New Cases of Solow-sustainability with Exhaustible Resources

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Abstract

We introduce distinct production funtions for investment and consumption goods to the Solow (1974) model of sustainable depletion of a finite oil stock. This endogenizes the price of investment goods and leads to two new scenarios of the using up of a finite stock of oil, while the the economy's consumption level remains unchanging. The new cases involve investment in durable capital increasing or decreasing as time passes.

1 Introduction

We introduce distinct production funtions for investment and consumption goods to the Solow (1974) model of sustainable depletion of a finite oil stock. This endogenizes the price of investment goods and leads to a different scenario of using up of a finite stock of oil, while the economy's consumption level remains unchanging. Our model allows for the possibility of investment in produced capital declining or increasing over time, rather than remaining non-varying, as in Solow (1974).

Our model has then two, constant returns to scale, production functions

$$q_C = f(K_C, R_C), \quad (1)$$

$$\text{and } q_I = g(K - K_C, R - R_C). \quad (2)$$

for q_C and q_I current output of consumption and investment goods respectively. Inputs K_C and R_C are services of produced capital and oil input respectively to the consumption goods sector. There are two zero profit relations

$$
\begin{aligned}
q_C &= rK_C + vR_C \quad (3)\\
p_I q_I &= r[K - K_C] + v[R - R_C]. \quad (4)
\end{aligned}
$$

The price of consumption goods is set at unity. r is the rental price per unit of capital K and v is the price for a unit of oil, R. We have $K_I = K - K_C$ and $R_I = R - R_C$.

There are static efficiency conditions for input use in

$$
\frac{f_{K_C}}{f_{R_C}} = \frac{r}{v} \text{ and } \frac{g_{K_I}}{g_{R_I}} = \frac{r}{v}. \quad (5)
$$

f_{R_C} is the derivative of $f(.)$ with respect to R_C. In solving the model, we impose the condition that q_C is non-varying ($q_C = \overline{q}_C$), and current oil rents fund current investment. q_C non-varying is taken as a condition for the economy to be on a sustainability trajectory and funding current investment with current exhaustible resource rents is taken as a savings "behavior". That is

$$
vR = p_I q_I. \quad (6)
$$

Our model is seven equations in q_I, p_I, K_C, R_C, r, v and R. Initial "endowments" are K and S_0. As time passes from date t to $t+1$, K_t increases by q_I and S_t decreases by R.

We solved the 7 equation system for a sequence of dates with Matlab with each sector having a constant returns to scale, Cobb-Douglas production function. We set $p_C = q_C = 1$ and $q_C = [K_C]^{0.8}[R_C]^{0.2}$ and $q_I = [K - K_C]^{0.3}[R - R_R]^{0.7}$. The initial value of K is 5.0. K is increased across dates by q_I. See Table 1 for numerical results.

Table 1

	q_I	p_I	K_C	R_C	r	v	v1	R
t_0	0.0373	17.8687	4.0000	0.0039	0.2000	51.2000	51.7731	0.0130
t_1	0.0366	18.2038	4.0298	0.0038	0.1985	52.7450	53.3202	0.0126
t_2	0.0360	18.5363	4.0591	0.0037	0.1971	54.2947	54.8720	0.0123
t_3	0.0353	18.8668	4.0879	0.0036	0.1957	55.8521	56.4314	0.0119
t_4	0.0347	19.1944	4.1162	0.0035	0.1944	57.4115	57.9928	0.0116
t_5	0.0342	19.5196	4.1439	0.0034	0.1931	58.9760	59.5593	0.0113

K and p_I are increasing. q_I and R are decreasing. p_I cannot move separately from p_C in the Solow formulation; nor could q_C vary over time. $v1$ is the current value of v increased at the current interest rate. Observe that the actual v rises at somewhat more than the currently projected v because we are working in discrete rather than continuous time. The solution is "rotating" along the isoquant for q_C and the price of a unit of exhaustible resources is "high" relative to the rental rate, r for a unit of capital. Also the solution involves shifting down at each date to a "lower" isoquant for q_I.

We turn to an example with q_I increasing over time. We set $p_I = q_C = 1$ and now $q_C = [K_C]^{0.4}[R_C]^{0.6}$ and $q_I = [K - K_C]^{0.9}[R - R_R]^{0.1}$. The initial value of K is 5.0. K is increased across dates by q_I. $q_C = p_C = 1$. Results of our Matlab solving are in Table 2.

