*, \tag{72}$$

which contradicts (55).

Therefore, we must conclude that $\hat{\tau}(\,)$ cannot be optimal since by replacing $\hat{\tau}(\,)$ with $\hat{\tau}_{\lambda,\bar{y}_R}()$ we not only enhance efficiency while preserving revenue, but also increase the value of the objective functional specified with equation (44). In other words, a given tax function cannot be optimal unless it is flat in a neighborhood of the highest income earned to start with. Naturally, originating from y_H and moving to the left we can repeat - see Dudek [2] - the above argument starting from any point y that is actually chosen, at which the optimal tax function stops being flat, and consequently, argue that any optimal tax function must be a step function at least over the range of incomes that are actually chosen.

The solution to the above optimal taxation problem can appear to be very special as it involves either marginal taxes of zero or implicit infinite marginal taxes. Such a solution may seem extreme; however, solutions of this form are not atypical in optimal control theory. Specifically, it may be the case that it is optimal to follow so-called bang-bang approach, i.e., bouncing between the extremes, in order to attain optimality. Therefore, our solution should not be considered implausible even though it relies on an extreme variation in the level of marginal tax rates.

So far we have solved our optimal taxation problem in a very special case when $G(\,) = h^{-1}(\,)$. Naturally, our approach is applicable in more general cases that do not require such a peculiar choice of $G(\,)$. In fact, one can easily identify an entire class of optimal taxation problems with the solution in the form of a step function. Such a class is described in the text below.

Let us assume that $G(\,)$ and $h(\,)$ are both increasing functions, $G(\,)$ is *concave*, $h(\,)$ is *convex*, and $G(h(\,))$ is *convex*. Furthermore, let us assume that $\hat{\tau}(\,)$, with the corresponding choice of income, y_a^*, solves the original optimal taxation problem when $G(h(\,))$ is convex. Naturally, the realized value of the social welfare function in this case is given by

$$W^* = \int_{a_L}^{a_H} G(h(y_a^* - \hat{\tau}(y_a^*) - \frac{1}{2}(\frac{y_a^*}{a})^2))f(a)da. \tag{73}$$

In addition, let $\hat{\tau}_{\lambda,\bar{y}_R}(\,)$, with the induced choice y_a^{**}, be the corresponding revenue neutral variation of $\hat{\tau}(\,)$. Naturally, given the feasibility of $\hat{\tau}_{\lambda,\bar{y}_R}(\,)$ for the original optimal taxation problem, we must have

$$\int_{a_L}^{a_H} G(h(y_a^{**} - \hat{\tau}_{\lambda,\bar{y}_R}(y_a^{**}) - \frac{1}{2}(\frac{y_a^{**}}{a})^2))f(a)da \leq \int_{a_L}^{a_H} G(h(y_a^* - \hat{\tau}(y_a^*) - \frac{1}{2}(\frac{y_a^*}{a})^2))f(a)da. \tag{74}$$

However, given the assumed convexity of $G(h(\,))$ and the specific distortions –

increase tax liabilities for low productivity agents and reduce the tax liabilities for high productivity agents – applied to $\hat{\tau}()$ to obtain $\hat{\tau}_{\lambda,\bar{y}_R}()$ we can conclude, Appendix B provides a formal proof, that the following must be true

$$\int_{a_L}^{a_H} G(h(y_a^* - \hat{\tau}(y_a^*) - \frac{1}{2}(\frac{y_a^*}{a})^2))f(a)da < \int_{a_L}^{a_H} G(h(y_a^{**} - \hat{\tau}_{\lambda,\bar{y}_R}(y_a^{**}) - \frac{1}{2}(\frac{y_a^{**}}{a})^2))f(a)da, \tag{75}$$

which naturally contradicts (74).

Therefore, we can conclude that $\hat{\tau}(\ )$ cannot be optimal unless it is locally flat in a neighborhood of the highest earned income and, consequently, see Dudek [2], that optimality requires the solution to be in the form of a step function.

We must concede, however, that our reasoning is applicable to cases when $G(\ )$ is concave, as assumed in the literature, but $G(h(\ ))$ is, at the same time, convex. Nevertheless, we believe that our assumption of convexity of $h(\,)$, and consequently of $G(h(\ ))$, does not stand in conflict with economic theory and more importantly with the mainstream of the literature. Furthermore, we believe that the existence of the above described solution, if anything, undermines the utilitarian approach to optimal taxation.

5 Relevance

Most readers may find our results presented in the previous section unappealing, restrictive, and without practical merit. Furthermore, many researchers may conclude that we simply assume the problem away by making a convenient specification of the utility function. We strongly disagree with such potential claims and argue that we study a legitimate class of problems that merit formal consideration. Moreover, we argue that one must not dismiss problems of the form studied in the previous section despite their triviality. In fact, we suggest that the problems and their solutions make us question the relevance of the utilitarian approach to optimal taxation.

The biggest objection that can be raised, given the state of the literature, pertains to the mathematical properties of our function $h(\ )$, which is used to describe the underlying utility of agents. While most readers agree that $h()$ should be an increasing function, and it is, very few would be willing to accept our assumption, critical in our proof, that $h(\,)$ is a convex function. Such objections would, however, be baseless from the theoretical perspective since for any $h(\ )$ the utility function of the form,

$U = h(c - \frac{1}{2}L^2)$, constitutes a legitimate representation of preferences as long as $h(\,)$ is an increasing function. Therefore, if we are willing to accept interpersonal comparisons of marginal utility, as the literature does, we should not, in principle, outright dismiss problems with $h(\,)$ convex rather than concave, as there is nothing fundamentally improper about the convexity of $h(\,)$. We should in fact allow for this new class of problems to be formally considered, which we do in this paper, and, thus, augment the existing findings.

We agree that working with a convex $h()$ may appear unnatural, but formally one cannot raise objections against the convexity of $h(\,)$. Furthermore, as the example below shows, rejecting convex $h(\,)'$s can cause some unintended inconsistencies.

Let us consider the following optimal taxation problem

$$max_{\{\tau(\,)\}} W = \int_{a_L}^{a_H} G(U_a) f(a) da, \tag{76}$$

where

$$U_a = h_1(h_2(y_a - \tau(y_a) - \frac{1}{2}(\frac{y_a}{a})^2)), \tag{77}$$

subject to

$$\forall a \epsilon [a_L, a_H] | \frac{dU_a}{dy_a} = 0 \tag{78}$$

and

$$\int_{a_L}^{a_H} \tau(y_a) f(a) da = R. \tag{79}$$

Furthermore, let us assume that functions $G()$, $h_1()$, and $h_2()$ are increasing, and that functions $G(\,)$ and $h_1(\,)$ are concave, and that function $h_2(\,)$ is convex, but the composition, $h_1(h_2(\,))$, is concave. Under these assumptions the problem described with (76), (77), (78), and (79) is standard and can be dealt with using the approach of Mirrlees [7]. Let $\tau_M(\,)$ denote the solution to the above problem.

