The Realized RSDC model

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Abstract

We introduce a new model for the joint dependence between returns and realized measures of volatility for multiple assets. This model borrows the methodology of the Realized GARCH of Hansen, Huang, and Shek (2012) to model the conditional variance of the returns. To model the dependence between the assets' returns, we introduce a regime switching model where (i) the correlations are constant within the regimes but different across regimes and (ii) both the returns and realized measures are subject to this regime switching process. Contrary to the RSDC model of Pelletier (2006), we can estimate the correlation process of the returns using the realized measures on top of the returns. An empirical application using transaction level data illustrates the performance of the model.

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1 Introduction

It is well known that the variance and covariance of most financial time series are time-varying. Modeling time-varying variance matrices has important impacts in terms of asset allocation, asset pricing, computation of Value-at-Risk (VaR). Despite a deep literature on univariate modeling of volatility using (daily) low-frequency observations with the GARCH models [*e.g.* Engle (1982) and Bollerslev (1986)], it took until the Dynamic Conditional Correlation models of Engle (2002) and Tse and Tsui (2002) for the topic to really develop.

Over the last years, starting with Andersen and Bollerslev (1998) the focus of volatility modeling has shifted from returns computed with low-frequency observations (such as daily returns) to high-frequency transaction data. Under some general assumptions, we can get very precise measurements of volatility by taking intra-day returns (say 5 minutes), squaring them and taking the sum. A growing literature has studied issues such as the asymptotic properties of these estimators [*e.g.*, Barndorff-Nielsen and Shephard (2002)], optimal sampling frequency [*e.g.* Bandi and Russell (2008)], robustness to jumps [*e.g.* Barndorff-Nielsen and Shephard (2004)], robustness to market microstructure noise [*e.g.* Hansen and Lunde (2006)]. The focus of this literature has mostly been on the measurement of volatility, less so on the modeling and forecasting of volatility.

In a recent paper, Hansen, Huang, and Shek (2012) have introduced a new framework, the Realized GARCH, for using realized measures to improve the modeling of the returns. The idea is to supplement a GARCH-type equation for the conditional variance of the return where we use lagged realized measures instead of lagged squared returns with a measurement equation linking contemporaneously the realized measure and the conditional variance.

Our goal in this paper is to extend the Realized GARCH framework of Hansen, Huang, and Shek (2012) to vectors of returns and corresponding realized (co-)measures. We do this by building on the Regime Switching Dynamic Correlation (RSDC) model of Pelletier (2006). The idea is to decompose the conditional covariance matrix of the returns into standard deviations and correlations as with DCC-type models and link the realized variance-covariance matrix to conditional variances and correlations through a measure equation. The correlation matrices evolve according to a regime switching process: they are constant within a regime but different across regimes. The setup of the model allows a two-step estimation procedure where the univariate Realized GARCH models are estimated first followed by the correlation model. The correlation model can be estimated with only the information from the returns, only the information from the realized measures, or with both. An empirical application with transaction data suggest that realized measures as more informative about correlations than returns.

The paper is organized as follows. Section 2 introduces the Realized RSDC model. The estimation of the model is discussed in Section 3. An overview of the computation of realized measures is given in Section 4. Section 5 presents an empirical application with transaction-level data for three stocks. Concluding remarks are given in Section 6.

2 The Realized RSDC Model

Contrary to the basic GARCH literature, it is assumed that not only we observe a $(n \times 1)$ vector $Y_t = [y_{1,t}, \ldots, y_{n,t}]'$ of daily log-returns but we also observe a corresponding $(n \times n)$ realized covariance matrix X_t computed from high-frequency intra-day returns. We will refer to the elements on the diagonal of X_t as $x_{i,t}$ for $i = 1, \ldots, n$ and element on row i and column j with $i \neq j$ of X_t as $x_{ij,t}$. Details about the computation of X_t and its properties are discussed below. We denote the conditional variance covariance matrix of the returns Y_t by H_t. We start by assuming that the returns are generated by

$$Y_t = H_t^{1/2} U_t, \tag{1}$$

with U_t being i.i.d. $N(0, I_n)$.

As with many models in the DCC literature (*e.g.* Bollerslev (1990), Engle (2002), Pelletier (2006), Tse and Tsui (2002)), we are interested in decomposing the conditional covariance matrix H_t into standard deviations and correlations: $H_t = D_t \Gamma_t D_t$ where $D_t = diag(\sqrt{h_{1,t}}, \sqrt{h_{2,t}}, ..., \sqrt{h_{n,t}})$ and Γ_t is a correlation matrix.

We assume that the conditional variances $h_{i,t}$ follow a univariate Realized GARCH model similar to Hansen, Huang, and Shek (2012):

$$h_{i,t} = \omega_i + \beta_i h_{i,t-1} + \alpha_i x_{i,t-1}. \tag{2}$$

Although we consider the linear specification of the Realized GARCH, we could also use some of the other formulations proposed by Hansen, Huang, and Shek (2012).

As in the RSDC model of Pelletier (2006) the correlation matrix Γ_t follows a regime switching model. There are M possible regimes and within a regime the correlation matrix is constant:

$$\Gamma_t = \sum_{i=1}^{M} \Gamma^{(i)} \mathbb{1}_{\{s_t = i\}}. \tag{3}$$

The unobserved variable s_t represents the state at time t and the indicator function $\mathbb{1}_{\{s_t=i\}}$ is equal to when the condition is true. It is assumed that at least one correlation in regime is different than in regime j so that $\Gamma^{(i)} \neq \Gamma^{(j)}$ if $i \neq j$. We denote by Π the associated transition probability matrix. The probability of going from regime i in period t to regime j in period $t+1$ is denoted by $\pi_{i,j}$ and the limiting probability of being in regime i is π_i. The element on row j and column i of Π is $\pi_{i,j}$. We make the standard assumptions on the Markov chain [aperiodic, irreducible and ergodic. See Ross (1993, Chapter 4)]

As in the Realized GARCH model, we also introduce a measurement equation linking contemporaneously the (scaled) realized measure X_t and the latent variances and correlations. We decompose the realized covariance matrix similarly to the approach of the DCC model:

$$\begin{aligned}
X_t &= \tilde{D}_t C_t \tilde{D}_t, & (4)\\
\tilde{D}_t &= diag\left(\sqrt{\tilde{h}_{1,t}}, \ldots, \sqrt{\tilde{h}_{n,t}}\right), & (5)\\
\tilde{h}_{i,t} &= \gamma_i^{(0)} + \gamma_i^{(1)} h_{i,t}, & (6)\\
C_t &\sim Wishart\left(\Gamma_t / v, v\right). & (7)
\end{aligned}$$

We can start by discussing Equation (6). It is similar to the linear specification of the Realized GARCH model of Hansen, Huang, and Shek (2012). In their specification, the contemporaneous link between the realized variance $x_{i,t}$ and the latent conditional variance $h_{i,t}$ of the daily return (neglecting a leverage effect term) is $x_{i,t} = \gamma_i^{(0)} + \gamma_i^{(1)} h_{i,t} + \epsilon_{i,t}$. We have a similar specification but instead of having an additive error term, we have a multiplicative error term: The realized variance is equal to $\tilde{h}_{i,t}$ times the element (i,i) of a random matrix C_t.

In turn, this random matrix C_t follows a Wishart distribution with location and scale parameters Γ_t/v and v respectively. Conditional on $\{\Gamma_t\}$, $\{C_t\}$ are independent. Two implications of Equation (7) are $E[C_t|\Gamma_t] = \Gamma_t$ and $Var[C_{ij,t}|\Gamma_t] = (\gamma_{ij,t}^2 + \gamma_{ii,t}\gamma_{jj,t})/v$. It also follows that the marginal distribution of $x_{i,t}$ is

$$c_{i,t} = \frac{x_{i,t}}{\tilde{h}_{i,t}} \sim \frac{\chi^2_{(v)}}{v}. \tag{8}$$

An additional implication of Equation (7) is that the scaled realized covariance matrix C_t is driven by the regime switching process. In fact, it's only the off-diagonal elements of C_t that are affected by the regime switching. Finally, it is assumed that C_t is independent of U_t.