Table 2

	q_I	p_I	K_C	R_C	r	v	v1	R
t_0	2.0602	0.3236	2.0000	0.6300	0.2000	0.9524	1.5411	0.7000
t_1	2.7466	0.2427	2.8241	0.5005	0.1416	1.1988	1.8983	0.5561
t_2	3.6117	0.1846	3.9227	0.4020	0.1020	1.4924	2.3168	0.4467
t_3	4.6902	0.1421	5.3674	0.3262	0.0745	1.8393	2.8037	0.3625
t_4	6.0212	0.1107	7.2435	0.2671	0.0552	2.2462	3.3665	0.2968
t_5	7.6484	0.0872	9.6520	0.2206	0.0414	2.7200	4.0132	0.2451
t_6	9.6209	0.0693	12.7113	0.1836	0.0315	3.2680	4.7520	0.2040

The above system of equations can be solved without explicit prices as four equations in four unknowns: q_I, K_C, R_C, and R.

Result 1: The above system of equations implies the Hotelling Rule,

$$\frac{\dot{g}_{R_I}}{g_{R_I}} = i \text{ for } i = r/p_I = g_{K_I}.$$

The demonstration involves taking the time-derivative of q_C and of $q_I = Rg_{R_I}$. When we differentiate $q_I = Rg_{R_I}$ we get

$$g_{K_I}[\dot{K} - \dot{K}_C] + g_{R_I}[\dot{R} - \dot{R}_C] - \dot{R}g_{R_I} - R\dot{g}_{R_I} = 0.$$

We use the static efficiency condition $\frac{g_{K_I}}{g_{R_I}} = \frac{f_{K_C}}{f_{R_C}}$ and $\dot{K} = q_I = Rg_{R_I}$ to reduce this to

$$-g_{R_I}\{\frac{f_{K_C}}{f_{R_C}}\dot{K}_C + \dot{R}_C\} + \dot{K}\left[g_{K_I} - \frac{\dot{g}_{R_I}}{g_{R_I}}\right] = 0.$$

The term in braces equals zero because $\dot{q}_C = 0$. Hence $\frac{\dot{g}_{R_I}}{g_{R_I}} = g_{K_I}$, which is the familiar Hotelling Rule (exhaustible resource price rises at the rate of interest). ■

With Cobb-Douglas production functions, $q_C = [K_C]^{ac}[R_C]^{1-ac}$ and $q_I = [K - K_C]^{ai}[R - R_R]^{1-ai}$, our four equation system allows us to express q_I in terms of K and other parameters, excluding S. We start with $Rg_{R_I} = q_I$ and for Cobb-Douglas, we get

$$R_C = ai * R. \tag{7}$$

Since q_C is a parameter, we have from $q_C = [K_C]^{ac}[R_C]^{1-ac}$

$$R_C = \left[\frac{q_C}{(K_C)^{ac}}\right]^{1/(1-ac)}. \tag{8}$$

Input-use efficiency yields

$$\left[\frac{ac}{1-ac}\right]\left[\frac{1-ai}{ai}\right]\frac{R_C}{K_C} = \frac{R - R_C}{K - K_C}. \tag{9}$$

These three equations allow us to express $R - R_C$ and $K - K_C$ in terms of K and parameters of the production functions. Thus using $q_I = \dot{K} = (K - K_C)^{ai}(R - R_C)^{1-ai}$ we get

$$\dot{K} = (1-ac)^{ai}\left(\frac{1-ai}{ai}\right)^{1-ai}\left(\frac{q_C}{(ac)^{ac}}\right)^{\frac{1-ai}{1-ac}} K^{\left[\frac{ai-ac}{1-ac}\right]}. \tag{10}$$

We observe below that $K(t)$ is increasing over time. Equation (10) suggests then three possibilities: (1) $\left[\frac{ai-ac}{1-ac}\right] = 0$ and $\dot{K}$ is unchanging (Solow (1974),

(2) $\left[\frac{ai-ac}{1-ac}\right] > 0$ and K and $\dot{K}$ are each increasing as time passes, and

(3) $\left[\frac{ai-ac}{1-ac}\right] < 0$, K is increasing and $\dot{K}$ is decreasing as time passes.

This last case was illustrated in Table 1 above. More on these possibilities below.

Result 2: Equation (10) solves as $K_t^{\left[\frac{1-ai}{1-ac}\right]} = A * [t + C1]$ for $A = \left[\frac{1-ai}{1-ac}\right](1-ac)^{ai}\left(\frac{1-ai}{ai}\right)^{1-ai}\left(\frac{q_C}{(ac)^{ac}}\right)^{\left[\frac{1-ai}{1-ac}\right]}$ and $C1$ a positive constant defined by the endowment K_0.[1]

[1] Quasi-arithmetic growth in $K(t)$ was central to the solution of the Solow Model with "extra savings" and endogenous population growth (Asheim, Buchholtz, Hartwick, Mitra and Withagen (2007)).