Let us now consider a related problem. Specifically, the social planner now wants to maximize

$$max_{\{\tau(\,)\}} W = \int_{a_L}^{a_H} G(h_1(V_a)) f(a) da, \tag{80}$$

where

$$V_a = h_2(y_a - \tau(y_a) - \frac{1}{2}(\frac{y_a}{a})^2), \tag{81}$$

subject to

$$\forall a \epsilon [a_L, a_H] | \frac{dV_a}{dy_a} = 0 \tag{82}$$

and

$$\int_{a_L}^{a_H} \tau(y_a) f(a) da = R. \tag{83}$$

Observe that now the preferences of the policy maker are represented with $G(h_1())$ rather than with just $G()$. Moreover, the preferences of agents were previously represented with a utility function that was described with $h_1(h_2())$, which by assumption was concave. However, now the utility function is just affected by $h_2(\,)$, which by assumption is convex. Clearly, the two optimal taxation problems are described differently: the first problem belongs to the standard category of optimal taxation problems, and the second not necessarily, as the function defining the utility function is convex. Nevertheless, from a purely mathematical perspective the two problems appear identical. Consequently, we can conclude that $\tau_M()$ solves the second problem as well[8]. In other words, it is mathematically possible to solve the problem described with (80), (81), (82), and (83), and, thus, there is no technical merit that would qualify the problem as inadmissible. Furthermore, the actual economic decisions are identical as well, as the two problems differ only with respect to the form of representation of preferences. Therefore, there is no formal economic justification that would allow us to consider the problem described with (76), (77), (78), and (79) as admissible, and at the same time to consider the problem identified with (80), (81), (82), and (83) as inadmissible.

To reiterate let us consider the following canonical optimal taxation problem

$$max_{\{\tau(\,)\}} W = \int_{a_L}^{a_H} G(U_a) f(a) da, \tag{84}$$

where

$$U_a = h(y_a - \tau(y_a) - \frac{1}{2}(\frac{y_a}{a})^2), \tag{85}$$

subject to

$$\forall a \epsilon [a_L, a_H] | \frac{dU_a}{dy_a} = 0 \tag{86}$$

[8]Note that we just like Mirrlees [7] do not allow for randomization.

and

$$\int_{a_L}^{a_H} \tau(y_a) f(a) da = R, \tag{87}$$

where both $G(\,)$ and $h(\,)$ are increasing and concave functions.

Most researchers would consider the optimal taxation problem defined with relationships (84), (85), (86), and (87), the the restrictions on $G(\,)$ and $h(\,)$ as legitimate. In particular, most would argue that the problem captures in a non trivial manner the motive for income redistribution. However, as pointed out by Kaplow [4] this motive is embedded in $\Gamma(\,) = G(h(\,))$. In other words, the actual form of the solution depends on $\Gamma(\,)$, and not on individual properties of $G(\,)$ or $h(\,)$. Can we say then the legitimacy of the problem depends on $\Gamma(\,)$ or must we pay attention to the actual properties of $G(\,)$ and $h(\,)$? Note, that we can arrive at given $\Gamma(\,)$ in a number of ways. In particular, we could arrive at given concave$\Gamma(\,)$ with a concave $G(\,)$ and convex $h(\,)$. Naturally, economic theory does not prevent us from working with convex $h(\,)$, but we would like to know whether the standard approach to optimal taxation does. The answer is not clear. Operationally, occasionally researchers, see the contrast between the exposition of Diamond [1] and that of Saez [9], are willing to dismiss the distinction between $G(\,)$ and $h(\,)$ and focus on $\Gamma(\,)$ directly. However, if we believe that $\Gamma(\,)$ matters then we must allow for problems that involve convex $h(\,)$ as long as $\Gamma(\,) = G(h(\,))$ is concave. But, if we do so, we must conclude that we do not fundamentally dispute problems with convex $h(\,)$. However, if we do not then, knowing that $G(\,)$ captures the preferences of a policy maker, and as such is a parameter, we cannot exclude *concave* $G(\,)$ and *convex* $h(\,)$ that lead to *convex* $\Gamma(\,)$, i.e., to problems manageable with tools outlined in this paper. On the other hand if we choose to ignore problems with convex $h(\,)$ even if they lead to concave $\Gamma(\,)$ then we are bound to ignore optimal taxation problems with identical mathematical representation to those that we actually choose to tackle.

Therefore, we can conclude that if we choose, for some reason, to focus only on problems that do not allow for the convexity of $h(\,)$, then we are bound to arrive at contradictions and internal inconsistencies since we will be forced, as our examples indicate, to disregard optimal taxation problems with identical mathematical representations to those that we actually choose to solve. However, once we decide that there is nothing improper about the convexity of $h(\,)$, we have to conclude that problems that describe the preferences of the government with a concave function,

$G(\ )$, become admissible, even if $G(h(\ ))$ is convex. Therefore, we can conclude that problems that we choose to study in this paper are legitimate and belong to the class of optimal taxation problems and as such deserve attention and formal treatment.

We must admit, however, that extending the class of optimal taxation problems in the way we do in this paper creates some uneasiness as it appears that the forms of the solutions of optimal taxation problems can be very sensitive to the underlying mathematical specifications even though the key economic fundamentals are identical. Naturally, it is mathematically possible that problems that look a like can have very different solutions. Consequently, it may be the case that the solution to an optimal taxation problem when $G(h())$ is *just* concave can be obtained with Mirrlees's method and can look very different from the solution to an optimal taxation problem when $G(h(\ ))$ is *just* convex. However, this would imply that the framework of optimal taxation proposed by Mirrlees [7] cannot be used as a basis for a practical policy guideline. Consequently, it may be necessary to search for other approaches as argued by Saez and Piketty [11], and by Saez and Stantcheva [12].