3 Estimation

It is in theory possible to estimate all the parameters of the model at once by maximizing the joint log-likelihood of $\{Y_t, X_t\}_{t=1,\ldots,T}$. To maximize the likelihood we need to evaluate

$$\mathcal{L}(\theta; \boldsymbol{Y}, \boldsymbol{X}) = \sum_{t=1}^{T} \ln f(Y_t, X_t|\underline{Y_{t-1}}, \underline{X_{t-1}}) \tag{9}$$

where $\underline{Y_{t-1}} = \{Y_{t-1}, Y_{t-2}, \ldots\}$, $\underline{X_{t-1}} = \{X_{t-1}, X_{t-2}, \ldots\}$ and $\boldsymbol{\theta}$ is the vector of parameter values. To do this we use Hamilton's filter [Hamilton (1989), Hamilton (1994, chapter 22)] because the Markov chain s_t is unobserved. Inference on the state of the Markov chain is given by the following equations:

$$\hat{\xi}_{t|t} = \frac{(\hat{\xi}_{t|t-1} \odot \eta_t)}{\mathbf{1}'(\hat{\xi}_{t|t-1} \odot \eta_t)}, \tag{10}$$

$$\hat{\xi}_{t+1|t} = \Pi\hat{\xi}_{t|t}, \tag{11}$$

$$\eta_t = \begin{bmatrix} f(Y_t, X_t|\underline{Y_{t-1}}, \underline{X_{t-1}}, s_t = 1; \boldsymbol{\theta}) \\ \vdots \\ f(Y_t, X_t|\underline{Y_{t-1}}, \underline{X_{t-1}}, s_t = M; \boldsymbol{\theta}) \end{bmatrix}, \tag{12}$$

where $\hat{\xi}_{t|t}$ is an $(M \times 1)$ vector which contains the probability of being in each regime at time t conditional on the observations up to time t. The $(M \times 1)$ vector $\hat{\xi}_{t+1|t}$ gives these probabilities at time $t+1$ conditional on observations up to time t. The m-th element of the $(M \times 1)$ vector η_t is the density of (Y_t, X_t) conditional on past observations and being in regime m at time t, $\mathbf{1}$ is an $(M \times 1)$ vector of 1s, and $\odot$ denotes elements-by-elements multiplication.

Given a starting value $\hat{\xi}_{1|0}$ and parameter values $\boldsymbol{\theta}$, one can iterate over (10) and (11) for $t = 1, \ldots, T$. The log-likelihood is obtained as a by-product of this algorithm:

$$\mathcal{L}(\boldsymbol{\theta}) = \sum_{t=1}^{T} \log \left(\mathbf{1}'(\hat{\xi}_{t|t-1} \odot \eta_t) \right). \tag{13}$$

Smoothing inference on the state of the Markov chain can also be computed using an algorithm developed by Kim (1994b). The probability of being in each regime at time t conditional on observations up to time T is given by the following equation:

$$\hat{\xi}_{t|T} = \hat{\xi}_{t|t} \odot \left\{ \Pi' \left[\hat{\xi}_{t+1|T} \, (\div) \, \hat{\xi}_{t+1|t} \right] \right\} \tag{14}$$

where $(\div)$ denotes element-by-element division. One would start iterating over (14) with $t = T$, where $\hat{\xi}_{T|T}$ is given by (10).

Unfortunately the model would be susceptible to a curse of dimensionality as the number of parameters that would have to be estimated through a nonlinear optimization grows (quadraticly) with the sample size.

3.1 Estimation of the univariate Realized GARCH models

As in the CCC model of Bollerslev (1990), the DCC model of Engle (2002) or the RSDC model of Pelletier (2006), we can consider a two-step estimation so as to break this curse of dimensionality.

We first begin by introducing elements of notation. The complete parameter space $\boldsymbol{\theta}$ is split into $\boldsymbol{\theta_1}$ for the parameters in the univariate Realized GARCH models ($\boldsymbol{\theta_1} = \{\omega_i, \alpha_i, \beta_i, \gamma_i^{(0)}, \gamma_i^{(1)}, v\}_{i=1\ldots,n}$) and $\boldsymbol{\theta_2}$ for the parameters in the regime switching model ($\boldsymbol{\theta_2} = \{\Gamma^{(1)}, \ldots, \Gamma^{(n)}, \Pi\}$). We denote by

$\mathcal{L}_1$ the log-likelihood where the correlation matrix is taken to be an identity matrix:

$$\begin{aligned}\mathcal{L}_1(\boldsymbol{\theta_1}; \boldsymbol{Y}, \boldsymbol{X}) &= -\frac{1}{2}\sum_{t=1}^{T}\sum_{i=1}^{n}\left\{(\ln(2\pi) + \ln(h_{i,t}) + u_{i,t}^2) + (2-v)\ln\left(\frac{vx_{i,t}}{\tilde{h}_{i,t}}\right)\right. \\ &\qquad \left. + \frac{vx_{i,t}}{\tilde{h}_{i,t}} - v\ln(2) - \ln\left(\Gamma\left(\frac{v}{2}\right)\right) + \ln\left(\frac{v}{\tilde{h}_{i,t}}\right)\right\}. \quad (15)\end{aligned}$$

In a first step, we can get a consistent estimate of $\boldsymbol{\theta_1}$ by maximizing $\mathcal{L}_1$. Since it is the sum of n independent log-likelihood, we can simply estimate the univariate Realized GARCH models one at a time. Evaluation of these log-likelihoods is straightforward since they do not the use of Hamilton's filter.

3.2 Estimation of the Regime switching model

We denote by $\mathcal{L}_2$ the log-likelihood given $\boldsymbol{\theta_1}$ where we have concentrate out $h_{i,t}$ and $\tilde{h}_{i,t}$, $\mathcal{L}_2(\boldsymbol{\theta}_2; \hat{\boldsymbol{U}}, \hat{\boldsymbol{C}}, \boldsymbol{\theta}_1)$. We would apply Hamilton's filter to get the log-likelihood with

$$\eta_t = [f(\hat{U}_t, \hat{C}_t | s_t = m, \boldsymbol{\theta})]_{m=1,\ldots,M}, \quad (16)$$

$$f(\hat{U}_t, \hat{C}_t | s_t = m, \boldsymbol{\theta}) = f_N(\hat{U}_t | 0, \Gamma^{(m)}) f_W(\hat{C}_t \big| \Gamma^{(m)}/v, v), \quad (17)$$

where $f_N(\bullet|0,\Gamma)$ is the density of a multivariate normal distribution with mean zero and variance Γ, and $f_W(\bullet|\Gamma, v)$ is the density of a Wishart distribution with location Γ and scale v.

At a cost of an additional loss of efficiency, we could also consider the estimation of the correlations and the regime switching process using either only the standardized returns ($\mathcal{L}_{2,y}$ with $\eta_t = [f_N(\hat{U}_t|0, \Gamma^{(m)})]_{m=1,\ldots,M}$) or the rescaled realized covariance matrices ($\mathcal{L}_{2,x}$ with $\eta_t = [f_W(\hat{C}_t|\Gamma^{(m)}/v, v)]_{m=1,\ldots,M}$)

In the following, we show that by using the EM algorithm to maximize the likelihood we can break the curse of dimensionality in that the correlation matrices $\Gamma^{(1)}, \ldots, \Gamma^{(M)}$ can be estimated through simple weighted sums. The algorithm consist in a set of updating equations which perform alternatively the Expectation and Maximization parts of the EM algorithm.

The smoothed joint and conditional probabilities needed for the updating equations are computing using a forward-backward inference algorithm as explained in Hamilton (1990) or Kim (1994a) and pioneered by Baum, Petrie, Soules, and Weiss (1970). As shown in Dempster, Laird, and Rubin (1977), the likelihood function is locally maximized after the above iterations converge.

The EM algorithm guaranties, given some regularity conditions, that the likelihood function is locally maximized after the above iterations converge (see section 4 in the appendix).

The two iterative steps of the EM algorithm are:

- **Expectation:** We compute the expected value of the log-likelihood with respect to the unknown states given the data and the current guess of the parameter values ($\theta_2^{(k)}$): $Q(\theta_2^{(k+1)}|\theta_2^{(k)})$.
- **Maximization:** We find $\theta_2^{(k+1)}$ such that $Q(\theta_2^{(k+1)}|\theta_2^{(k)})$ is maximized.

The two above steps are repeated until the log-likelihood is (locally) maximized.

If we do the estimation using only the information in the log-returns, then the correlation model becomes the RSDC model of Pelletier (2006). In this case, for a given value of the parameters $\theta_2^{(k)}$, we run Hamilton's filter with $\eta_t = [f_N(\hat{U}_t|0, \Gamma^{(m)})]_{m=1,\ldots,M}$ to get $P(s_t = j, s_{t-1} = i|\hat{\mathbf{U}}, \theta_2^{(k)})$ and $P(s_t = j|\hat{\mathbf{U}}, \theta_2^{(k)})$, then we have an explicit form for $\theta_2^{(k+1)}$:

$$
\begin{aligned}
\hat{\pi}_{ij} &= \frac{\sum_{t=1}^{T} P(s_t = j, s_{t-1} = i|\hat{\mathbf{U}}, \theta_2^{(k)})}{\sum_{t=1}^{T} P(s_{t-1} = i|\mathbf{U}, \theta_2^{(k)})}, && (18)\\
\hat{\Gamma}^{(m)} &= \frac{\sum_{t=1}^{T} \hat{U}_t \hat{U}_t' P(s_t = m|\hat{\mathbf{U}}, \theta_2^{(k)})}{\sum_{t=1}^{T} P(s_t = m|\hat{\mathbf{U}}, \theta_2^{(k)})}. && (19)
\end{aligned}
$$

