Identification by Laplace Transform in Nonlinear Panel or Time Series Models with Unobserved Stochastic Effects

Patrick GAGLIARDINI and Christian GOURIEROUX

Montreal, November 21, 2014

1. INTRODUCTION

This paper is about...

... identification in a general class of (semi-)parametric nonlinear models for time series or panel data with **unobserved dynamic effects**

These models are characterized by an **exponential affine specification** for the nonlinear regression of data on lagged endogenous variables and unobserved dynamic effects

1. INTRODUCTION

Exponential affine specifications are implied by standard distributional assumptions and are popular in asset pricing because of their analytical tractability

The unobservable dynamic components capture time effects in panel data, stochastic volatilities and covolatilities of asset returns, systematic factors in risk analysis, etc.

They complicate identification and estimation by standard methods such as Maximum Likelihood (ML)

1. INTRODUCTION

The goal of the paper is to ...

... provide general procedures to obtain **continuum sets of nonlinear moment restrictions** for affine specifications based on the (conditional) Laplace transform

Study when these sets of moment restrictions identify

- the nonlinear regression parameters, and
- the parametric or nonparametric specification for the distribution of dynamic effects

The identification strategy naturally leads to Generalized Method of Moments (GMM) estimation

OUTLINE

1. Introduction
2. The model: Examples and general specification
3. First-order moment restrictions: Nonlinear cross-differencing, moment restrictions from Laplace transform
4. Higher-order nonlinear moment restrictions
5. Application to linear and nonlinear latent factor models

2. THE MODEL

2.1 Examples

Example 1: Count panel data with stochastic time effect

A Poisson dynamic panel regression model:

$$y_{i,t} \sim \mathcal{P}(f_t + \alpha x_{i,t} + c y_{i,t-1}), \qquad i = 1, ..., n,\ t = 1, ..., T,\ (n \text{ and } T \text{ large})$$

with **unobserved stochastic time effect** f_t

Observed individual heterogeneity in the basic specification

Extension to individual dynamic effects $f_{i,t}$, e.g. “effort” processes for moral hazard in car insurance [Gourieroux, Jasiak (2004)]

2. THE MODEL

2.1 Examples

Example 2: Linear factor model with lagged endogenous variables

The vector y_t of index returns for n markets

$$y_t = Bf_t + Cy_{t-1} + \varepsilon_t, \quad t = 1, ..., T \quad (T \text{ large})$$

where the idiosyncratic shocks are $\varepsilon_t \sim IIN(0, \Sigma)$ and matrix Σ is diagonal

The vector f_t of K latent common factors represents **systematic risks** [see e.g. the Arbitrage Pricing Theory (APT), Ross (1976), Chamberlain, Rothschild (1983)]

Lagged returns vector y_{t-1} accounts for **contagion** (e.g. spillover) effects across markets

Example 3: Systematic risk and contagion in time series count data

Multivariate time series count data $y_t = (y_{1,t}, ..., y_{m,t})'$

Application to hedge fund survival: data are monthly liquidation counts in m management styles [Darolles, Gagliardini, Gourieroux (2011)]

Conditionally on past counts and common factor path:

$$y_{j,t} \sim \mathcal{P}(a_j + b_j' f_t + c_j' y_{t-1}), \quad j = 1, ..., m, \; t = 1, ..., T, \; (T \text{ large})$$

Systematic risk factor f_t with loading matrix $B = [b_1, ..., b_m]'$

Contagion effects within and between groups through lagged counts and coefficients matrix $C = [c_1, ..., c_m]'$

Example 4: Conditionally Gaussian factor model with stochastic volatility in the factor

$$y_t = \beta h_t^{1/2}\eta_t + \varepsilon_t, \quad t = 1, ..., T \quad (T \text{ large})$$

with $(n, 1)$ loading vector β and bivariate latent factor $f_t = (h_t, \eta_t)'$