Moreover, while it is true that most researchers choose to work with a concave function, $h(\ )$, this is probably done out of convenience as it ensures that a specific criterion voluntarily adopted by the literature is met. In particular, it is customary to assume, see Tresh [16], that the third Atkinson criterion is met. Specifically, it is requested that the marginal utility of income of the social welfare function be decreasing. While we share the view that such restrictions are not supported by economic theory, we want to emphasize that our assumption of convexity of $h(\ )$ need not violate Atkinson criterion and consequently our choices of $h(\ )$ should be considered as valid even by mainstream researchers. Specifically, if the social welfare function expressed in terms of individual incomes, y_a, where $a\epsilon[a_L, a_H]$, is of the form

$$W = \int_{a_L}^{a_H} G(h(y_a - \frac{1}{2}(\frac{y_a}{a})^2))f(a)da, \tag{88}$$

then the social marginal utility of income of an agent whose productivity is equal to a is given by

$$\frac{dW}{dy_a} = \frac{d}{dy_a}(G(h(y_a - \frac{1}{2}(\frac{y_a}{a})^2))f(a)), \tag{89}$$

which reduces, assuming that $G(\,) = h^{-1}(\,)$, to

$$\frac{dW}{dy_a} = \frac{d}{dy_a}(y_a - \frac{1}{2}(\frac{y_a}{a})^2)f(a) = (1 - \frac{y_a}{a^2})f(a), \tag{90}$$

which is a decreasing function as $\frac{d^2W}{dy_a^2} = -\frac{1}{a^2}f(a) < 0$.

Consequently, our choice of $h(\,)$ does not contradict the basic assumptions made in the literature. Furthermore, we want to emphasize that by assuming that $h(\,)$ is convex we do not undermine the underlying premise voiced by Atkinson that the social welfare function should encompass a motive for income redistribution. Clearly, social welfare functions that we consider in our paper exhibit properties that are consistent with Atkinson criterion. In addition, it is worth noting that from a purely mathematical perspective the convexity of $h(\,)$ does not preclude the marginal utility of income of individual utilities from declining. Specifically, if $h(x) = (1-nx^\alpha)^\beta$, then $h(\,)$ is increasing and convex, but at the same time the marginal utility of income is declining, thus ensuring that Atkinson criterion is met even for individual utilities.

6 Conclusions

In this paper, we use the utilitarian approach to optimal taxation and we argue that the solution to the optimal taxation problem need not be of the form typically identified in the literature. We explicitly identify optimal taxation problems with solutions in the form of a step function. We want to emphasize that we do not undermine the existing literature at the technical level. In fact, we consider our contribution to be complementary to the existing findings as it is applicable to a certain class of optimal taxation problems. Nevertheless, there are two drastically different ways to interpret our results.

First of all, it is possible to assert that our results are driven by our convenient choice of the utility function, which is true, and consequently, have no practical bearing. Such opinions, however, would be unjustified as there is nothing improper in our approach from the theoretical point of view. Moreover, the form of the utility function that we choose to work with, while special in mathematical sense, is fully admissible from the perspective of economic theory. Therefore, there are no formal grounds to disregard our results unless one is willing to use the simplicity of our approach as the basis for a rejection of our method.

Secondly, while it is true that from the formal perspective our contribution is technical it, nevertheless, casts some doubts on the approach to the optimal taxation utilized in the literature. Specifically, it is very much expected, given the consensus that it is admissible to embark on interpersonal comparisons of marginal utility, that solutions to optimal taxation problems can be very sensitive to the underlying mathematical specification of the model. However, as our results indicate the sensitivity manifests itself both at the quantitative and qualitative level. In particular, it can be the case that two economies described with identical fundamentals could be subjected to drastically different optimal taxation policies − one involving mostly positive marginal taxes and one relying mostly on zero marginal taxes − only because the preferences of the policy maker are marginally different or the representations of individual preferences are marginally distinct. Naturally, we can accept such situations simply as a mathematical fact, but even if we do, we should, it seems, revisit our stand on the applicability of the Mirrlees's [7] approach as a basis for a practical policy discussion.

Naturally, it is possible to interpret our findings in a dual manner and argue that our results just invalidate our initial assumption with regard to the mathematical properties of $h(\ )$. Consequently, it may be necessary to modify standard consumer theory by augmenting it with stricter restrictions on the forms of representation of preferences.

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A Appendix

It is our goal, in this Appendix, to establish that the variational procedures considered in this paper lead to a continuous change in the amount of revenue collected by the government. We start from the case when all values of income are chosen in a neighborhood of the highest income earned y_H. Moreover, let us consider a tax function, $\tau(\ )$, of the form given in figure (1) and let us assume that

$$\forall y \in [y_L, y_H] |\ |\tau(y)| < T \tag{91}$$

and

$$\forall a \in [a_L, a_H] |\ |f(a)| < \bar{f}. \tag{92}$$

Let us consider range of income levels $[y_M, y_H]$ and let us assume that $\tau(\ )$ is concave on $[y_M, y_H]$. Moreover, let us assume that all income levels from $[y_M, y_H]$ are actually chosen. Let us consider an agent whose productivity is equal to a who chooses $y_a^* \in [y_M, y_H]$. Naturally, efficiency requires that

$$1 - \tau'(y_a^*) = \frac{y_a^*}{a^2} \tag{93}$$

and

$$-\tau''(y_a^*) - \frac{1}{a^2} < 0. \tag{94}$$

Naturally, inequality (94) together with the assumption of concavity of $\tau()$ on $[y_M, y_H]$ imply that

$$-\frac{1}{a^2} < \tau''(y_a^*) < 0. \tag{95}$$

Consequently, the series of inequalities (95) allows us to establish that

$$\forall a |\ y_a^* \in [y_M, y_H] \Rightarrow |\tau''(y_a^*)| < \frac{1}{a^2}. \tag{96}$$

Now given our assumption that all incomes from $[y_M, y_H]$ are actually chosen we can rewrite (96) as

$$\forall y \in [y_M, y_H] | \, |\tau''(y)| < \frac{1}{a_L^2}, \tag{97}$$

i.e., the second derivative of $\tau(\,)$ remains bounded from above in absolute value.

Furthermore, given equation (93) we can write inequality (94) as

$$0 < \tau''(y_a^*) + \frac{1 - \tau'(y_a^*)}{y_a^*}. \tag{98}$$

Naturally, given that $\tau''(\,)$ and $\tau'(\,)$ exist the following function

$$m(y) = \tau''(y) + \frac{1 - \tau'(y)}{y} \tag{99}$$

is continuous. Therefore, we can write that

$$\forall y \in [y_M, y_H] | \, 0 < M_L \leq \tau''(y) + \frac{1 - \tau'(y)}{y} \leq M_H \tag{100}$$

since any continuous function reaches its bounds on a closed interval and inequality (98) holds for all $y_a^* \in [y_M, y_H]$.