If we do the estimation using only the rescaled realized covariance matrices C_t, we run Hamilton's filter with $\eta_t = [f_W(\hat{C}_t|\Gamma^{(m)}/v, v)]_{m=1,\ldots,M}$ to get $P(s_t = j, s_{t-1} = i|\hat{\mathbf{U}}, \theta_2^{(k)})$ and $P(s_t = j|\hat{\mathbf{U}}, \theta_2^{(k)})$ and we have an an explicit form for $\theta_2^{(k+1)}$:

$$
\begin{aligned}
\hat{\pi}_{ij} &= \frac{\sum_{t=1}^{T} P(s_t = j, s_{t-1} = i|\hat{\mathbf{U}}, \theta_2^{(k)})}{\sum_{t=1}^{T} P(s_{t-1} = i|\mathbf{U}, \theta_2^{(k)})}, && (20)\\
\hat{\Gamma}^{(m)} &= \frac{\sum_{t=1}^{T} C_t P(s_t = m|\hat{\mathbf{U}}, \theta_2^{(k)})}{\sum_{t=1}^{T} P(s_t = m|\hat{\mathbf{U}}, \theta_2^{(k)})}. && (21)
\end{aligned}
$$

If we use both log-returns and rescaled realized covariances to estimate the correlation model, then Hamilton's filter would be used with η_t as described in Equations (16) and (17). As for the updating the correlation matrices, taking partial derivatives of the expected log-likelihood with respect to $\Gamma^{(m)}$, gives the following third order polynomial equation in $\Gamma^{(m)}$:

$$\Gamma^{(m)^2} v \left(\Gamma^{(m)} \sum_{t=1}^{T} P(s_t = m) - \sum_{t=1}^{T} \hat{C}_t P(s_t = m) \right) = - \sum_{t=1}^{T} \hat{U}_t \hat{U}_t' P(s_t = m) + \Gamma^{(m)} \sum_{t=1}^{T} P(s_t = m) \tag{22}$$

We see that it has a particular form. If the right-hand side of Equation (22) was set to zero, the resulting $\Gamma^{(m)}$ would be the same as Equation (19). If it is the left-hand side of (22) that is set to zero, then the resulting $\Gamma^{(m)}$ would be the same as Equation (21). It is well known that scalar cubic polynomial equations have an explicit solution but for the moment we have not explored using both returns and realized covariances to estimate the correlation model in the empirical section.

4 Computation of realized covariance matrix

In this section we explain who the realized covariance matrices are computed. We employ the multivariate realized kernel of Barndorff-Nielsen, Hansen, Lunde, and Shephard (2011). An exposition of their method is presented below.

As it is common in the literature, we assume that the vector of efficient log prices is modeled as a Brownian semimartingale defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$:

$$Y(t) = \int_0^t a(s)ds + \int_0^t \sigma(s)dB(s) \tag{23}$$

Where $a(t)$ and $\sigma(t)$ are the predictable processes for the drift and the instantaneous volatility matrices. $B(t)$ is a vector of independent Brownian motions. The integrated covariance process is then

$$[Y](t) = \int_0^t \sigma(s)\sigma(s)'ds \tag{24}$$

Consider the quadratic variation, $v_n(t) = \sum_{i=0}^{n-1}(Y(t_{i+1}^n) - Y(t_{i+1}^n))(Y(t_{i+1}^n) - Y(t_{i+1}^n))'$ for a partition $\{t_i^n\}$ of $[0, t]$. It can be shown that $v_n(t)$ converges in probability to $[Y](t)$. See for example Klebaner (2005, Section 8.5) and Protter (2004, Chapter 2, Section 6). Therefore by increasing the sampling frequency over a day worth of data we can approximate the integrated covariance by the quadratic variation. However, the high frequency observed prices or quotes are not the ones described in equation (23) because of the increased microstructure noises. Additionally, in a multivariate framework, the data is irregularly spaced and non-synchronuous causing a bias toward zero, the so called Epps effect [see Epps (1979)]. We use the procedure proposed by Barndorff-Nielsen, Hansen, Lunde, and Shephard (2011), (BNHLS henceforth), to construct the matrices of realized covariances that correct for non-synchronuous transactions and presence of noise.

The steps for computing the matrices of realized covariances are the following:

1. Synchronizing the data: The time clock adopted for each multivariate observation is the time at which the last asset is traded for the ith trade. In BNHLS(2011), this is referred as the refresh time. More precisely, let τ_t be the refresh time then: $\tau_1 = max(t_1^{(1)}, t_1^{(2)}, ..., t_1^{(n)})$ and subsequent refresh times are $\tau_{j+1} = max(t^{(1)}_{N_{\tau_j}+1}, t^{(2)}_{N_{\tau_j}+1}, ..., t^{(3)}_{N_{\tau_j}+1})$ where N_{τ_j} is a counts the transaction up to refresh time τ_j. We obtain then N sampling times.

2. Removing end effects: The asymptotics results of BNHLS(2011) requires that the first m and last m prices be averaged (jittering end conditions). They suggests using $m = 2$.

3. Construct vectors of returns : based on the refresh time and the removal of the end effects, the return vectors are constructed. if m observations are used to remove the end effects as done above, the sample size of return is n such that $n - 1 + 2m = N$

4. We compute the realized covariances by kernel estimation:

$$X_t = \sum_{h=0}^{H} K\left(\frac{h}{H}\right)(\gamma_h + \gamma_h') \tag{25}$$

   Where, $K(h/H)$ is the Parzen kernel function[1] and H is the bandwidth. The bandwidth for each day is selected following the prescriptions of Barndorff-Nielsen, Hansen, Lunde, and Shephard (2009) and Barndorff-Nielsen, Hansen, Lunde, and Shephard (2009).

[1] $K(x) = 1 - 6x^2 + 6x^3$ if $0 \leq x \leq 0.5$, $= 2(1-x)^3$ if $0.5 \leq x \leq 1$, $= 0$ if $x > 1$.

5 Empirical results

5.1 Data description

To illustrate our model we apply it to milli-second transaction data for three stocks (Dell, IBM and Microsoft) over the period January 3, 2006 to March 30, 2012. A very important step in the use of high frequency is the data cleaning process. We start by removing days where the transaction period is less than the normal 6.5 hours (Independence Day, the days before Thanksgiving and Christmas) are dropped. We are left with 1,580 trading days after removing between 12 and 13 days. For the remaining trading days, we clean the dataset following the recommendations of Barndorff-Nielsen, Hansen, Lunde, and Shephard (2009). This includes restricting analysis only to the regular trading hours, that is from 9:30am to 4:00pm EST. Furthermore, we only consider transaction records with a correction indicator equal to zero or one. We also exclude the transactions with a sale condition equal to Z, which means that the transaction is reported to the tape at the time later than it occurred. We occasionally see multiple trades recorded with the same timestamp. They are trades from multiple buyers or sellers or split-transactions occurring when the volume of an order on one side of the market is larger than the available opposing orders. We follow the common practice of aggregating these trades as a single transaction and use the volume-weighted price as the price for this aggregated transaction.

As mentioned above the synchronization scheme adopted is the refresh time procedure. A measure of the data retention by the synchronizing process is $p = \frac{n\,N}{\sum_{i=1}^{n} n^i}$ where n is the number of assets, n^i the initial number of transactions for the i^{th} asset and N is the final common number of transactions selected after synchronization. Therefore, p is the degree to which we keep data. In this application, the minimum of that statistic is 37% while the maximum is 80.25% and the mean over the time period is 54%.

After the data jittering and the construction of the log-returns, we proceed with the bandwidth selection. As recommended by Barndorff-Nielsen, Hansen, Lunde, and Shephard (2008) and Barndorff-Nielsen, Hansen, Lunde, and Shephard (2011), we apply a univariate optimal mean square error bandwidth selection to each asset and we choose the maximum of the bandwidth each day for the kernel estimation. The minimum, maximum and mean of the bandwidths are respectively 54.49, 115.34 and

76.76. Since we are using the Parzen kernel, this also corresponds to the number of autocovariances included in the computation of the realized covariance matrices.

Descriptive statistics for the daily log-returns and realized volatility are presented in Table 1. As expected, the mean daily log-return is very close to zero. The worst single-day returns are about -10% for IBM and Microsoft during this period and -15.8% for Dell, while the highest single-day gains are between 7% (IBM) and 13% (Dell). As for the realized volatility, we see that the least and most volatile stocks over this period are respectively IBM and Dell. In terms of daily realized volatility, we see that for each stock the most volatile day can be 10 times or more than the least volatile day. Plots of the daily log-returns, daily annualized realized volatility and daily realized correlations can be found in Figures 1, 6 and 6 respectively.

Table 1: Summary of daily returns and realized volatility (realized standard error) from January, 03 2006 to March,30 2012

Stocks	Log-returns			
	Mean	Std. dev	Min	Max
$DELL$	-0.0004	0.0250	-0.1558	0.1369
IBM	0.0006	0.0145	-0.0983	0.0744
$MSFT$	0.0001	0.0176	-0.1074	0.1314
	Annualized realized volatility			
$DELL$	35.74	14.92	15.10	186.15
IBM	19.35	12.68	6.32	146.94
$MSFT$	24.53	10.37	12.19	122.02

5.2 Estimation results: Univariate Realized GARCH

The individual estimation of the univariate Realized GARCH models, which are based on a combination of a Gaussian term and a Chi-square term, are presented in Table 2. For the parameters of the GARCH equation, compared to some of the results in Hansen, Huang, and Shek (2012) we get estimates β that have more persistence in the variance and estimates of α corresponding to lesser impacts of lagged realized variance on the variance of the return. Also, as in Hansen, Huang, and Shek (2012), our estimates of $\gamma^{(1)}$ are also less than one, suggesting that the variance measured while the market is

Table 2: Parameter estimates for the univariate Realized GARCH models.