The factor stochastic volatility process (h_t), the common shock $\eta_t \sim IIN(0, 1)$ and the idiosyncratic shocks vector $\varepsilon_t \sim IIN(0, \Sigma)$ are mutually independent

Process (h_t) is an Autoregressive Gamma (ARG) Markov process with noncentral gamma transition:

$$E[\exp(uh_t)|h_{t-1}] = \exp\left\{\frac{\rho u}{1-cu}h_{t-1} - \delta\log(1-cu)\right\},$$

where $\delta > 0$ is the degree of freedom parameter, $c > 0$ is a scale parameter, and $\rho \in (0, 1)$ is the first-order autocorrelation

2. THE MODEL

2.1 Examples

Example 5: Multivariate stochastic volatility model

Vector of returns y_t for n assets

$$y_t = a + \begin{pmatrix} Tr(B_1\Sigma_t) \\ \vdots \\ Tr(B_n\Sigma_t) \end{pmatrix} + \Sigma_t^{1/2}\varepsilon_t, \quad t = 1, ..., T \quad (T \text{ large})$$

where $\varepsilon_t \sim IIN(0, Id_n)$

Stochastic process Σ_t of symmetric positive-definite (n, n) volatility-covolatility matrices: assume autoregressive Wishart dynamics [Bru (1991), Gourieroux (2006), Gourieroux, Jasiak, Sufana (2009)]

Volatility risk premia parametrized by symmetric matrices B_1, ..., B_n

See Gagliardini, Gourieroux (2014) for identification and estimation

2. THE MODEL

2.2 General specification

Endogenous variable y_t, with dimension n

Unobservable dynamic effect f_t, with dimension K

Observable covariate process x_t

Information sets: let $\underline{y_{t-1}} = (y_{t-1}, y_{t-2}, ...)$ and similarly for $\underline{f_t}$ and $\underline{x_t}$

Assumption A.1: Exponentially affine nonlinear regression model

$$E[\exp(u'y_t)|\underline{y_{t-1}}, \underline{f_t}, \underline{x_t}] = \exp\left\{a(u, x_t, \theta)'[Bf_t + Cy_{t-1} + d(x_t, \theta)] + b(u, x_t, \theta)\right\}$$

where u is the multidimensional argument of the Laplace transform, θ, B, C are parameters (possibly constrained) and a, b, d are known functions.

Assumption A.2: Exogeneity, strict stationarity and Markov property of unobservable dynamic effects and observable regressors

The joint process $(f_t', x_t')'$ *is:*

i) *exogenous and Markov, that is, the conditional distribution of* $(f_t', x_t')'$ *given* $\underline{y_{t-1}}$, $\underline{f_{t-1}}$ *and* $\underline{x_{t-1}}$ *depends on* f_{t-1}, x_{t-1} *only;*

ii) *strictly stationary.*

The dynamics of the unobservable dynamic effect can be either specified parametrically, or let nonparametric

2. THE MODEL

2.2 General specification

Nonlinear (semi-)affine state space specification as in Bates (2006), but suitable for the analysis of multivariate cross-sectional measurements (potentially of large dimension)

The identification conditions in our moment-based setting are different from those in the likelihood-based framework of Bates (2006)

The moment restrictions in this paper also differ from those underlying simulation based methods such as Simulated Method of Moments [SMM, McFadden (1989), Pakes, Pollard (1989)], Simulated Nonparametric Moments [SNM, Creel, Kristensen (2012)], etc.