Now, let us focus on the following modification of $\tau(\,)$

$$\tau_{\lambda,\bar{y}}(y) = \begin{cases} \tau(y) + \lambda & \text{for } y \leq \bar{y} \\ \tau(\bar{y}) + \lambda & \text{for } \bar{y} < y \end{cases}, \tag{101}$$

where $\bar{y}$ is assumed to be close enough to y_H.

Furthermore, let y_a^* denote the income level chosen by an agent whose productivity is equal to a when she faces $\tau(\,)$ and y_a^{**} be the level of income chosen by an agent whose productivity is equal to a when she faces $\tau_{\lambda,\bar{y}}()$. Naturally, in the latter case the revenue by the government collected is given by $R(\bar{y}) = \int_{a_L}^{a_H} \tau_{\lambda,\bar{y}}(y_a^{**}) f(a) da$. Let $\tilde{a}$ denote the level of productivity at which agents are just indifferent between remaining on the steep part of $\tau_{\lambda,\bar{y}}(\,)$ or on the flat part of $\tau_{\lambda,\bar{y}}(\,)$. Naturally, we must have

$$h(y_{\tilde{a}}^* - \tau(y_{\tilde{a}}^*) - \lambda - \frac{1}{2}(\frac{y_{\tilde{a}}^*}{\tilde{a}})^2) = h(\frac{1}{2}\tilde{a}^2 - \tau(\bar{y}) - \lambda), \tag{102}$$

which obviously reduces to

$$y_{\tilde{a}}^* - \tau(y_{\tilde{a}}^*) - \frac{1}{2}(\frac{y_{\tilde{a}}^*}{\tilde{a}})^2 = \frac{1}{2}\tilde{a}^2 - \tau(\bar{y}). \tag{103}$$

Naturally, $y_{\tilde{a}}^*$ is an optimal choice, so it must be the case that

$$1 - \tau'(y_{\tilde{a}}^*) = \frac{y_{\tilde{a}}^*}{\tilde{a}^2} \tag{104}$$

and

$$-\tau''(y_{\tilde{a}}^*) - \frac{1}{\tilde{a}^2} < 0. \tag{105}$$

Observe that we can express the revenue collected in this case as

$$R(\bar{y}) = \int_{a_L}^{\tilde{a}} \tau(y_a^*) f(a)da + \tau(\bar{y}) \int_{\tilde{a}}^{a_H} f(a)da + \lambda \int_{a_L}^{a_H} f(a)da. \tag{106}$$

Let us now consider a simple variation of $\tau_{\lambda,\bar{y}}(\ )$ of the form

$$\tau_{\lambda,\bar{y}-\varepsilon}(y) = \begin{cases} \tau(y) + \lambda & \text{for } y \leq \bar{y} - \varepsilon \\ \tau(\bar{y} - \varepsilon) + \lambda & \text{for } \bar{y} - \varepsilon < y \end{cases}. \tag{107}$$

Naturally, $\tau_{\lambda,\bar{y}-\varepsilon}(\ )$ is just the same as $\tau(\ ) + \lambda$ with its end flattened starting from $\bar{y} - \varepsilon$.

Let $\tilde{a}_\varepsilon$ be the value of productivity at which agents are just indifferent between paying a positive marginal tax and zero marginal tax when they face $\tau_{\lambda,\bar{y}-\varepsilon}(\ )$. Obviously, we must have

$$y_{\tilde{a}_\varepsilon}^* - \tau(y_{\tilde{a}_\varepsilon}^*) - \frac{1}{2}(\frac{y_{\tilde{a}_\varepsilon}^*}{\tilde{a}_\varepsilon})^2 = \frac{1}{2}\tilde{a}_\varepsilon^2 - \tau(\bar{y} - \varepsilon). \tag{108}$$

Furthermore, optimality of $y_{\tilde{a}_\varepsilon}^*$ requires that

$$1 - \tau'(y_{\tilde{a}_\varepsilon}^*) = \frac{y_{\tilde{a}_\varepsilon}^*}{\tilde{a}_\varepsilon^2}, \tag{109}$$

and

$$-\tau''(y_{\tilde{a}_\varepsilon}^*) - \frac{1}{\tilde{a}_\varepsilon^2} < 0. \tag{110}$$

The level of revenue collected in this case is given by

$$R(\bar{y}-\varepsilon) = \int_{a_L}^{\tilde{a}_\varepsilon} \tau(y_a^*) f(a) da + \tau(\bar{y}-\varepsilon) \int_{\tilde{a}_\varepsilon}^{a_H} f(a) da + \lambda \int_{a_L}^{a_H} f(a) da. \tag{111}$$

We can using (106) and (111) estimate the size of the difference between the two levels of revenue collected. In particular, we have

$$|R(\bar{y})-R(\bar{y}-\varepsilon)| = |\int_{\tilde{a}_\varepsilon}^{\tilde{a}} (\tau(y_a^*)-\tau(\bar{y}-\varepsilon)) f(a) da + (\tau(\bar{y})-\tau(\bar{y}-\varepsilon)) \int_{\tilde{a}}^{a_H} f(a) da|, \tag{112}$$

which can be bounded from above as follows

$$|R(\bar{y})-R(\bar{y}-\varepsilon)| \leq \int_{\tilde{a}_\varepsilon}^{\tilde{a}} |\tau(y_a^*)-\tau(\bar{y}-\varepsilon)| f(a) da + |\tau(\bar{y})-\tau(\bar{y}-\varepsilon)| \int_{\tilde{a}}^{a_H} f(a) da. \tag{113}$$

Furthermore, given that $\tau(\,)$ is bounded, restriction (91), and $f(\,)$ is bounded, restriction (92), we can write

$$|R(\bar{y}) - R(\bar{y}-\varepsilon)| \leq |\tilde{a} - \tilde{a}_\varepsilon| 2T\bar{f} + |\tau(\bar{y}) - \tau(\bar{y}-\varepsilon)| \int_{\tilde{a}}^{a_H} f(a) da. \tag{114}$$

Now using equations (104) and (109) we can establish that

$$|\frac{1}{\tilde{a}_\varepsilon^2} - \frac{1}{\tilde{a}^2}| = |\frac{1-\tau'(y^*_{\tilde{a}_\varepsilon})}{y^*_{\tilde{a}_\varepsilon}} - \frac{1-\tau'(y^*_{\tilde{a}})}{y^*_{\tilde{a}}}| = |\frac{-\tau''(y_c)y_c - (1-\tau'(y_c))}{y_c^2}||y^*_{\tilde{a}_\varepsilon} - y^*_{\tilde{a}}|, \tag{115}$$

where $y_c \in [y^*_{\tilde{a}_\varepsilon}, y^*_{\tilde{a}}]$.