Parameters	DELL	IBM	MSFT
$\hat{\omega}$	0.0000	0.0000	0.0000
	(0.0000)	(0.0000)	(0.0000)
$\hat{\beta}$	0.7058	0.7278	0.5749
	(0.0206)	(0.0067)	(0.0043)
$\hat{\alpha}$	0.3658	0.3075	0.4655
	(0.0158)	(0.0045)	(0.0023)
$\hat{\gamma}^{(0)}$	0.0000	0.0000	0.0000
	(0.0000)	(0.0000)	(0.0000)
$\hat{\gamma}^{(1)}$	0.8492	0.9159	0.8731
	(0.0186)	(0.0111)	(0.0054)
$\hat{v}$	20.3071	8.5886	20.6307
	(0.5371)	(0.1505)	(0.2635)

open corresponds to between 84% to 92% of the daily variance.

As for the scale parameter v of the underlying Wishart distribution, it is rather precisely estimated. We can see that two of the three stocks (Dell and Microsoft) give very similar estimates (about 20), while the estimate obtained with IBM is much smaller (about 8.6). Remembering that with our specification, the variance of the Wishart is inversely proportional to v, implying that the scaled realized variances for IBM are more volatile than for Dell and Microsoft. Since the model implies that this parameter v should be the same for each series, this suggest that the measurement equation ($\tilde{h}_{i,t}$) for IBM could be incorrectly specified.

5.3 Estimation Results: Correlations

Given the estimates of the univariate Realized GARCH models, we can compute the standardized returns $\hat{U}_t$ and the scaled realized covariance matrices $\hat{C}_t$, and estimate the regime switching correlation model. As discussed previously, we can do it with just the information for the standardized returns, or with just the information from the scaled covariance matrices, or with both series. We have not yet implemented the later one (looked into solving cubic equations of correlation matrices).

Before discussing the estimation results, we need to discuss the time series properties of $\hat{C}_t$. In Figure 6 we present the elements on the diagonal. Figure 6 presents the elements off the diagonal.

Table 3: Estimation results for the correlation model applied to the scaled daily returns

Parameters	Regime 1	Regime 2
$\hat{\Gamma}_{DELL,IBM}$	0.6016	0.1880
	(0.0322)	(0.0365)
$\hat{\Gamma}_{DELL,MSFT}$	0.5837	0.1761
	(0.0327)	(0.0554)
$\hat{\Gamma}_{IBM,MSFT}$	0.6549	0.1852
	(0.0284)	(0.0603)
$\hat{\pi}_{11}$	0.9016	
	(0.0308)	
$\hat{\pi}_{22}$	0.7885	
	(0.0800)	

First, we can see that there are outliers; very infrequent observations that take much larger values than the rest of the observations. The most serious outliers are two observations for IBM that take values well above 20 (compared to an average value of one). This might certainly be a reason why $\hat{\upsilon}$ for IBM is much smaller than for the other two stocks. The help visualize the time series behavior of these series, in Figures 6 and 6, we shrink the range of the vertical axis so the outliers don't dominate the rest of the observations. As expected, the series for the off-diagonal elements appears more persistent (Figure 6) than for the elements on the diagonal (Figure 6), since the regime switching only affects the mean value off the diagonal.

We present first the results for the standardized returns assuming there are two regimes ($M = 2$). The parameters estimates can be found in Table 3 and the filtered and smoothed probabilities of being in each regime can be found in Figures 6 and 6 respectively. We see that based on the returns, we get one regime with high correlations and one regime with low correlations (similar to the results of Pelletier (2006) with exchange rate data). The regime switching process is only moderately persistent as can be seen in the figures.

When estimating the regime switching model with the daily rescaled covariance matrices, we also have a regime of high correlation and a regime of low correlation, although the "high correlation" are not as high as when the model is estimated with returns. The estimates and standard errors can be found in Table 4. Also in contrast to results with returns, we found much more persistence in the

Table 4: Estimation results for the correlation model applied to the scaled daily realized covariance matrices

Parameters	Regime 2	Regime 1
$\hat{\Gamma}_{DELL,IBM}$	0.3156	0.1648
	(0.0607)	(0.0147)
$\hat{\Gamma}_{DELL,MSFT}$	0.3062	0.1614
	(0.0635)	(0.0116)
$\hat{\Gamma}_{IBM,MSFT}$	0.3989	0.2087
	(0.0512)	(0.0249)
$\hat{\pi}_{22}$	0.9749	
	(0.0075)	
$\hat{\pi}_{11}$	0.9946	
	(0.0056)	

correlations. The estimates $\hat{\pi}_{11}$ and $\hat{\pi}_{22}$ are much closer to zero, and in Figures 6 and 6 the probabilities stay close to 1/0 for much longer stretches. This suggest that the scaled realized covariance matrices are much more informative about the evolution of the correlations than the daily returns, which is not surprising since we know that realized variance is much better measurement of the latent variance than the (squared) returns.

6 Conclusion

In this paper we present the Realized RSDC model, a new model for vectors of daily returns and corresponding realized measures. This model borrows from the univariate Realized GARCH model of Hansen, Huang, and Shek (2012) by having a measurement equation linking the realized measure and the latent conditional variance matrix, as well as the RSDC model of Pelletier (2006) by modeling the conditional correlation matrix as a regime switching model. This model does not have a curse of dimensionality since it can be easily estimated in two steps as with the DCC-type models. In an empirical application, we show that correlations do appear to switch over time. We also can see that realized measures are more informative about correlations than the returns. This paper is quite incomplete and many things more things should be done, including an out-of-sample forecasting study to evaluate the relative performance of this model.

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Figure 1: Daily log-returns from January 3, 2006 to March 30, 2012.

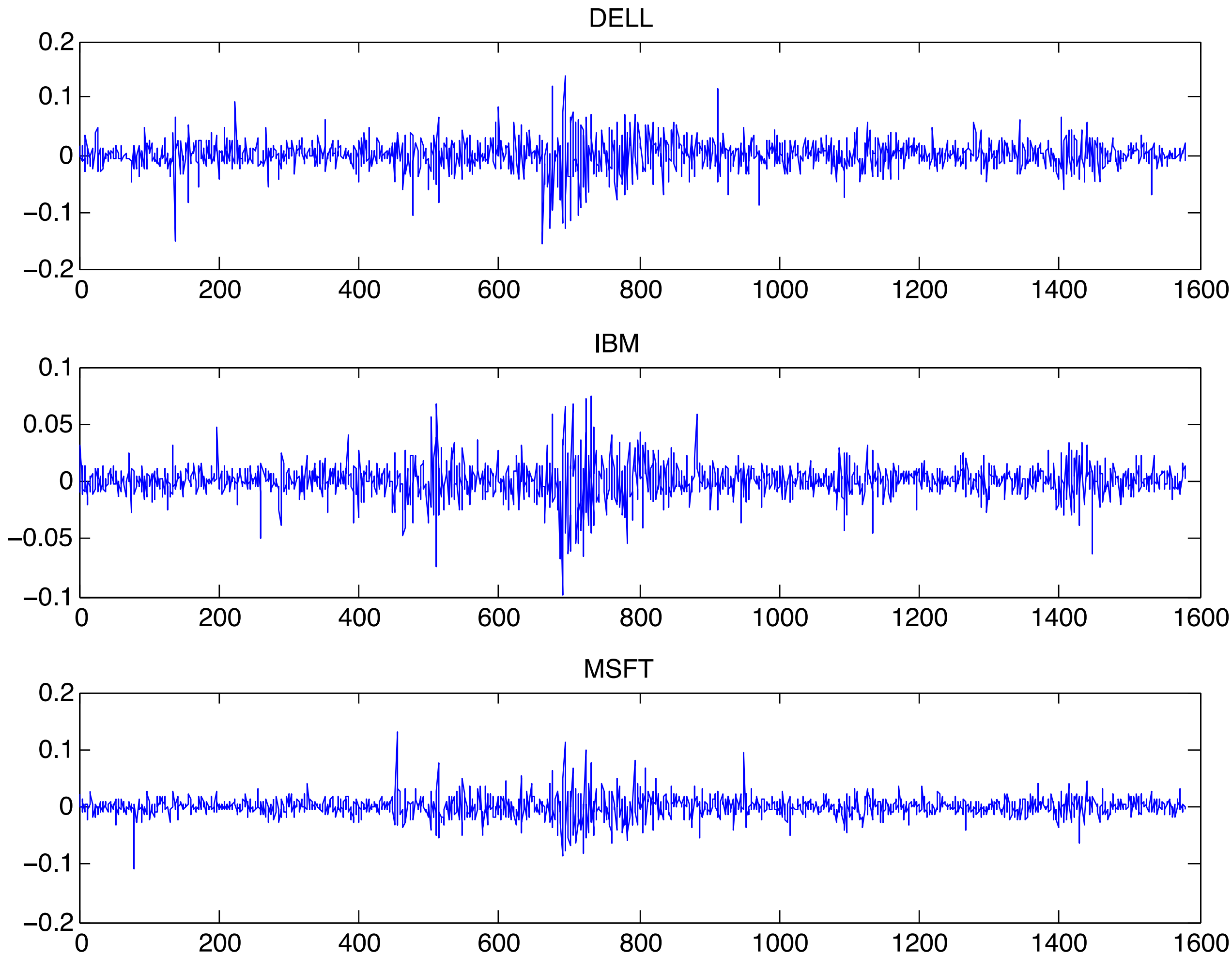


Figure 2: Daily annualized realized volatility from January 3, 2006 to March 30, 2012.