The (semi-)affine and (semi-)parametric framework in Assumption A.1 is more structural than the nonparametric setting in Hu, Shum (2012), Hu, Shiu (2013), allowing for more informative identifying restrictions

3. FIRST-ORDER NL MOMENT RESTRICTIONS

3.1 Nonlinear cross-differencing for panel data

Assume that variables $y_{i,t}$ and f_t are one-dimensional, and individual histories $(y_{i,t}, x_{i,t})$, i varying, are independent conditional on factor path (f_t). Then:

$$E[\exp(uy_{i,t})|\underline{y_{t-1}}, \underline{f_t}, \underline{x_t}] = \exp\{a(u, x_{i,t}, \theta)[f_t + cy_{i,t-1} + d(x_{i,t}, \theta)] + b(u, x_{i,t}, \theta)\}$$

i.e.

$$\begin{aligned} & E\left[\exp\{uy_{i,t} - a(u, x_{i,t}, \theta)[cy_{i,t-1} + d(x_{i,t}, \theta)] - b(u, x_{i,t}, \theta)\}|\underline{y_{t-1}}, \underline{f_t}, \underline{x_t}\right] \\ = \; & \exp[a(u, x_{i,t}, \theta)f_t], \quad \forall i, \; u, \; f_t \end{aligned} \tag{1}$$

Assumption A.3*: *The function $u \to a(u, x_{i,t}, \theta)$ is continuous and strictly monotonous w.r.t. argument u, for any given $x_{i,t}$ and θ.*

Then, we can define:

$u(v, x_{i,t}, \theta)$ as the solution of the equation $a(u, x_{i,t}, \theta) = v$,

and replace argument u by $u(v, x_{i,t}, \theta)$ in equation (1)

3. FIRST-ORDER NL MOMENT RESTRICTIONS

3.1 Nonlinear cross-differencing for panel data

We get:

$$
\begin{aligned}
& E\left[\exp\{u(v,x_{i,t},\theta)y_{i,t} - v[cy_{i,t-1} + d(x_{i,t},\theta)] - b[u(v,x_{i,t},\theta),x_{i,t},\theta]\} \middle| \underline{y_{t-1}}, \underline{f_t}, \underline{x_t}\right] \\
= \quad & \exp(vf_t)
\end{aligned}
$$

By differencing the equations for any pair of individuals i and j, with $i \neq j$, and computing conditional expectations given $\underline{y_{i,t-1}}, \underline{y_{j,t-1}}, \underline{x_t}$, we get

First-order nonlinear moment restrictions with cross-differencing:

$$
\begin{aligned}
& E\left[\exp\{u(v,x_{i,t},\theta)y_{i,t} - v[cy_{i,t-1} + d(x_{i,t},\theta)] - b[u(v,x_{i,t},\theta),x_{i,t},\theta]\} \middle| \underline{y_{i,t-1}}, \underline{y_{j,t-1}}, \underline{x_t}\right] \\
= & E\left[\exp\{u(v,x_{j,t},\theta)y_{j,t} - v[cy_{j,t-1} + d(x_{j,t},\theta)] - b[u(v,x_{j,t},\theta),x_{j,t},\theta]\} \middle| \underline{y_{j,t-1}}, \underline{y_{i,t-1}}, \underline{x_t}\right], \\
& \forall v,\ \forall i,j,\ i \neq j
\end{aligned}
$$

$\Rightarrow$ A continuum of nonlinear moment restrictions for any pair of individuals

3. FIRST-ORDER NL MOMENT RESTRICTIONS

3.1 Nonlinear cross-differencing for panel data

Proposition 1: *For a nonlinear panel data model with unobserved dynamic effects, under Assumptions A.1-A.3*, generically, the first-order nonlinear moment restrictions identify:*

- **i)** *the regression parameters B, C, θ by nonlinear cross-differencing. Then*
- **ii)** *the unconditional distribution of the dynamic effect f_t is nonparametrically identified*

A nonlinear cross-sectional extension of the quasi-differencing approach usually applied to panel data with individual fixed effects and based on the first-order moments only [see Mullahy (1997), Wooldridge (1997), (1999)]

3. FIRST-ORDER NL MOMENT RESTRICTIONS

3.1 Nonlinear cross-differencing for panel data

Example 1: Count panel data with stochastic time effect (cont.)