Now, invoking inequality (97) and relying on $0 \leq |\tau'(\,)| \leq 1|$, finding due to Seade [13, 14], and noting that $y_c \in [y_L, y_H]$ we can use equation (115) to establish the following

$$|\frac{1}{\tilde{a}_\varepsilon^2} - \frac{1}{\tilde{a}^2}| < \frac{\frac{1}{a_L^2} y_H + 1}{y_L^2} |y^*_{\tilde{a}_\varepsilon} - y^*_{\tilde{a}}|, \tag{116}$$

which is equivalent to

$$|\tilde{a}_\varepsilon - \tilde{a}| < \frac{(\tilde{a}_\varepsilon \tilde{a})^2}{\tilde{a}_\varepsilon + \tilde{a}} \frac{\frac{1}{a_L^2} y_H + 1}{y_L^2} |y^*_{\tilde{a}_\varepsilon} - y^*_{\tilde{a}}|, \tag{117}$$

which in turn implies that

$$|\tilde{a}_\varepsilon - \tilde{a}| < \frac{a_H^4}{2a_L} \frac{\frac{1}{a_L^2} y_H + 1}{y_L^2} |y^*_{\tilde{a}_\varepsilon} - y^*_{\tilde{a}}|. \tag{118}$$

Observe that we can use equations (103) and (104) to establish that

$$y^*_{\tilde{a}} - \tau(y^*_{\tilde{a}}) - \frac{1}{2} y^*_{\tilde{a}} (1 - \tau'(y^*_{\tilde{a}})) - \frac{1}{2} \frac{y^*_{\tilde{a}}}{1 - \tau'(y^*_{\tilde{a}})} = -\tau(\bar{y}). \tag{119}$$

Similarly, using equations (108) and (109) we can show that

$$y^*_{\tilde{a}_\varepsilon} - \tau(y^*_{\tilde{a}_\varepsilon}) - \frac{1}{2} y^*_{\tilde{a}_\varepsilon} (1 - \tau'(y^*_{\tilde{a}_\varepsilon})) - \frac{1}{2} \frac{y^*_{\tilde{a}_\varepsilon}}{1 - \tau'(y^*_{\tilde{a}_\varepsilon})} = -\tau(\bar{y} - \varepsilon). \tag{120}$$

Now, by subtracting equation (119) from equation (120) and using Taylor expansion we can establish that

$$\frac{1}{2} |1 - \tau'(y^*_D) + y^*_D \tau''(y^*_D)| |1 - \frac{1}{(1 - \tau'(y^*_D))^2}| |y^*_{\tilde{a}_\varepsilon} - y^*_{\tilde{a}}| = |\tau(\bar{y}) - \tau(\bar{y} - \varepsilon)|, \tag{121}$$

where $y^*_D \in [y^*_{\tilde{a}_\varepsilon}, y^*_{\tilde{a}}]$.

We can rearrange equation (121) to

$$\frac{1}{2} |\frac{1 - \tau'(y^*_D)}{y^*_D} + \tau''(y^*_D)| |\frac{y^*_D}{1 - \tau'(y^*_D)}| |\tau^{'}(y^*_D)| |1 + \frac{1}{1 - \tau'(y^*_D)}| |y^*_{\tilde{a}_\varepsilon} - y^*_{\tilde{a}}| = |\tau(\bar{y}) - \tau(\bar{y} - \varepsilon)|. \tag{122}$$

Now, equation (122) implies, given (100), that

$$\frac{1}{2} M_L |\frac{y^*_D}{1 - \tau'(y^*_D)}| |\tau^{'}(y^*_D)| |1 + \frac{1}{1 - \tau'(y^*_D)}| |y^*_{\tilde{a}_\varepsilon} - y^*_{\tilde{a}}| \le |\tau(\bar{y}) - \tau(\bar{y} - \varepsilon)|. \tag{123}$$

Recall that by assumption $\tau''(y)$ is negative for $y \in [y_M, y_H]$ therefore $\tau'(\,)$ is an decreasing function on $[y_M, y_H]$. Thus, we can write

$$1 > \tau'(y^*_D) > \tau'(\bar{y}) > 0 \tag{124}$$

since $y^*_D \in [y^*_{\tilde{a}_\varepsilon}, y^*_{\tilde{a}}]$, and $y^*_{\tilde{a}} < \bar{y}$, and optimal marginal taxes are smaller than 1, see Seade [13, 14].

Naturally (124) is equivalent to

$$0 < 1 - \tau'(y_D^*) < 1 - \tau'(\bar{y}) < 1, \tag{125}$$

which in turn implies that

$$1 < \frac{1}{1 - \tau'(\bar{y})} < \frac{1}{1 - \tau'(y_D^*)}. \tag{126}$$

Obviously, the above inequalities lead to

$$2 < |1 + \frac{1}{1 - \tau'(y_D^*)}|. \tag{127}$$

Thus we can inequality (123) can be rewritten as

$$M_L|\frac{y_D^*}{1 - \tau'(y_D^*)}||\tau^{'}(y_D^*)||y_{\tilde{a}_\varepsilon}^* - y_{\tilde{a}}^*| \leq |\tau(\bar{y}) - \tau(\bar{y} - \varepsilon)|. \tag{128}$$

Now, invoking inequalities (124) we can rewrite the above inequality as

$$M_L|\frac{y_D^*}{1 - \tau'(y_D^*)}||\tau^{'}(\bar{y})||y_{\tilde{a}_\varepsilon}^* - y_{\tilde{a}}^*| \leq |\tau(\bar{y}) - \tau(\bar{y} - \varepsilon)|. \tag{129}$$

Furthermore, given our initial assumption, y_D^* is actually chosen by someone, i.e., we have

$$1 - \tau'(y_D^*) = \frac{y_D^*}{a_D^2}. \tag{130}$$

Thus,

$$a_D^2 = \frac{y_D^*}{1 - \tau'(y_D^*)} \tag{131}$$

and, hence, inequality (129) becomes

$$M_L a_D^2 |\tau^{'}(\bar{y})||y_{\tilde{a}_\varepsilon}^* - y_{\tilde{a}}^*| \leq |\tau(\bar{y}) - \tau(\bar{y} - \varepsilon)|, \tag{132}$$

which in turn leads to, as $a_L < a_D$,

$$|y_{\tilde{a}_\varepsilon}^* - y_{\tilde{a}}^*| \leq \frac{|\tau(\bar{y}) - \tau(\bar{y} - \varepsilon)|}{M_L a_L^2 |\tau^{'}(\bar{y})|}. \tag{133}$$