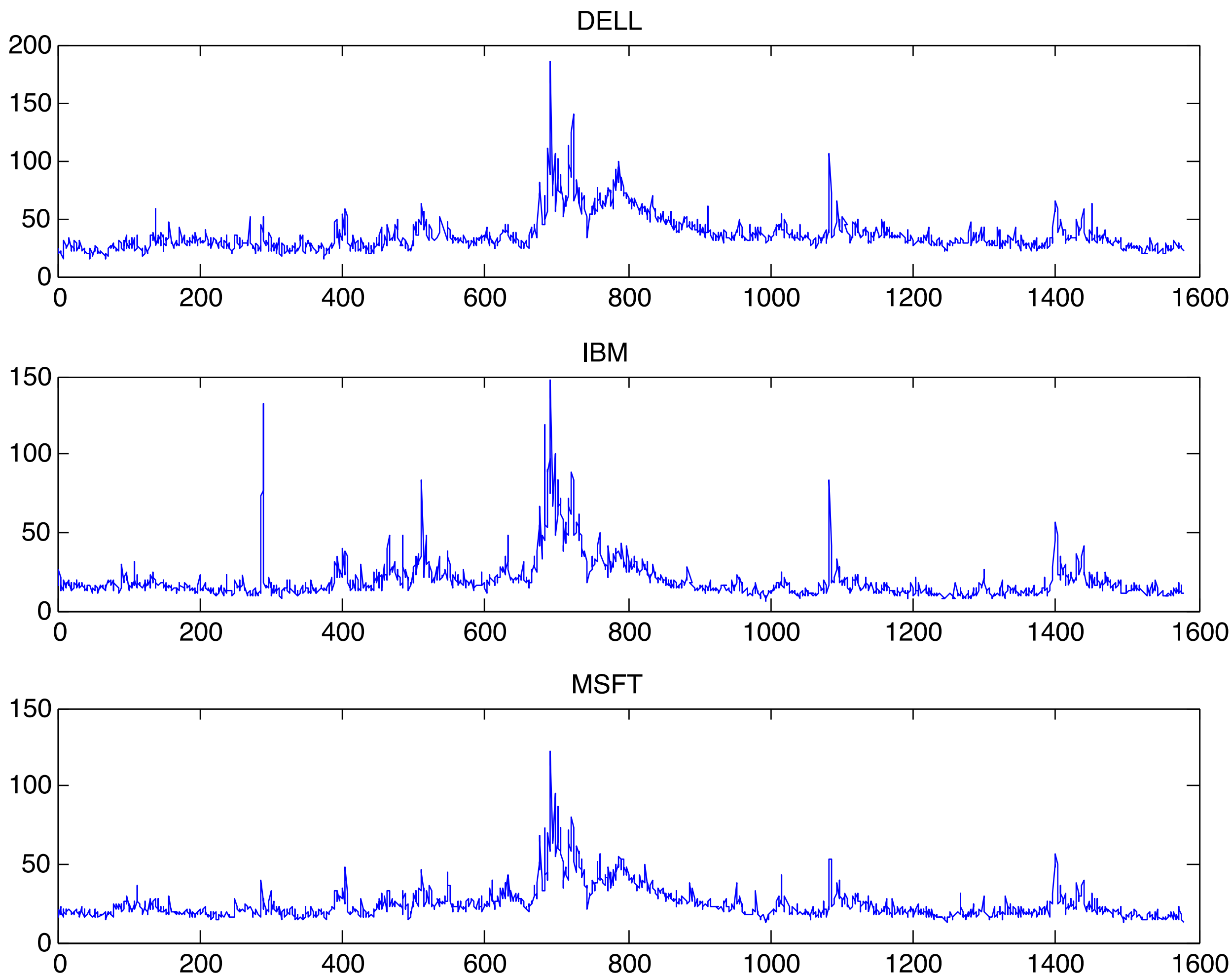


Figure 3: Daily realized correlations from January 3, 2006 to March 30, 2012.

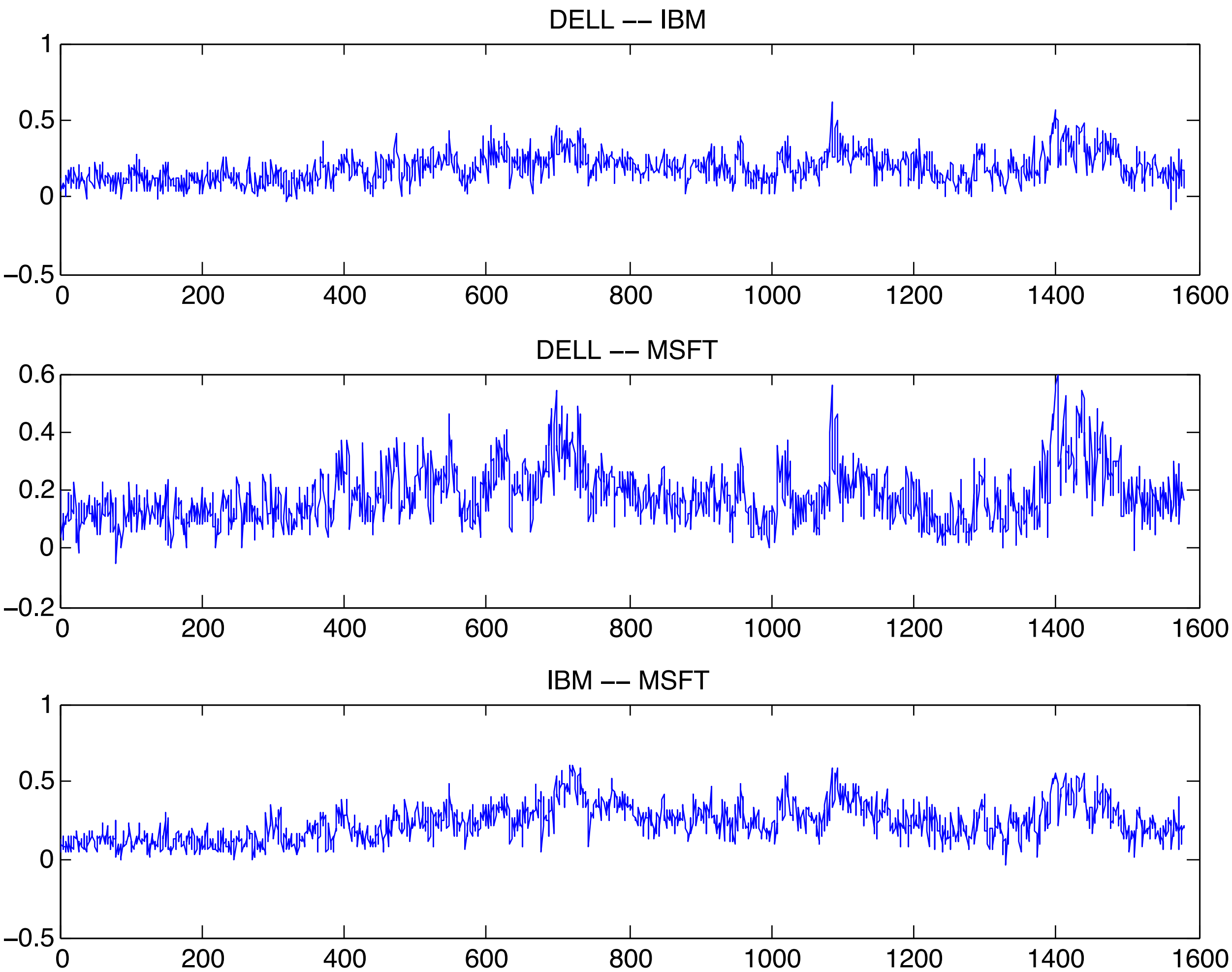


Figure 4: Daily standardized realized variance ($\hat{C}_{i,t}$) from January 3, 2006 to March 30, 2012.