Obviously, $K(t)$ is increasing in time since $\left[\frac{1-ai}{1-ac}\right] > 0$. Our solution implies that $\dot{K}/K = B/(t+C1)$ for $B > 0$. Hence $\dot{K}/K \to 0$ as time passes. This latter behavior appears in the Solow formulation also.

Consider now the finiteness of oil use over infinite time. We know that

$$R = \frac{1}{ai}\left[\frac{q_C}{(ac * K)^{ac}}\right]^{1/(1-ac)}.$$

Hence convergence turns on the finiteness of $\int_0^\infty R dt$ or the finiteness of

$$\frac{1}{ai}\left[\frac{q_C}{(ac)^{ac}}\right]^{1/(1-ac)} \int_0^\infty \frac{1}{K^{ac/(1-ac)}} dt$$

for $K = A^{\left[\frac{1-ac}{1-ai}\right]} * [t + C1]^{\left[\frac{1-ac}{1-ai}\right]}$. That is, we are concerned about the finiteness of

$$\frac{1}{ai}\left[\frac{q_C}{(ac)^{ac}}\right]^{1/(1-ac)} \frac{1}{[A]^{ac/(1-ai)}} \int_0^\infty \frac{1}{[t + C1]^Z} dt. \tag{11}$$

$$\text{for } Z = \frac{ac}{(1-ai)}. \tag{12}$$

The integral

$$\begin{aligned}\int_0^T \frac{1}{[t + C1]^Z} dt &= \frac{-1}{Z+1}[t + C1]^{(-Z+1)} \Big|_0^T \\ &= (\frac{-1}{Z-1})\{(T + C1)^{(-Z+1)} - (C1)^{(-Z+1)}\}.\end{aligned}$$

For $T \to \infty$, we get

$$(\frac{1}{Z-1})\{(C1)^{(-Z-1)}\} \text{ for } Z - 1 > 0 \text{ or } ac - (1 - ai) > 0.$$

Hence (Result 3) for the limiting value of the integral to be finite and positive, we require that $1 - ai < ac$. If one takes this result to the expression in (11), one observes that this condition is one on terms in the "sum" $\int_0^\infty \frac{1}{[t+C1]^{ac/(1-ai)}} dt$ that get smaller relatively quickly as time passes (that is, $ac/(1 - ai) > 1$). ■

We return to $\dot{K} = (1-ac)^{ai}\left(\frac{1-ai}{ai}\right)^{1-ai}\left(\frac{q_C}{(ac)^{ac}}\right)^{\frac{1-ai}{1-ac}} K^{\left[\frac{ai-ac}{1-ac}\right]}$. We know that finiteness of oil use implies that $\frac{ai}{1-ac}$ must be greater than unity. The case of $\dot{K}$ declining is particularly novel: output from the investment goods sector is shrinking to zero in the limit while the use of oil in the consumption goods sector is also declining to zero in the limit. One presumes that the level of sustainable consumption q_C is very small for this new case. See Table 1 for some solved time-steps in a trajectory for this case.

APPENDIX:

```
function f=stol(x)
ac=0.4;ai=0.9;
K=5;qc=1.0;
% define terms needed in solving...
pc=1;
v1=x(1);
qi=x(2);
pi=x(3);
Kc=x(4);
% Nc=x(5);
Rc=x(5);
r=x(6);
Ki=K-Kc;
% Ni=Nb-Nc;
v=x(7);
R=x(8);
Ri=R-Rc;
%
f(1)=qc-(Kc^ac*Rc^(1-ac));
f(2)=qi-(Ki^ai*Ri^(1-ai));
%
f(3)=ac*Rc*v-(1-ac)*Kc*r;
f(4)=ai*Ri*v-(1-ai)*Ki*r;
f(5)=pc*qc-(r*Kc+v*Rc);
f(6)=pi*qi-(r*Ki+v*Ri);
f(7)=v*R-pi*qi;
% hotelling ...
f(8)=v1-(1+(r/pi))*v;
% Base has K=5.0 and qc=0.8172... outputs R=0.5 and qi=1.9921... then re-do
% K+qi and get new shrinking R under qc CONSTANT... i=r/pi ...
% ... Hotelling relations...pr and pr1 for approx i % motion (discrete time)
```

% 3.9843, qi=1.9921, pi=0.2735, Kc=2.0000, Nc=0.4500, r=0.1634, v=1.0896, R= 0.5000, pr1=6.3654