From the Laplace transform of the Poisson distribution:

$$\begin{aligned} & E\left[\exp\{uy_{it} + (1-\exp u)(\alpha x_{i,t} + c y_{i,t-1})\} | \underline{y_{i,t-1}}, \underline{y_{j,t-1}}, \underline{x_t}\right] \\ = \; & E\left[\exp\{uy_{j,t} + (1-\exp u)(\alpha x_{j,t} + c y_{j,t-1})\} | \underline{y_{i,t-1}}, \underline{y_{j,t-1}}, \underline{x_t}\right], \; \forall i \neq j, \; \forall u \end{aligned}$$

By considering the first-order expansion w.r.t. argument u at $u = 0$, the equations become:

$$E[y_{i,t} - \alpha x_{i,t} - c y_{i,t-1} | \underline{y_{i,t-1}}, \underline{y_{j,t-1}}, \underline{x_t}] = E[y_{j,t} - \alpha x_{j,t} - c y_{j,t-1} | \underline{y_{i,t-1}}, \underline{y_{j,t-1}}, \underline{x_t}], \quad \forall i \neq j,$$

which are the analogue of the moment restrictions considered in Windmeijer (2000), Blundell et al. (2002)

3. FIRST-ORDER NL MOMENT RESTRICTIONS

3.2 (Non-)parametric identification of the marginal distribution of the dynamic effect

In the framework of Assumptions A.1 and A.2 we have:

$$\begin{aligned} & E[\exp\{u'y_t - a(u,x_t,\theta)'[Cy_{t-1} + d(x_t,\theta)] - b(u,x_t,\theta)\}|\underline{y_{t-1}}, \underline{f_t}, \underline{x_t}] \\ & = \exp\{a(u,x_t,\theta)'Bf_t\} \end{aligned} \tag{2}$$

Assumption A.3: *There exist a change of argument from* $u \in \mathbb{R}^n$ *to* $v = (v_1', v_2')' \in \mathbb{R}^{n-K} \times \mathbb{R}^K$ *such that* $u(v, x_t, \theta, B)$*, say, satisfies:*

$$a[u(v,x_t,\theta,B),x_t,\theta]'B = v_1', \quad \forall v, B, \theta, x_t.$$

We get the continuum of **nonlinear first-order moment restrictions for the marginal distribution of** (f_t) (conditional on $\underline{x_t}$):

$$\begin{aligned} & E[\exp\{u(v,x_t,\theta,B)'y_t - a[u(v,x_t,\theta,B),x_t,\theta]'[Cy_{t-1} + d(x_t,\theta)] - b[u(v,x_t,\theta,B),x_t,\theta]\}|\underline{x_t}] \\ & = E[\exp(v_1'f_t)|\underline{x_t}], \quad \forall v \end{aligned}$$

3. FIRST-ORDER NL MOMENT RESTRICTIONS

3.2 (Non-)parametric identification of the marginal distribution of the dynamic effect

Proposition 2: *Under Assumptions A.1-A.3, generically, the first-order nonlinear moment restrictions identify:*

- **i)** *The unconditional distribution of process (f_t), once the parameters B, C, θ are known;*
- **ii)** *The regression parameters B, C, θ and the parameters characterized by the marginal of (f_t), when the distribution of (f_t) is specified parametrically.*

The parameters characterizing the dynamics of (f_t) are not identifiable from the first-order moment restrictions

3. FIRST-ORDER NL MOMENT RESTRICTIONS

3.3 Examples: Factor models

Example 3: Linear model with common factor and contagion (cont.)

$$\begin{cases} y_t = Bf_t + Cy_{t-1} + \varepsilon_t, & \varepsilon_t \sim IIN(0, \Sigma) \\ f_t = \Phi f_{t-1} + \eta_t, & \eta_t \sim IIN(0, Id - \Phi\Phi') \end{cases}$$

under the identification constraint that matrix $B'B$ is diagonal

First-order nonlinear moment restrictions:

$$\begin{aligned} & E_0[\exp(u'(y_t - Cy_{t-1}))] = \exp\left\{\frac{1}{2}u'(\Sigma + BB')u\right\}, \quad \forall u \\ \Leftrightarrow \quad & V_0(y_t - Cy_{t-1}) = \Sigma + BB' \\ \Leftrightarrow \quad & \Gamma_0(0) - \Gamma_0(1)C' - C\Gamma_0(1)' + C\Gamma_0(0)C' = \Sigma + BB', \end{aligned} \tag{3}$$

where $\Gamma_0(0) = V_0(y_t)$ and $\Gamma_0(1) = Cov_0(y_t, y_{t-1})$

Equation (3) is a linear combination of the first two Yule-Walker equations for (y_t)