Therefore, combining inequality (118) with inequality (133) we can write the following

$$|\tilde{a}_{\varepsilon} - \tilde{a}| < \frac{a_H^4}{2a_L}\frac{\frac{1}{a_L^2}y_H + 1}{y_L^2}|y^*_{\tilde{a}_{\varepsilon}} - y^*_{\tilde{a}}| \leq \frac{a_H^4}{2a_L}\frac{\frac{1}{a_L^2}y_H + 1}{y_L^2}\frac{|\tau(\bar{y}) - \tau(\bar{y} - \varepsilon)|}{M_L a_L^2 |\tau'(\bar{y})|}. \tag{134}$$

Noting that $y_H = a_H^2$, $y_L = a_L^2$, and $1 < \frac{a_H^2}{a_L^2}$ we can rewrite the above inequality as

$$|\tilde{a}_{\varepsilon} - \tilde{a}| \leq \frac{a_H^6}{a_L^7}\frac{|\tau(\bar{y}) - \tau(\bar{y} - \varepsilon)|}{M_L |\tau'(\bar{y})|}. \tag{135}$$

Furthermore, we can now rewrite inequality (114) as

$$|R(\bar{y}) - R(\bar{y} - \varepsilon)| \leq \frac{a_H^6}{a_L^7}\frac{|\tau(\bar{y}) - \tau(\bar{y} - \varepsilon)|}{M_L |\tau'(\bar{y})|}2T\bar{f} + |\tau(\bar{y}) - \tau(\bar{y} - \varepsilon)| \int_{\tilde{a}}^{a_H} f(a)da, \tag{136}$$

which reduces, as $a_L \leq \tilde{a}$ and consequently $\int_{\tilde{a}}^{a_H} f(a)da \leq \int_{a_L}^{a_H} f(a)da$, to

$$|R(\bar{y}) - R(\bar{y} - \varepsilon)| \leq (\frac{a_H^6}{a_L^7}\frac{2T\bar{f}}{M_L |\tau'(\bar{y})|} + \int_{a_L}^{a_H} f(a)da)|\tau(\bar{y}) - \tau(\bar{y} - \varepsilon)| \tag{137}$$

Obviously, using Taylor expansion inequality (137) can be rewritten as

$$|R(\bar{y}) - R(\bar{y} - \varepsilon)| \leq (\frac{a_H^6}{a_L^7}\frac{2T\bar{f}}{M_L |\tau'(\bar{y})|} + \int_{a_L}^{a_H} f(a)da)|\tau^{'}(\bar{y} - \theta_B\varepsilon)|\varepsilon, \tag{138}$$

where $\theta_B \in [0, 1]$.

Finally, following Seade [13, 14] and assuming that marginal tax rates are bounded from above and below we can conclude that $|\tau^{'}(\bar{y} - \theta_B\varepsilon)| < 1$ and hence inequality (138) becomes

$$|R(\bar{y}) - R(\bar{y} - \varepsilon)| \leq \varepsilon(\frac{a_H^6}{a_L^7}\frac{2T\bar{f}}{M_L |\tau'(\bar{y})|} + \int_{a_L}^{a_H} f(a)da), \tag{139}$$

which confirms that $R(\bar{y})$ is a continuous function of $\bar{y}$ as the expression multiplying ε in (139) is a constant. Therefore, we can conclude that our variational procedure leads to a continuous change in the amount of revenue collected by the government if all values of income are chosen in the range, $[y_M, y_L]$.

We still need to consider the remaining case when some income levels are originally unchosen. Let (y_1, y_2) be the highest range of unchosen incomes. Naturally, we can expect that bunching occurs in this case at y_1. Furthermore, let us consider modifications of the original tax functions of the form

$$\tau_{\lambda,\bar{y}}(y) = \begin{cases} \tau(y) + \lambda & \text{for } y \leq y_1 \\ \infty & \text{for } y_1 < y \leq y_2 \\ \tau(\bar{y}) + \lambda & \text{for } y_2 < y \end{cases}, \tag{140}$$

where $\bar{y} \in (y_1, y_2]$.

Let us denotes the range of individual productivities that lead to bunching at y_1 with $[a_1, a_2]$. Naturally, we must, in particular, have

$$h(y_1 - \tau(y_1) - \lambda - \frac{1}{2}(\frac{y_1}{a_2})^2) = h(y_2 - \tau(\bar{y}) - \lambda - \frac{1}{2}(\frac{y_2}{a_2})^2), \tag{141}$$

which reduces to

$$y_1 - \tau(y_1) - \frac{1}{2}(\frac{y_1}{a_2})^2 = y_2 - \tau(\bar{y}) - \frac{1}{2}(\frac{y_2}{a_2})^2 \tag{142}$$

and, further, implies that

$$\frac{1}{a_2^2} = 2\frac{y_2 - y_1 + \tau(y_1) - \tau(\bar{y})}{y_2^2 - y_1^2}. \tag{143}$$

The amount of revenue collected when agents face $\tau_{\lambda,\bar{y}}(\,)$ described with (140) is given by

$$R(\bar{y}) = \int_{a_L}^{a_1} \tau(y_a^*) f(a) da + \tau(y_1) \int_{a_1}^{a_2} f(a) da + \tau(\bar{y}) \int_{a_2}^{a_H} f(a) da + \lambda \int_{a_L}^{a_H} f(a) da. \tag{144}$$

Let us now consider the following modification of $\tau_{\lambda,\bar{y}}(y)$

$$\tau_{\lambda,\bar{y}-\varepsilon}(y) = \begin{cases} \tau(y) + \lambda & \text{for } y \leq y_1 \\ \infty & \text{for } y_1 < y \leq y_2 \\ \tau(\bar{y} - \varepsilon) + \lambda & \text{for } y_2 < y \end{cases}. \tag{145}$$

Naturally, tax liabilities dictated by $\tau_{\lambda,\bar{y}-\varepsilon}(\,)$ above y_2 are smaller than those dictated by $\tau_{\lambda,\bar{y}}(\,)$. Therefore, the range of productivities for which agents choose to bunch at y_1 shrinks. Let us denote the new range with $[a_1, a_2^{\varepsilon}] \subset [a_1, a_2]$.