% 6.5888 2.6343 0.2068 2.7968 0.3598 0.1169 1.3626 R= 0.3998 10.3124

% 10.6437 3.4386 0.1584 3.8506 0.2908 0.0849 1.6864 R=0.3231 16.3467

% 16.8291 4.4353 0.1228 5.2260 0.2372 0.0625 2.0672 R=0.2635 25.3988

% 26.0895 5.6585 0.0963 7.0001 0.1952 0.0467 2.5119 R= 0.2169 38.7431

% 39.7166 7.1465 0.0762 9.2635 0.1619 0.0353 3.0277 R= 0.1799 58.1006

% 59.4533 8.9419 0.0609 12.1221 0.1354 0.0270 3.6223 R= 0.1504 85.7666

% 87.6215 11.0918 0.0491 15.6989 0.1139 0.0208 4.3037 R=0.1266 124.7662

% 127.2784 13.6484 0.0399 20.1356 0.0965 0.0162 5.0805 R= 0.1072 179.0420

% IRR mode... very small R (=0.5)

% 0.8172 1.9921 0.2735 2.0000 0.4500 0.1634 1.0896

% same with added pr

% 0.8172 1.9921 0.2735 2.0000 0.4500 0.1634 1.0896 pr=3.9842

% pr exog, pr=3.9842, and R endog

% 0.8172 1.9921 0.2735 2.0000 0.4500 0.1634 1.0896 R= 0.5000

% new pr as pr1, comes out

% 0.8172 1.9921 0.2735 2.0000 0.4500 0.1634 1.0896 0.5000 pr1=6.3652

% 0.5980 1.8910 0.2108 2.0000 0.2674 0.1196 1.3419 0.2971 pr1= 9.9762

% 0.4432 1.7989 0.1642 2.0000 0.1623 0.0886 1.6385 0.1803 pr1= 15.3601

% ....

% ac=0.2, s=.9

% 0.8173, 6.7922, pi=1.0830, 0.3448, 1.0141, r= 0.4740, v= 0.6448

% ac=0.25 .... s=.9

% 0.7809 6.7909 1.0350 0.4237 0.9574 0.4607 0.611

% ac=.8, ai=.9, s=.9

% 0.5820 4.7749 1.0969 0.4494 1.6364 1.0359 0.0711

****

Asheim, Geir, Wolfgang Buchholz, John Hartwick, Tapan Mitra, Cees Withagen (2007) "Constant Savings rates and quasi-arithmetic Population Growth under Exhaustible Resource constraints" *Journal of Environmental Economics and Management,* 64, pp. 213-229.

Solow, Robert M. (1974) "Intergenerational Equity and Exhaustible Resources", *The Review of Economic Studies*, Vol. 41, pp. 29-45.

2 Introduction

We emphasize that a pollutant such as a greenhouse gas is a public-good commodity, entering here equally in the production functions of all producers. This implies that the allocation analysis for an economy with global warming must keep public-good charges front and center. We make this case with an extension of the one-sector Stollery (1996) "growth" model, a model descending directly from the Solow (1974) sustainability model with oil extraction and stock depletion. Firms in our consumption goods and investment goods sectors each end up paying a Pigovian tax (a "price" for temperature) in their respective current zero profit relations, a tax which is a public goods charge of the Samuelson (1954) type (our public good is actually a public bad and is an intermediate good rather than a final good[2]). These firm-specific charges sum to the public good "price" for temperature which appears in the zero arbitrage profit condition for extractors. In brief, we disaggregate the Stollery model of global warming into two distinct sectors, pay close attention to the public goods nature of pigovian charges, and take up the possibility of solving the model for the case of each production function of the Cobb-Douglas form.

3 The Analysis

At each date the economy is producing q_C of consumption goods and q_I of investment goods, each with a distinct constant returns to scale production function. The inputs are capital K and oil R, and inverse-temperature Z. That is, Z is $1/T$ for temperature T and T reflects cumulative extraction of oil. For example, T might take the form $\mu[S_0 - S(t)]$. Each period involves more oil extraction, R and thus a rising temperature since cumulative extraction increases by R. Temperature is the direct "bad" imposing a current cost on producers. Population is assumed to be unchanging and we leave labor and population out of the production functions.

[2] There is a large literature on intermediate public good inputs. See for example ...., Sandmo, MacMillan, Manning, Markusen, etc.

Our model has then two production functions

$$q_C = f(K_C, R_C, Z), \tag{13}$$

$$\text{and } q_I = g(K - K_C, R - R_C, Z). \tag{14}$$

The static efficiency condition for use of K and R is

$$\frac{f_{K_C}}{f_{R_C}} = \frac{g_{K_I}}{g_{R_I}} \tag{15}$$

for $K_I = K - K_C$ and $R_I = R - R_C$.In solving the model, we impose the condition that q_C is non-varying ($q_C = \overline{q}_C$), and current oil rents fund current investment. That is

$$Rg_{R_I} = q_I. \tag{16}$$

Our model is four equations in q_I, K_C, R_C and R. Initial "endowments" are K and S_0. As time passes from date t to $t+1$, K_t increases by q_I and S_t decreases by R.