Special case I: $\Phi = 0$ ***and*** $C = 0$***, i.e. static factor and no contagion***

The first-order identification restriction becomes:

$$\Gamma_0(0) = \Sigma + BB' \quad \Rightarrow \quad \Sigma = \Sigma_0,\ B = B_0 \tag{4}$$

[see e.g. Anderson, Rubin (1956), Lawley, Maxwell (1971)]

Special case II: $\Phi = 0$***, i.e. static factor but possibly contagion***

The order condition for identification from restriction (3)

$$\frac{n(n+1)}{2} + \frac{K(K-1)}{2} \geq n(K+n+1)$$

is **not** satisfied for $n \geq K \geq 1$

Parameter (B_0, C_0, Σ_0) is **not point identifiable** from the first-order nonlinear moment restrictions, i.e. the set

$$\mathcal{E}_0 = \{(B, C, \Sigma) \ : \ \text{solution of the first-order restriction (3)} \ \}$$

is not a singleton

3.3 Examples: Factor models

However, the first-order nonlinear moment restrictions (3) can be rewritten as

$$\Sigma_0 + B_0 B_0' + (C - C_0)\Gamma_0(0)(C - C_0)' = \Sigma + BB'$$

$\Rightarrow$ Parameter (B_0, C_0, Σ_0) is **set identifiable** under (4) since:

$$\Sigma_0 + B_0 B_0' = \min\{\Sigma + BB' : (\Sigma, B, C) \in \mathcal{E}_0\},$$

where the minimum is with respect to the ordering on symmetric matrices

Set identification holds even if the order condition fails!

General case: dynamic factor and contagion

The parameters are neither point identifiable, nor set identifiable, from the first-order moment restrictions

Higher-order moment restrictions are needed: next section!

Darolles, Dubecq, Gourieroux (2014) show that the loadings matrix B is identifiable under a full-rank condition for a multivariate partial autocovariance of order 2 of the observable process

Their identification strategy relies on the fact that the linear combinations of the components of y_t, which are immune to the unobserved factor f_t, are uncorrelated with y_{t-2}, y_{t-3}, ..., conditional on y_{t-1}

Example 4: Conditionally Gaussian factor model with stochastic volatility in the factor (cont.)

$$\begin{cases} y_t = \beta h_t^{1/2} \eta_t + \varepsilon_t, & \varepsilon_t \sim IIN(0, \Sigma), \\ h_t \sim ARG(\delta, \rho, c), & \eta_t \sim IIN(0, 1), \end{cases}$$

under the identification constraint $c = 1 - \rho$, which implies $h_t \sim \gamma(\delta)$

Variance-covariance matrix Σ is not necessarily diagonal

The first-order nonlinear moment restrictions are:

$$E_0[\exp(u' y_t)] = \exp\left\{ \frac{u'\Sigma u}{2} - \delta \log\left(1 - \frac{(u'\beta)^2}{2}\right)\right\}, \quad \forall u$$

Parameters β, Σ and δ are identifiable since:

$$\frac{u'\Sigma u}{2} - \delta \log\left(1 - \frac{(u'\beta)^2}{2}\right) = \frac{u'\Sigma_0 u}{2} - \delta_0 \log\left(1 - \frac{(u'\beta_0)^2}{2}\right), \quad \forall u$$

$$\Leftrightarrow \qquad \Sigma = \Sigma_0, \ \beta = \beta_0, \ \delta = \delta_0$$

The autocorrelation ρ of the SV process is not identifiable from the first-order nonlinear moment restrictions

Parameter identification is *simpler* in this nonlinear framework compared to the Gaussian linear factor model!