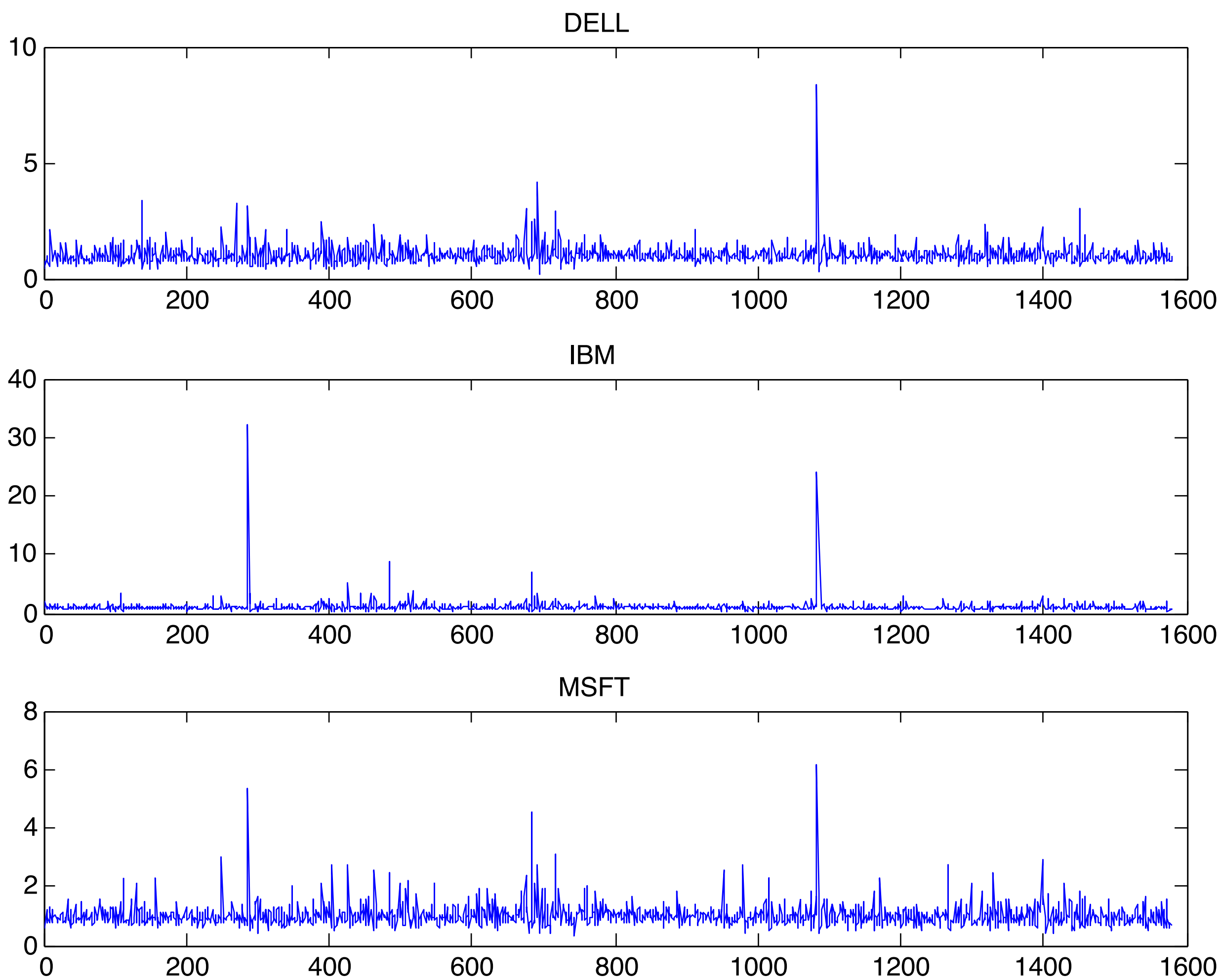


Figure 5: Daily standardized realized variance ($\hat{C}_{i,t}$) from January 3, 2006 to March 30, 2012.

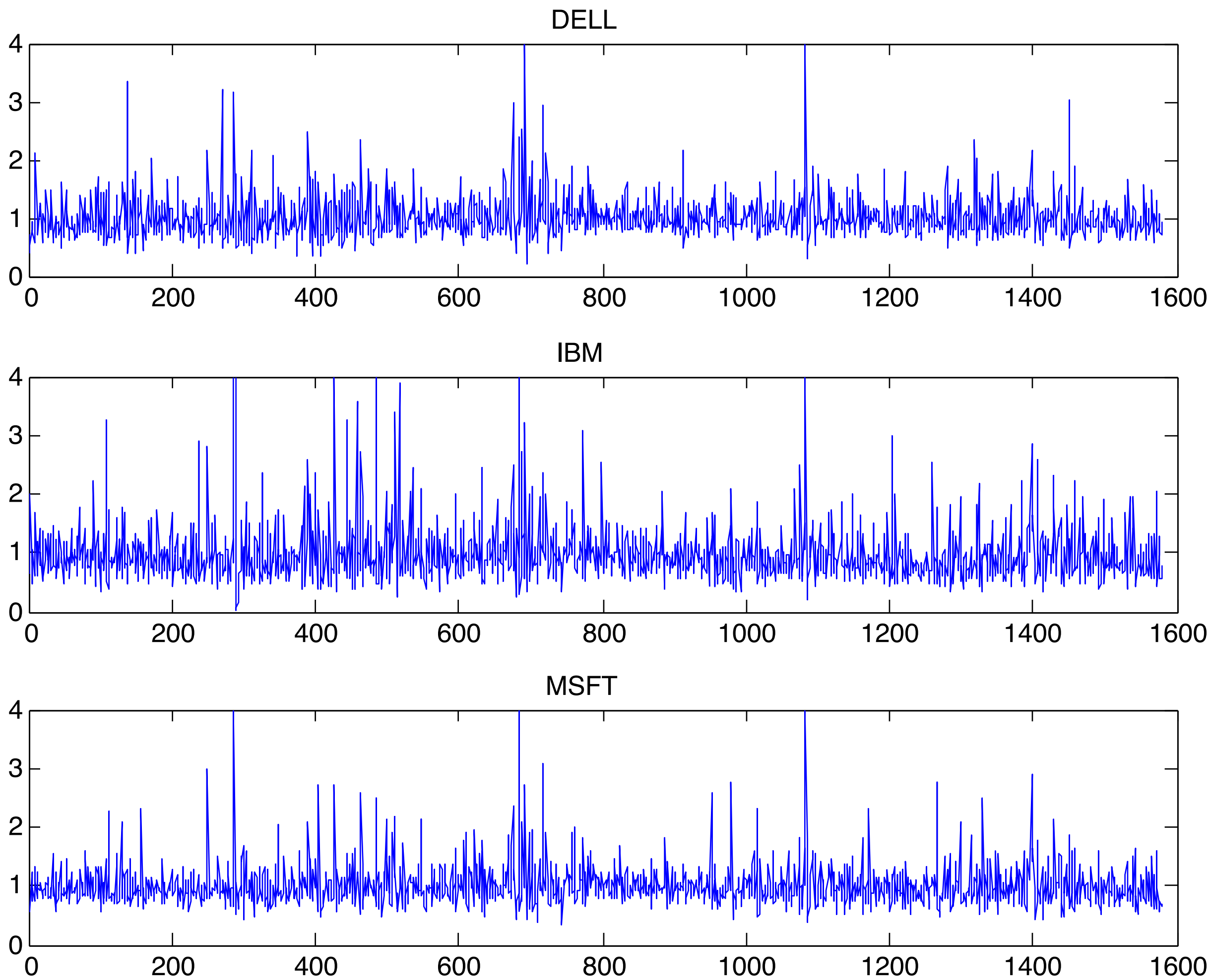


Figure 6: Daily standardized realized covariance ($\hat{C}_{ij,t}$) from January 3, 2006 to March 30, 2012.