We consider the analysis for the moment with Z as a generic externality to producers in each of the consumption and investment goods sectors. We would

of course like our externality to impinge on each firm in the economy but we cannot break out distinct firms, given contant returns to scale at the level of the industry. Hence from a public goods standpoint we have two distinct competitive firms, one doing consumption goods and one doing investment goods.

Result 1: The above system implies the amended Hotelling Rule,

$$\frac{\dot{g}_{R_I}}{g_{R_I}} = g_{K_I} + \left[\frac{\dot{Z}}{Z}\right]\left[\frac{Zg_Z}{Rg_{R_C}} + \frac{Zf_Z}{Rf_{R_C}}\right]. \tag{17}$$

The demonstration involves taking the time-derivative of (16) and then using $\dot{q}_C = 0$. When we differentiate (16) we get

$$g_{K_I}[\dot{K} - \dot{K}_C] + g_{R_I}[\dot{R} - \dot{R}_C] - \gamma g_Z \dot{S} - \dot{R}g_{R_I} - R\dot{g}_{R_I} = 0.$$

We use $R=-\dot{S}$, the static efficiency conditions (15) and (**??**), and $\dot{K}=q_I=Rg_{R_I}$ to reduce this to

$$-\{\frac{g_{R_I}f_{K_C}}{f_{R_C}}\dot{K}_C+\frac{g_{K_I}f_{R_C}}{f_{K_C}}\dot{R}_C\}+g_Z\dot{Z}+\dot{K}\left[g_{K_I}-\frac{\dot{g}_{R_I}}{g_{R_I}}\right]=0.$$

We add and subtract $\frac{g_{R_I}f_Z\dot{Z}}{f_{R_C}}$ to get

$$-\{\frac{g_{R_I}f_{K_C}}{f_{R_C}}\dot{K}_C+\frac{g_{K_I}f_{R_C}}{f_{K_C}}\dot{R}_C+\frac{g_{R_I}f_Z\dot{Z}}{f_{R_C}}\}+\frac{g_{R_I}f_Z\dot{Z}}{f_{R_C}}+g_Z\dot{Z}+\dot{K}\left[g_{K_I}-\frac{\dot{g}_{R_I}}{g_{R_I}}\right]=0.$$

The term in braces equals zero because $\dot{q}_C=0$. The remaining terms reduce to (17). ■

The term $\left[\frac{\dot{Z}}{Z}\right]\left[\frac{Zg_Z}{Rg_{R_C}}+\frac{Zf_Z}{Rf_{R_C}}\right]$ represents the "distortion" from the familiar Hotelling Rule (exhaustible resource price rises at the rate of interest). This negative term is a pigovian tax on extraction (a source tax) that induces extractors to slow extraction because a larger R induces currently a larger increase in temperature. Observe that each of Zg_Z and Zf_Z are own Pigovian taxes associated with the investment and consumption goods sectors respectively. They sum in (17) because they are charges for a public bad, namely temperature, Z. We can consider the case of prices for our system, with q_C with price unity and q_I with price p_I. Then we would have

$$\begin{aligned} f_{R_C} &= v=p_Ig_{R_I} \\ \text{and } f_{K_C} &= r=p_Ig_{K_I}. \end{aligned}$$

Our adjusted Hotelling Rule would then be

$$\frac{\dot{g}_{R_I}}{g_{R_I}}=g_{K_I}+\left[\frac{\dot{Z}}{Z}\right]\left[\frac{Zf_Z+Zp_Ig_Z}{vR}\right].$$

$Zf_Z+Zp_Ig_Z$ are the public goods charges or total tax revenue for "public good", Z.

Our Hotelling expression in (17) is a generic zero arbitrage profit condition for extractors and is derived in Appendix 1 in a fairly generic growth model (the Dasgupta-Heal (1974) model). Hence what we have essentially is that investing exhaustible resource rents implies (17), a dynamic efficiency condition. Some may find the "inverse" expression more congenial, namely unchanging consumption and efficiency conditions, including the dynamic "adjusted" Hotelling rule, implies the savings investment rule: invest current resource rents in current capital accumulation. Given the expression in (17), this proposition is easy to demonstrate (it makes use of procedures in our derivation above).

4 Numerical Investigation

We turn to solving the system for the case of production functions of the Cobb-Douglas form.