4. HIGHER-ORDER NL MOMENT RESTRICTIONS

4.1 Second-order nonlinear moment restrictions

Let us define $\psi_t(u,C,\theta) = a(u,x_t,\theta)'[Cy_{t-1} + d(x_t,\theta)] + b(u,x_t,\theta)$. Then:

$$\begin{aligned} & E\left[\exp\{u'y_t + \tilde{u}'y_{t-1} - \psi_t(u,C,\theta) - \psi_{t-1}(\tilde{u},C,\theta)\} | \underline{y_{t-2}}, \underline{f_t}, \underline{x_t}\right] \\ = \; & \exp\{a(u,x_t,\theta)'Bf_t + a(\tilde{u},x_{t-1},\theta)'Bf_{t-1}\}, \quad \forall\, u,\tilde{u} \end{aligned}$$

By the change of variable in Assumption A.3, and integrating out the unobservable effects (f_t, f_{t-1}), we get a continuum of **second-order nonlinear moment restrictions**:

$$E\left[\exp\{u_t'y_t + \tilde{u}_t'y_{t-1} - \psi_t(u_t,C,\theta) - \psi_{t-1}(\tilde{u}_t,C,\theta)\} | \underline{x_t}\right] = E\left[\exp\{v_1'f_t + \tilde{v}_1'f_{t-1}\} | \underline{x_t}\right],$$

for all admissible v, $\tilde{v}$, where $u_t = u(v,x_t,\theta,B)$ and $\tilde{u}_t = u(\tilde{v},x_t,\theta,B)$

4. HIGHER-ORDER NL MOMENT RESTRICTIONS

4.1 Second-order nonlinear moment restrictions

Proposition 3: *Under Assumptions A.1-A.3, the first- and second-order nonlinear moment restrictions allow for identifying (generically):*

- **i)** *The complete model, if the distribution of Markov process (f_t) is specified parametrically.*
- **ii)** *The stationary and transition distribution of Markov process (f_t), if the regression parameters are known and the distribution of process (f_t) is let unspecified.*

4. HIGHER-ORDER NL MOMENT RESTRICTIONS

4.2 Third-order nonlinear moment restrictions

Third-order nonlinear moment restrictions from the joint Laplace transform of y_t, y_{t-1}, y_{t-2} conditional on $\underline{y_{t-3}}, \underline{f_t}, \underline{x_t}$

Allow to identify the conditional transition densities of the unobservable effect $g_1(f_t|f_{t-1}; B, C, \theta)$ at horizon 1, and $g_2(f_t|f_{t-2}; B, C, \theta)$ at horizon 2, for given regression parameter values B, C, θ

Kolmogorov relationship implies an infinite number of restrictions:

$$g_2(f_t|f_{t-2}; B, C, \theta) = \int g_1(f_t|f_{t-1}; B, C, \theta) g_1(f_{t-1}|f_{t-2}, B, C, \theta) df_{t-1},$$
$$\forall\ f_t,\ f_{t-2},\ B,\ C, \theta,$$

which can be used to identify the regression parameters

Proposition 4: *Under Assumptions A.1-A.3, the first-, second- and third-order nonlinear moment restrictions (generically) identify the regression parameters and the unspecified transition of Markov process* (f_t).

Example 3: Linear model with common factor and contagion (cont.)