Observe that in this case indifference between choosing $y_1$and y_2 by agents whose productivity is equal to a_2^ε implies that, an analog of (143),

$$\frac{1}{(a_2^\varepsilon)^2} = 2\frac{y_2 - y_1 + \tau(y_1) - \tau(\bar{y} - \varepsilon)}{y_2^2 - y_1^2}. \tag{146}$$

The level of revenue collected by the government when agents face $\tau_{\lambda,\bar{y}-\varepsilon}()$ is given by

$$R(\bar{y}) = \int_{a_L}^{a_1} \tau(y_a^*)f(a)da + \tau(y_1)\int_{a_1}^{a_2^\varepsilon} f(a)da + \tau(\bar{y} - \varepsilon)\int_{a_2^\varepsilon}^{a_H} f(a)da + \lambda\int_{a_L}^{a_H} f(a)da. \tag{147}$$

Now, using (144) and (147) we can establish that

$$|R(\bar{y}) - R(\bar{y}-\varepsilon)| = |(\tau(y_1) - \tau(\bar{y}-\varepsilon))\int_{a_2^\varepsilon}^{a_2} f(a)da + (\tau(\bar{y}) - \tau(\bar{y}-\varepsilon))\int_{a_2}^{a_H} f(a)da|, \tag{148}$$

which implies

$$|R(\bar{y}) - R(\bar{y}-\varepsilon)| \leq |\tau(y_1) - \tau(\bar{y}-\varepsilon)|\int_{a_2^\varepsilon}^{a_2} f(a)da + |\tau(\bar{y}) - \tau(\bar{y}-\varepsilon)|\int_{a_2}^{a_H} f(a)da \tag{149}$$

and in turn

$$|R(\bar{y}) - R(\bar{y} - \varepsilon)| \leq |a_2 - a_2^\varepsilon| 2T\bar{f} + |\tau^{'}(\bar{y} - \theta_C\varepsilon)|\varepsilon\int_{a_2}^{a_H} f(a)da, \tag{150}$$

where $\theta_c \in (0,1)$, which further leads to, as marginal tax rates are bounded from above and below and $a_L < a_2$,

$$|R(\bar{y}) - R(\bar{y} - \varepsilon)| \leq |a_2 - a_2^\varepsilon| 2T\bar{f} + \varepsilon\int_{a_L}^{a_H} f(a)da. \tag{151}$$

Equations (143) and (146) imply that

$$\frac{1}{(a_2^\varepsilon)^2} - \frac{1}{a_2^2} = 2\frac{\tau(\bar{y}) - \tau(\bar{y} - \varepsilon)}{y_2^2 - y_1^2}, \tag{152}$$

which leads to

$$|a_2 - a_2^\varepsilon| = 2\frac{(a_2 a_2^\varepsilon)^2}{a_2 + a_2^\varepsilon}\frac{|\tau'(\bar{y} - \theta_D\varepsilon)|\varepsilon}{y_2^2 - y_1^2} \tag{153}$$

and, in turn, implies that

$$|a_2 - a_2^\varepsilon| \leq \frac{a_H^4}{a_L(y_2^2 - y_1^2)}\varepsilon. \tag{154}$$

Combining (151) and (154) we can establish that

$$|R(\bar{y}) - R(\bar{y} - \varepsilon)| \leq \varepsilon(\frac{a_H^4}{a_L(y_2^2 - y_1^2)}2T\bar{f} + \int_{a_L}^{a_H} f(a)da), \tag{155}$$

which confirms continuity of $R(\bar{y})$ since $\frac{a_H^4}{a_L(y_2^2-y_1^2)}2T\bar{f} + \int_{a_L}^{a_H} f(a)da$ is a constant.

Therefore, we have established that the simple variational procedures applied in this text lead to a continuous change in the amount of revenue collected. Consequently, we can invoke mean values theorems and argue that

$$\exists \bar{y}_R \in [y_L, y_H]| \ R(\bar{y}_R) = R. \tag{156}$$

B Appendix

In this appendix we show that the social welfare function as measured by the objective functional increases when $G(h(\ ))$ is convex and the optimal tax function, $\hat{\tau}(\ )$, is replaced with $\hat{\tau}_{\lambda,\bar{y}_R}(\ )$, which gives the same amount of revenue.

Observe that by revealed preference we can always write that

$$\forall a\epsilon[a_L, a_H]| \ h(aL_a^{**} - \hat{\tau}_{\lambda,\bar{y}_R}(aL_a^{**}) - \frac{1}{2}(L_a^{**})^2) \geq h(aL_a^* - \hat{\tau}_{\lambda,\bar{y}_R}(aL_a^*) - \frac{1}{2}(L_a^*)^2). \tag{157}$$

Recall, relationship (28), that

$$\hat{\tau}_{\lambda,\bar{y}_R}(aL_a^*) = \begin{cases} \hat{\tau}(aL_a^*) + \lambda & \text{for } aL_a^* \leq \bar{y}_R \\ \hat{\tau}(\bar{y}_R) + \lambda & \text{for } aL_a^* > \bar{y}_R \end{cases}. \tag{158}$$

Therefore, we can rewrite $\hat{\tau}_{\lambda,\bar{y}_R}(\ )$ as

$$\hat{\tau}_{\lambda,\bar{y}_R}(aL_a^*) = \hat{\tau}(aL_a^*) + \lambda_a, \tag{159}$$

where λ_a is given by

$$\lambda_a = \begin{cases} \lambda & \text{for } a \text{ such that } aL^*_a \leq \bar{y}_R \\ \lambda + \hat{\tau}(\bar{y}_R) - \hat{\tau}(aL^*_a) & \text{for } a \text{ such that } aL^*_a > \bar{y}_R \end{cases}. \tag{160}$$

Recall that $\bar{y}_R$, equation (30), was chosen to ensure revenue neutrality, i.e., we can write

$$\int_{a_L}^{a_H} \lambda_a f(a) da = 0. \tag{161}$$

Furthermore, given that incomes chosen are non-decreasing in a and $\hat{\tau}(\ )$ is a non-decreasing function we can conclude that $\lambda_a' s$ are non-increasing, i.e.,

$$\forall a_1, a_2 \epsilon [a_L, a_H] |\ a_1 \leq a_2 \Rightarrow \lambda_{a_1} \geq \lambda_{a_2}. \tag{162}$$

Furthermore, given the revenue neutrality condition (161), we can conclude that

$$\exists a_M \epsilon [a_L, a_H] |\ \lambda_a \geq 0 \text{ for } a \leq a_M \text{ and } \lambda_a \leq 0 \text{ for } a \geq a_M. \tag{163}$$