We have

$$\begin{aligned} q_C &= [K_C]^{ac}[R_C]^{acr}[Z]^{(1-ac-acr)} \\ \text{and } q_I &= [K-K_C]^{ai}[R-R_C]^{air}[Z]^{(1-ai-air)}. \end{aligned}$$

Since q_C is unchanging, we have

$$R_C = \left[\frac{q_C}{[K_C]^{ac}[Z]^{(1-ac-acr)}}\right]^{1/acr}.$$

The static efficiency condition gives us

$$\frac{ac*air}{acr*ai}*\frac{R_C}{K_C} = \frac{R-R_C}{K-K_C}.$$

The investment condition $Rg_{R_I} = q_I$ becomes

$$R_C = (1-air)R.$$

This allows us to reduce the efficiency condtion to

$$K_c = \frac{ac*(1-air)*K}{(acr*ai)+(ac*(1-air))}.$$

We proceed to substitute into $\dot{K} = [K-K_C]^{ai}[R-R_C]^{air}[Z]^{(1-ai-air)}$. First, we get

$$\begin{aligned} \dot{K} &= \left\{K\left[1-\frac{ac*(1-air)}{(acr*ai)+(ac*(1-air))}\right]\right\}^{ai}\{Z\}^{(1-ai-air)} \\ &\quad *\left\{\frac{ac*air*(K-K_C)}{ai*acr*K_C}\left[\frac{q_C}{[K_C]^{ac}[Z]^{(1-ac-acr)}}\right]^{1/acr}\right\}^{air}. \end{aligned}$$

Some additional substitution yields

$$\dot{K} = C1 * K^{(ai*acr-ac*air)/acr} * Z^{[(1-ai-air)(acr)-(1-ac-acr)(air)]/acr} \tag{18}$$

for

$$C1 = \left\{\frac{acr * ai}{(acr * ai) + (ac * (1 - air))}\right\}^{ai} \left\{\left[\frac{air}{(1 - air)}\right]\left[\frac{acr * ai + (1 - air) * ac}{ac(1 - air)}\right]^{\frac{ac}{acr}} q_C^{[1/acr]}\right\}^{air}.$$

Both K and Z on the right side enter with an exponent (i.e. relatively simply). Note that for the special case of each sector having the same production function (Stollery (1996)) we have $\dot{K}$ a constant, as in the original Solow model. For this case, the temperature externality is "significant" only in the $\dot{S}$ equation, below. Stollery was aware that his formulation involved $\dot{K}$ a constant ($K(t) = A + C1 * t$, with A a positive constant) and he exploited this property in obtaining an explicit solution of this system. See below for details.

The companion equation to our two equation system is

$$\begin{aligned}
\dot{S} &= -R = -\frac{1}{(1 - air)} R_C \\
&= -\frac{1}{(1 - air)}\left[\frac{q_C}{[\frac{ac*(1-air)*K}{(acr*ai)+(ac*(1-air))}]^{ac}[Z]^{(1-ac-acr)}}\right]^{1/acr} \\
&= -C2 * K^{-ac/acr} * Z^{-(1-ac-acr)/acr} \qquad (19)
\end{aligned}$$

for

$$C2 = \left[\frac{1}{(1 - air)}\right]\left[\frac{q_C}{[\frac{ac*(1-air)}{(acr*ai)+(ac*(1-air))}]^{ac}}\right]^{1/acr}.$$

We now make Z an explicit function of S and consider our two simultaneous non-linear differential equations in K and S.

Two possibilities are $T(t) = T_0 \exp(-\mu S(t))$ (μ and T_0 positive; Stollery (1996), and $T(t) = \beta[S_0 - S(t)]$ with $Z = 1/T$. We use the Stollery form in our calculations reported below.

Considerable information can be gleaned from numerical solving with parameters of reasonable values.

(1) Identical production functions for q_C and q_I. (Stollery (1996))

Equation (19) becomes

$$\dot{S} = -[A + C1 * t]^{-ac/acr} * C2 * [1/(T_0 \exp(-\mu S(t)))]^{-(1-ac-acr)/acr}$$

Numerical investigation is not trivial for this case. Well-behavedness requires that the exponent for Z in equation (19) be small in absolute value and the value of acr be larger than ac. Well-behavedness involves solutions with $S(t)$ convex in t and asymptotic to the y axis. Solving numerically with Maple software yielded such solutions.

(2) We then took up cases with the exponent for S in equation (18) equal to zero. Such cases can be described as those following Hartwick (2013), with a temperature externality added to each production function. We simply "translated" our system in (18) and (19) into discrete time and solved for time paths of K and S. We chose $K_0 = 10$ and $S_0 = 12$ for each case. We chose parameter values for the production functions that left the exponent on K in equation (18) close to zero (the exponent for S was set at zero).