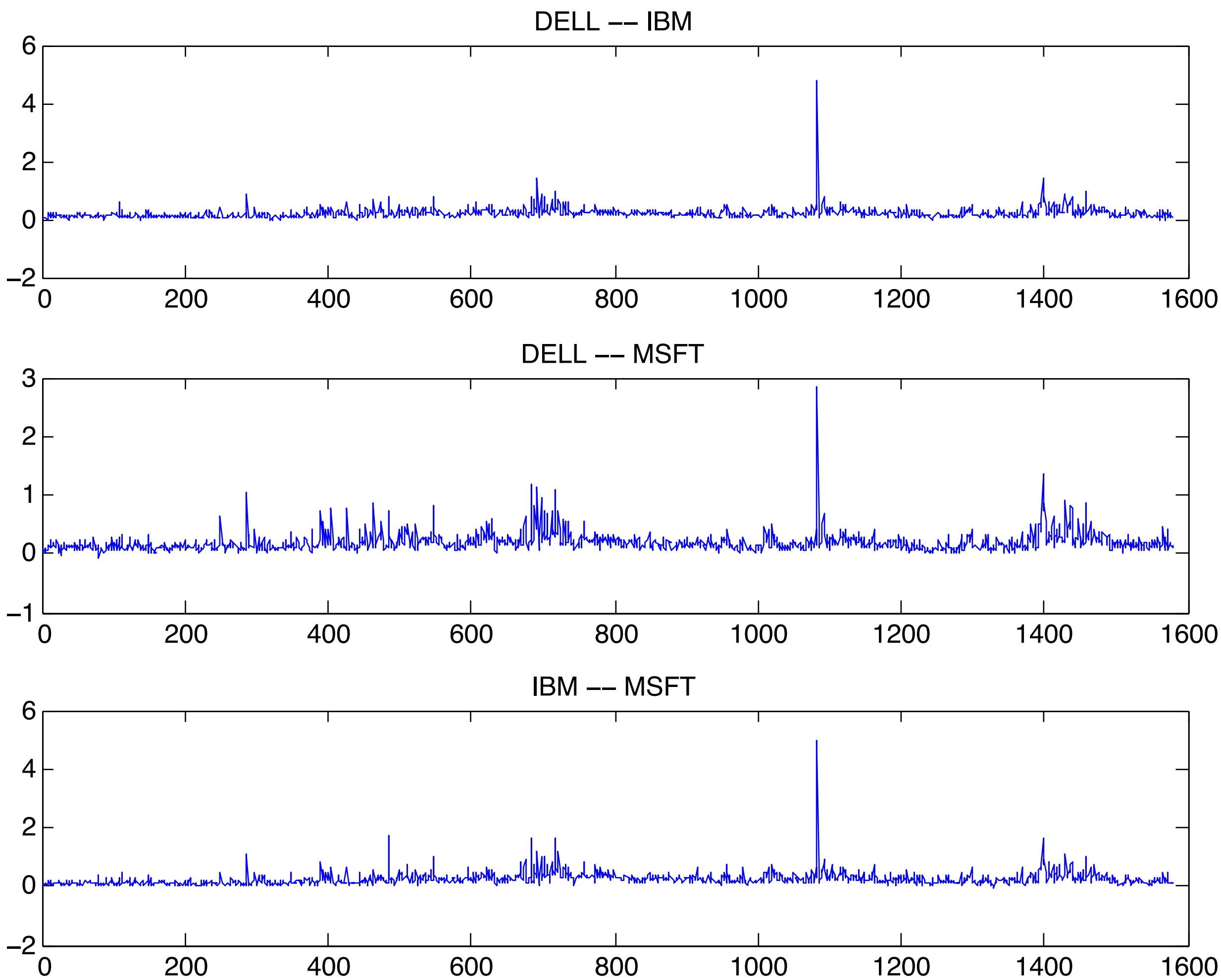


Figure 7: Daily standardized realized covariance ($\hat{C}_{ij,t}$) from January 3, 2006 to March 30, 2012.

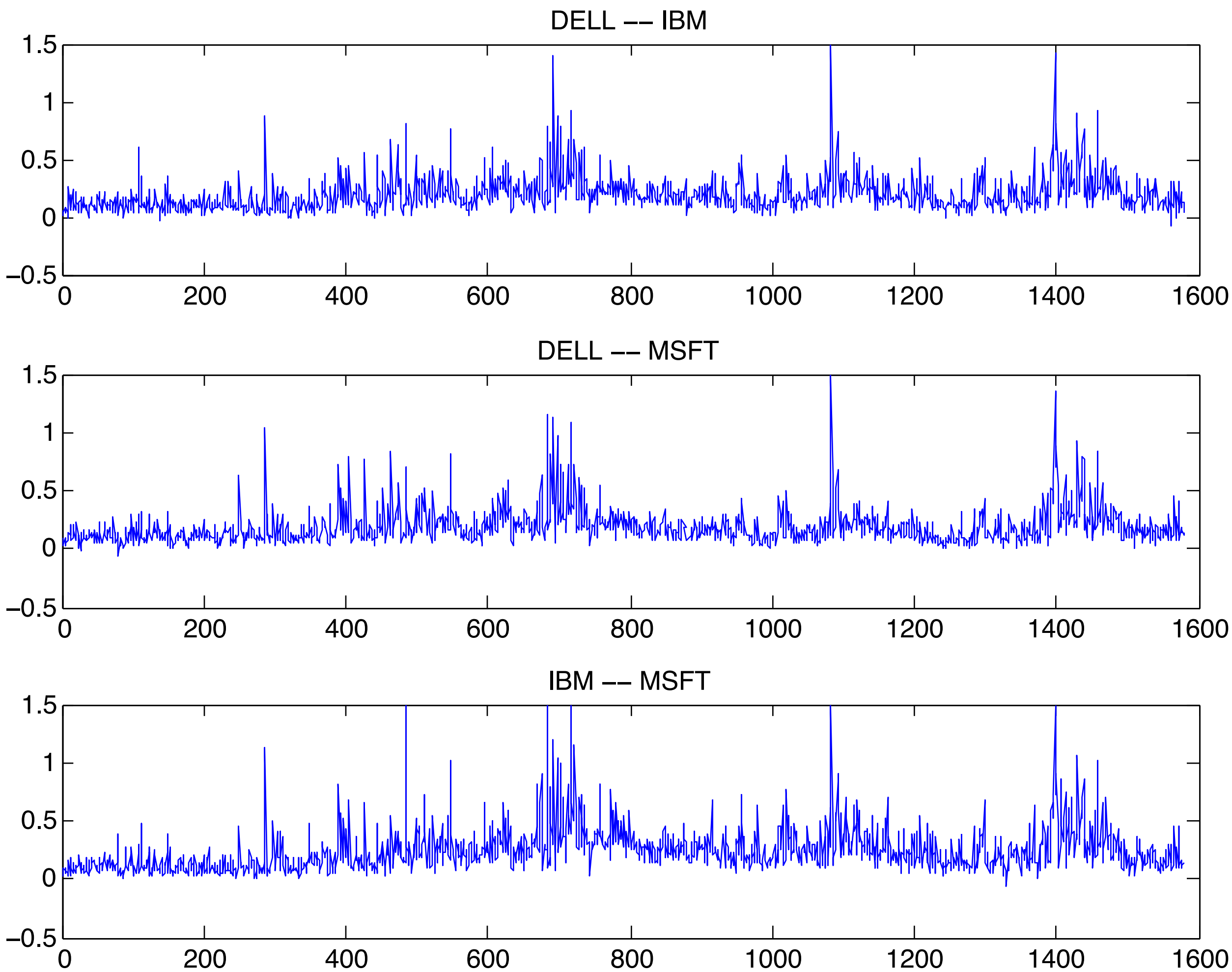


Figure 8: Filtered probabilities of being in each regime on a given day. The probabilities are obtained with the standardized returns.