Observe that we can rewrite (157) as

$$\forall a \epsilon [a_L, a_H] |\ h(aL^{**}_a - \hat{\tau}_{\lambda,\bar{y}_R}(aL^{**}_a) - \frac{1}{2}(L^{**}_a)^2) \geq h(aL^*_a - \hat{\tau}(aL^*_a) - \lambda_a - \frac{1}{2}(L^*_a)^2), \tag{164}$$

which, in particular, implies, as $G(\ )$ is an increasing function,

$$\forall a \epsilon [a_L, a_H] |\ G(h(aL^{**}_a - \hat{\tau}_{\lambda,\bar{y}_R}(aL^{**}_a) - \frac{1}{2}(L^{**}_a)^2)) \geq G(h(aL^*_a - \hat{\tau}(aL^*_a) - \lambda_a - \frac{1}{2}(L^*_a)^2)) \tag{165}$$

and consequently,

$$\int_{a_L}^{a_H} G(h(aL^{**}_a - \hat{\tau}_{\lambda,\bar{y}_R}(aL^{**}_a) - \frac{1}{2}(L^{**}_a)^2)) f(a) da \geq \int_{a_L}^{a_H} G(h(aL^*_a - \hat{\tau}(aL^*_a) - \lambda_a - \frac{1}{2}(L^*_a)^2)) f(a) da. \tag{166}$$

Let us denote $G(h(\ ))$ with $B(\ )$. Now, we can expand the right hand side of inequality (166) can be expanded as

$$\int_{a_L}^{a_H} B(aL^*_a - \hat{\tau}(aL^*_a) - \lambda_a - \frac{1}{2}(L^*_a)^2) f(a) da = \int_{a_L}^{a_H} (B(x^*_a) - B^{'}(x^*_a)\lambda_a + \frac{1}{2}B^{''}(x^*_a)\theta^2_a \lambda^2_a) f(a) da, \tag{167}$$

where $x_a^* = aL_a^* - \hat{\tau}(aL_a^*) - \frac{1}{2}(L_a^*)^2$ and $\theta_a \epsilon [0,1]$.

Naturally, it must be the case that agents whose productivity is higher must be better of than agents whose productivity is lower when they face the optimal tax function $\hat{\tau}(\,)$, which, in particular, implies

$$\forall a_1, a_2 \epsilon [a_L, a_H] |\; a_1 \geq a_2 \Rightarrow x_{a_1}^* \geq x_{a_2}^* \tag{168}$$

and in turn, given that $B(\,)$ by assumption is increasing and convex,

$$\forall a_1, a_2 \epsilon [a_L, a_H] |\; a_1 \geq a_2 \Rightarrow B^{'}(x_{a_1}^*) \geq B^{'}(x_{a_2}^*). \tag{169}$$

Furthermore, in particular, we can write

$$\forall a \epsilon [a_L, a_M] |\; B^{'}(x_{a_M}^*) \geq B^{'}(x_a^*) \tag{170}$$

and

$$\forall a \epsilon [a_M, a_H] |\; B^{'}(x_a^*) \geq B^{'}(x_{a_M}^*). \tag{171}$$

Recall, relationship (163), that λ_a is positive for $a \leq a_M$, so we can rewrite (170) as

$$\forall a \epsilon [a_L, a_M] |\; \lambda_a B^{'}(x_{a_M}^*) \geq \lambda_a B^{'}(x_a^*) \tag{172}$$

and λ_a is negative for $a \geq a_M$, so we can rewrite (171) as

$$\forall a \epsilon [a_M, a_H] |\; \lambda_a B^{'}(x_{a_M}^*) \geq \lambda_a B^{'}(x_a^*). \tag{173}$$

Now, combining equations (172) and (173) we can write

$$\forall a \epsilon [a_L, a_H] |\; \lambda_a B^{'}(x_{a_M}^*) \geq \lambda_a B^{'}(x_a^*), \tag{174}$$

which, in turn, implies

$$\int_{a_L}^{a_H} \lambda_a B^{'}(x_{a_M}^*) f(a) da \geq \int_{a_L}^{a_H} \lambda_a B^{'}(x_a^*) f(a) da. \tag{175}$$

Naturally, we can rewrite inequality (175) as

$$B^{'}(x^*_{a_M})\int_{a_L}^{a_H}\lambda_a f(a)da \geq \int_{a_L}^{a_H}\lambda_a B^{'}(x^*_a)f(a)da, \tag{176}$$

which, given (161), becomes

$$0 \geq \int_{a_L}^{a_H}\lambda_a B^{'}(x^*_a)f(a)da. \tag{177}$$

Now, we can establish a lower bound on the right hand side of (167)

$$\begin{aligned}
&\int_{a_L}^{a_H}(B(x^*_a) - B^{'}(x^*_a)\lambda_a + \frac{1}{2}B^{''}(x^*_a)\theta_a^2\lambda_a^2)f(a)da = \\
&\int_{a_L}^{a_H}B(x^*_a)f(a)da - \int_{a_L}^{a_H}B^{'}(x^*_a)\lambda_a f(a)da + \int_{a_L}^{a_H}\frac{1}{2}B^{''}(x^*_a)\theta_a^2\lambda_a^2 f(a)da \geq \\
&\int_{a_L}^{a_H}B(x^*_a)f(a)da + \int_{a_L}^{a_H}\frac{1}{2}B^{''}(x^*_a)\theta_a^2\lambda_a^2 f(a)da > \\
&\int_{a_L}^{a_H}B(x^*_a)f(a)da,
\end{aligned} \tag{178}$$

where the last inequality holds due to the assumed concavity of $B(\,)$. Therefore, we can use (167) and (178) to establish that

$$\int_{a_L}^{a_H}B(aL^*_a - \hat{\tau}(aL^*_a) - \lambda_a - \frac{1}{2}(L^*_a)^2)f(a)da > \int_{a_L}^{a_H}B(aL^*_a - \hat{\tau}(aL^*_a) - \frac{1}{2}(L^*_a)^2)f(a)da, \tag{179}$$

which when combined with (166) implies that, recall that $B(\,) = G(h(\,))$.

$$\int_{a_L}^{a_H}G(h(aL^{**}_a - \hat{\tau}_{\lambda,\bar{y}_R}(aL^{**}_a) - \frac{1}{2}(L^{**}_a)^2))f(a)da > \int_{a_L}^{a_H}G(h(aL^*_a - \hat{\tau}(aL^*_a) - \frac{1}{2}(L^*_a)^2))f(a)da. \tag{180}$$

Naturally, inequality (180) contradicts our initial assumption that $\hat{\tau}()$ was optimal as welfare could be increased by defining tax liabilities according to $\hat{\tau}_{\lambda,\bar{y}_R}(\,)$ rather than $\hat{\tau}(\,)$. Therefore, we have just established that a given tax function cannot be optimal unless it is flat in a neighborhood of the highest earned income when both $G(\,)$ and $h(\,)$ are increasing, $G(\,)$ is concave, $h(\,)$ is convex, and $G(h(\,))$ is convex.