(a) With the exponent on $K > 0$ in equation (18), $S(t)$ declined with the absolute value of ΔS declining, and $K(t)$ increased with ΔK increasing.

(b) With the exponent on $K < 0$ in equation (18), $S(t)$ declined with the absolute value of ΔS declining, and $K(t)$ increased with ΔK decreasing.

(3) We then took up cases with the exponent for K in equation (18) equal to zero.

(a) With the exponent on $S > 0$ in equation (18), $S(t)$ declined with the absolute value of ΔS declining, and $K(t)$ increased with ΔK decreasing.

(b) With the exponent on $S < 0$ in equation (18), $S(t)$ declined with the absolute value of ΔS declining, and $K(t)$ increased with ΔK increasing.

(4) With the exponent on Z in equation (18) negative and the exponent on K positive, we observed K increasing with ΔK increasing and S declining with the absolute value of ΔS declining.

APPENDIX 1: THE DASGUPTA-HEAL MODEL with two distinct Sectors

The Dasgupta-Heal (1974) model involves Ramsey saving (present value of the utility of consumption is maximized with a constant discount rate ρ) and oil use in current production, where current oil used is drawn from a finite stock, S. Population (the labor force) is taken to be unchaning. We simply restate their model and its necessary conditions for the case of consumption goods and investment goods each having a distinct, constant returns to scale production function.

$$\begin{aligned} q_C &= f(K_C, R_C, Z) \\ q_I &= g(K - K_C, R - R_C, Z) \\ Z &= \gamma * [S_0 - S_t], \text{ ... temperature, } Z\text{...} \end{aligned}$$

$[S_0 - S_t]$ is cumulative extraction. $q_I = \dot{K}$ and $R = -\dot{S}$. The current-value Hamiltonian is

$$\begin{aligned} H &= u(q_C) + \lambda[q_I] - \phi R \\ \text{for } q_I &= \dot{K} \text{ and } R = -\dot{S}. \end{aligned}$$

Necessary conditions are

$$\begin{aligned} \frac{\partial H}{\partial K_C} &= 0 \Rightarrow u_{q_C} f_{K_C} = \lambda g_{K_I}, \text{ for } p_I = \lambda / u_{q_C} \text{ and } r = f_{K_C} = p_I g_{K_I}, && \text{(eleven)} \\ \frac{\partial H}{\partial R_C} &= 0 \Rightarrow u_{q_C} f_{R_C} = \lambda g_{R_I}, \text{ for } v = f_{R_C} = p_I g_{R_I} && \text{(thirteen)} \\ \frac{\partial H}{\partial R} &= 0 \Rightarrow \lambda g_{R_I} = \phi, && \text{(fourteen)} \\ -\frac{\partial H}{\partial K} &= \dot{\lambda} - \rho\lambda \text{ or } -\lambda g_{K_I} = \dot{\lambda} - \rho\lambda, && \text{(fifteen)} \\ -\frac{\partial H}{\partial S} &= \dot{\phi} - \rho\phi \text{ or } -[u_{q_C} f_Z + \lambda g_Z]\frac{dZ}{dS} = \dot{\phi} - \rho\phi. && \text{(sixteen)} \end{aligned}$$

Combining (15) and (16) yields

$$\frac{\dot{g}_{R_I}}{g_{R_I}} = g_{K_I} + [\frac{Z f_Z}{R f_{R_C}} + \frac{Z g_Z}{R g_{R_I}}]\frac{dZ}{dS}\frac{R}{Z} \tag{20}$$

This is the amended Hotelling Rule. The form of (20) differs slightly from the corresponding term in the text because in the above planning formulation, the planner takes account explicitly of the dependence of the externality Z on stock S whereas in the formulation in the text, firms react to the externality without linking the externality to current stock size S. In the text, firms are treating Z as a parameter whereas above Z is being treated as a function of stock size S.

APPENDICES WITH BACK-UP MATERIAL

... example with "arbitrary" initial values (K=10, S=12) and parameters...

```
c1=1;c2=1;ac=.3;acr=.5;ai=.4;air=.45;T0=1.0;mu=0.5;
% (ai*acr-ac*air)>0 in first equation...
% ((1-ai-air)*(acr)-(1-ac-acr)*air)<0 in first equation
% delt K (pos) is getting larger and delt S (neg) is getting smaller.
% K=10, S=12
% 11.1267 11.9772
% 12.2696 11.9557
% 13.4275 11.9354
% 14.5994 11.9161
% 15.7845 11.8976
% 16.9820 11.8799
% 18.1912 11.8629
% 19.4116 11.8465
% 20.6426 11.8307
% 21.8838 11.8154
% 23.1348 11.8006
% 24.3951 11.7863
% 25.6644 11.7724
% 26.9424 11.7589
% 28.2287 11.7457
% 29.5231 11.7328
% 30.8253 11.7202
% 32.1351 11.7079
% 33.4523 11.6959
% 34.7766 11.6842
% 36.1078 11.6727
% 37.4458 11.6614
% 38.7903 11.6504
% 40.1412 11.6396
% 41.4984 11.6290
% 42.8617 11.6186
```