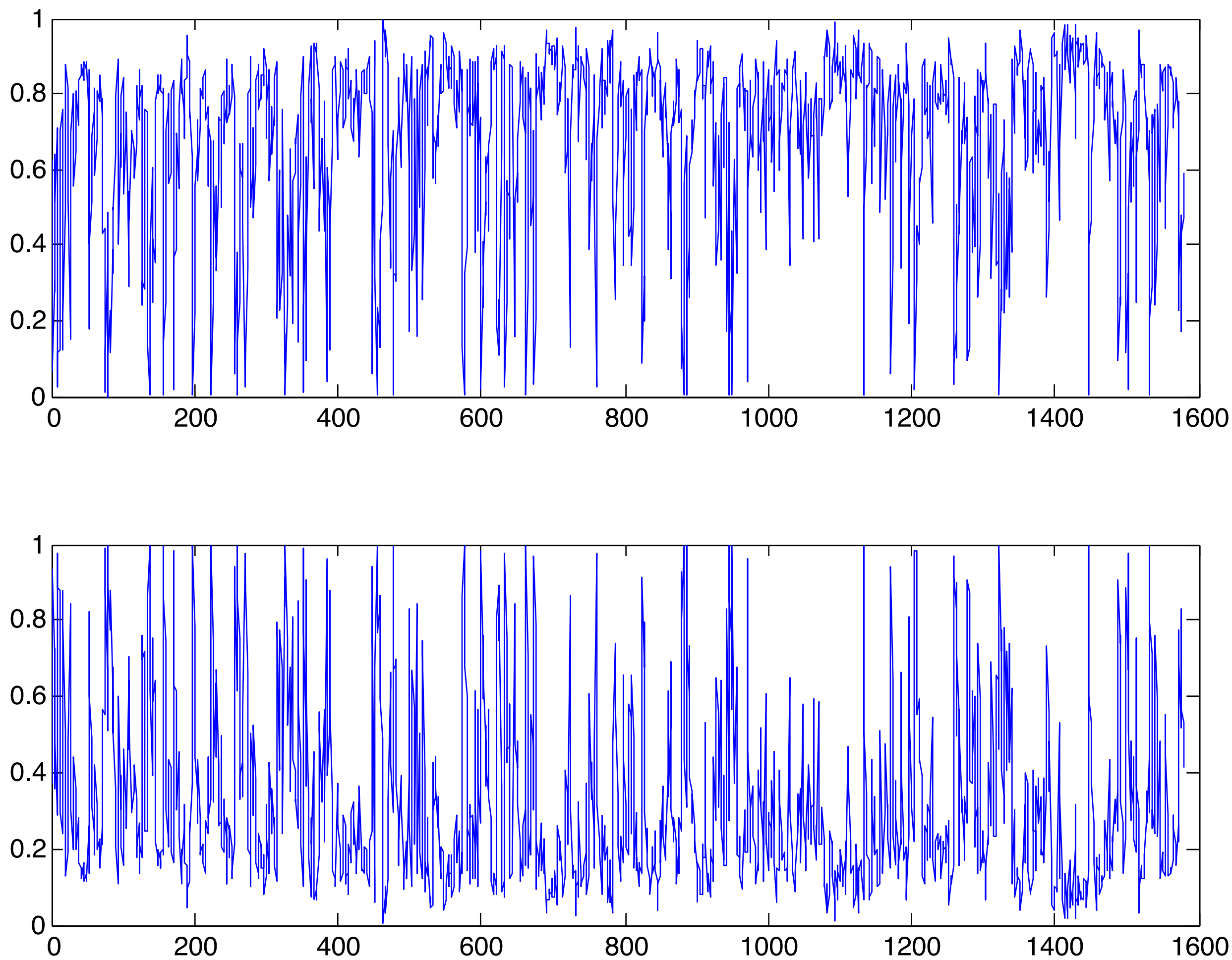
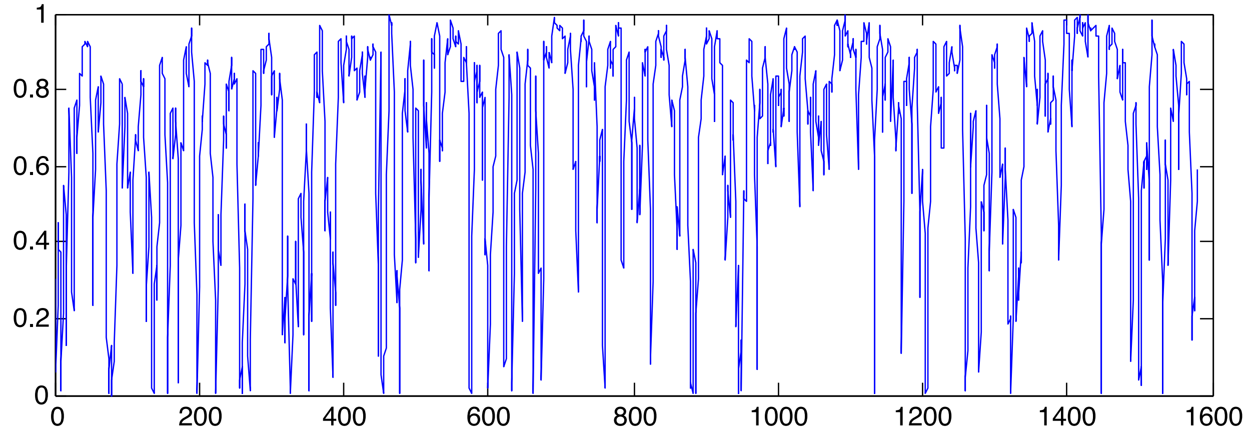


Figure 9: Smoothed probabilities of being in each regime on a given day. The probabilities are obtained with the standardized returns.



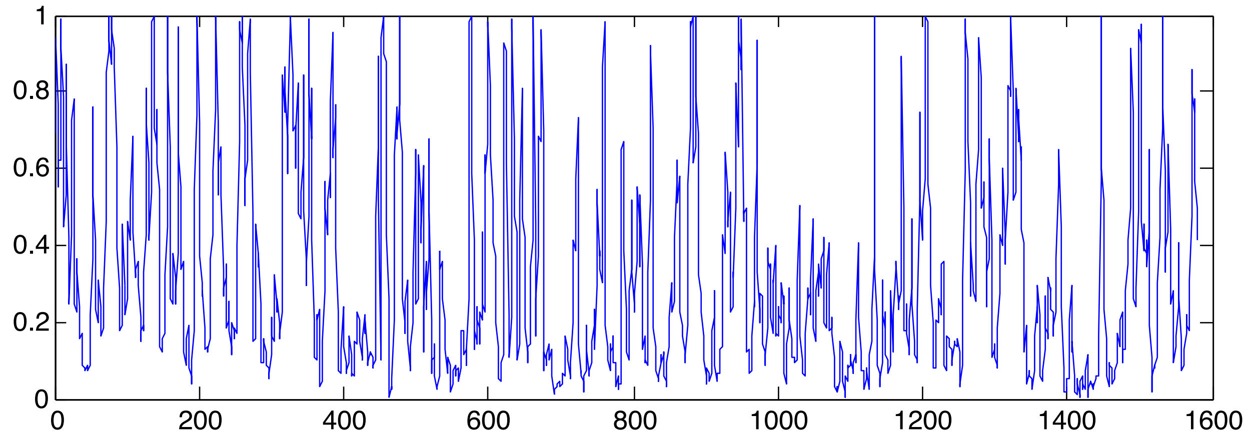


Figure 10: Filtered probabilities of being in each regime on a given day. The probabilities are obtained with the scaled realized variance matrix.

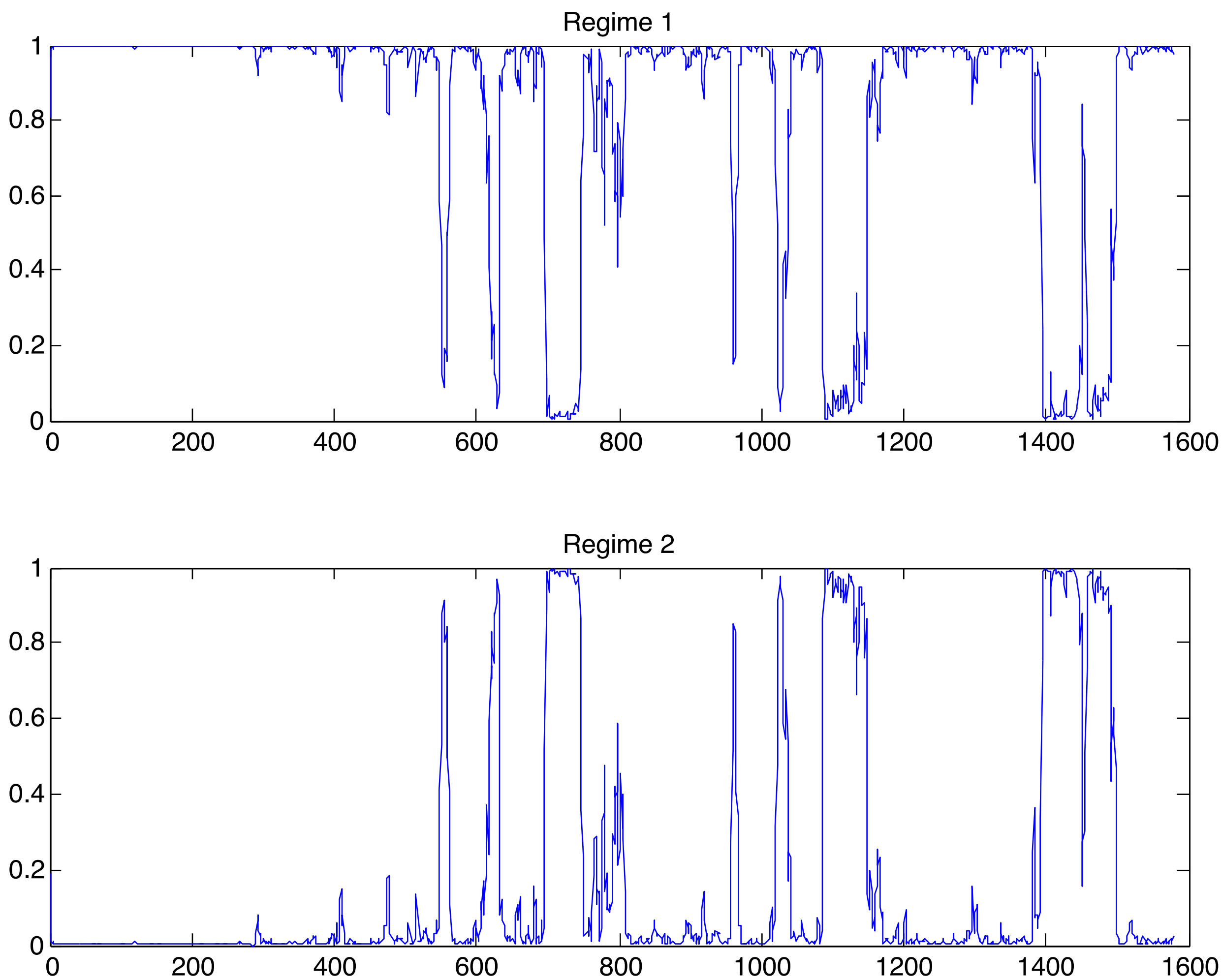


Figure 11: Smoothed probabilities of being in each regime on a given day. The probabilities are obtained with the scaled realized variance matrix.