% 44.2309 11.6083

% 45.6060 11.5982

% 46.9868 11.5883

ΔK increasing and ΔS decreasing in abs value

/////////

example with exponent of K in first equation set at zero (start K=10,S=12)

c1=1;c2=1;ac=.45;acr=.5;ai=.4;air=2/4.5;T0=1.0;mu=0.5;

% (ai*acr-ac*air)=0 in first equation...(yields declining ΔK)

% expon on S : ((1-ai-air)*(acr)-(1-ac-acr)*air)=(.07777-.02222)>0 in first equation

% K=10; S=12

% 11.9477 11.9309

% 13.8880 11.8718

% 15.8219 11.8201

% 17.7503 11.7740

% 19.6737 11.7323

% 21.5927 11.6942

% 23.5076 11.6591

% 25.4188 11.6265

% 27.3265 11.5961

% 29.2310 11.5676

% 31.1325 11.5407

% 33.0312 11.5153

% 34.9272 11.4912

% 36.8206 11.4682

% 38.7116 11.4462

% 40.6003 11.4252

% 42.4868 11.4051

% 44.3712 11.3857

% 46.2536 11.3671

% 48.1340 11.3491

% 50.0126 11.3317

% 51.8893 11.3149

% 53.7643 11.2987

% 55.6376 11.2830

% 57.5093 11.2677

% 59.3794 11.2529

ΔK decreasing (toward zero?) and ΔS decreasing in abs value.

////////

Other example with exponent on K at zero (reverse coefficients from above case)

c1=1;c2=1;ac=.4;acr=2/4.5;ai=.45;air=.5;T0=1.0;mu=0.5;

% (ai*acr-ac*air)=0 in first equation...

% expon on S : ((1-ai-air)*(acr)-(1-ac-acr)*air)=.02222-.07777<0 in first equation

% K=10; S=12

% 10.4724 11.9846

% 10.9452 11.9698

% 11.4185 11.9555

% 11.8922 11.9417

% 12.3663 11.9284

% 12.8408 11.9155

% 13.3157 11.9030

% 13.7909 11.8909

% 14.2665 11.8791

% 14.7424 11.8677

% 15.2187 11.8566

% 15.6953 11.8458

% delt K increasing and delt S decreasing in abs value

(this delt K is different from previous example)

//////////

expon on S at zero

c1=1;c2=1;acr=2/4.5;ac=.155555;T0=1.0;mu=0.5;ai=.05;air=.5;

K= 13.6873;S= 11.9904;

K1=x(1);

S1=x(2);

```
Z=1/(T0*exp(-mu*S));
f(1)=K1-(K+c1*K^((ai*acr-ac*air)/acr)*Z^(((1-ai-air)*(acr)-(1-ac-acr)*air)/acr));
f(2)=S1-(S-c2*K^(-ac/acr)*Z^(-(1-ac-acr)/acr));
% ((1-ai-air)*(acr)-(1-ac-acr)*air)=0 in first eqn
% ai*acr-ac*air=.02222-.077777<0 (expon on K) in first equation
% K=10; S=12
% 10.7499 11.9980
% 11.4930 11.9960
% 12.2300 11.9941
% 12.9613 11.9922
% 13.6873 11.9904
% 14.4083 11.9886
% delt K declining; absval delt S declining
///////
expon on S=0
c1=1;c2=1;T0=1.0;mu=0.5;
acr=.5;ac=.05;ai=.155555;air=2/4.5;
K= 17.9861;S= 11.9789;
K1=x(1);
S1=x(2);
Z=1/(T0*exp(-mu*S));
f(1)=K1-(K+c1*K^((ai*acr-ac*air)/acr)*Z^(((1-ai-air)*(acr)-(1-ac-acr)*air)/acr));
f(2)=S1-(S-c2*K^(-ac/acr)*Z^(-(1-ac-acr)/acr));
% ((1-ai-air)*(acr)-(1-ac-acr)*air)=0 in first eqn
% ai*acr-ac*air=.07777-.022222>0 (K exponent) in first equation
% K=10; S=12
% 11.2916 11.9964
% 12.6007 11.9928
% 13.9259 11.9893
% 15.2659 11.9858
% 16.6196 11.9823
% 17.9861 11.9789
% 19.3647 11.9755
```

... delt K increasing; absval delt S decreasing

/////////////