The nonlinear moment restrictions at order 1, 2 and 3 are :

$$\begin{aligned}
\Gamma_0(0) + C\Gamma_0(0)C' - \Gamma_0(1)C' - C\Gamma_0(1)' &= BB' + \Sigma, \text{ for order } 1, \\
\Gamma_0(1) + C\Gamma_0(1)C' - \Gamma_0(2)C' - C\Gamma_0(0) &= B\Phi B', \text{ to be added for order } 2, \\
\Gamma_0(2) + C\Gamma_0(2)C' - \Gamma_0(3)C' - C\Gamma_0(1) &= B\Phi^2 B', \text{ to be added for order } 3,
\end{aligned}$$

where $\Gamma_0(h) = Cov_0(y_t, y_{t-h})$, $h = 0, 1, 2, 3$

Let us focus on second-order moment restrictions

If symmetric matrix Σ is unrestricted, the order condition for identification is not satisfied

If Σ is assumed diagonal, the order condition is satisfied if

$$K \text{ is sufficiently small}$$

In both cases, the Gaussian linear model with common unobservable factor and contagion is **not** identifiable

Thus, the order condition is neither sufficient, nor necessary, for identification

Example 4: Conditionally Gaussian factor model with stochastic volatility in the factor (cont.)

The second-order nonlinear moment restrictions are

$$E_0[\exp(u'y_t + \tilde{u}'y_{t-1})] = \exp\left(\frac{1}{2}u'\Sigma u + \frac{1}{2}\tilde{u}'\Sigma\tilde{u}\right) E\left[\exp\left(\frac{(u'\beta)^2}{2}h_t + \frac{(\tilde{u}'\beta)^2}{2}h_{t-1}\right)\right],$$

for all $u, \tilde{u}$. Let us assume $\beta_{0,1} \neq 0$, and consider argument vectors $u = (\sqrt{2v}/\beta_{0,1}, 0, \ldots, 0)'$ and $\tilde{u} = (\sqrt{2\tilde{v}}/\beta_{0,1}, 0, \ldots, 0)'$. We get:

$$E_0\left[\exp\left(\frac{\sqrt{2v}}{\beta_{0,1}}y_{1,t} + \frac{\sqrt{2\tilde{v}}}{\beta_{0,1}}y_{1,t-1}\right)\right] = E\left[\exp\left(v(h_t + \lambda_0) + \tilde{v}(h_{t-1} + \lambda_0)\right)\right], \ \forall v, \tilde{v},$$

where $\beta_{0,1}$ and $\lambda_0 = \sigma^2_{0,11}/\beta^2_{0,1}$ are identified from the first-order restrictions

The ARG dynamics of (h_t) is identifiable from the second-order restrictions

Consider the semiparametric model with common factor and contagion

$$y_t = B_0 f_t + C_0 y_{t-1} + \varepsilon_t$$

where the unobservable processes (ε_t) and (f_t) are mutually independent and

- (ε_t) is a n-dim. stationary strong white noise with unconditional Laplace transform $E_0[\exp(u'\varepsilon_t)] = \exp[\psi_\varepsilon^0(u)]$
- (f_t) is a K-dim. stationary first-order Markov process with joint Laplace transform $E_0[\exp(u'f_t + w'f_{t-1})] = \exp[\psi_f^0(u, w)]$

Parameter $(B_0, C_0, \psi_\varepsilon^0, \psi_f^0)$ has both finite- and infinite-dimensional components

i) Identification of K, B_0 *and* C_0

Consider the identifiable function:

$$h_0(u, \tilde{u}, C) := \log E_0 \left[\exp \left(u'(y_t - Cy_{t-1}) + \tilde{u}'(y_{t-1} - Cy_{t-2}) \right) \right]$$

It is such that:

$$h_0(u, \tilde{u}, C_0) \quad = \quad \psi_\varepsilon^0(u) + \psi_\varepsilon^0(\tilde{u}) + \psi_f^0(B_0'u, B_0'\tilde{u})$$

Main idea: cross-terms in u, $\tilde{u}$ are informative on common factor dynamics and factor loadings!

The matrix of second-order cross partial derivatives of function $h_0(., ., C_0)$

$$\frac{\partial^2 h_0(u, \tilde{u}, C_0)}{\partial u \partial \tilde{u}'} = B_0 \frac{\partial^2 \psi_f^0(B_0'u, B_0'\tilde{u})}{\partial v \partial w'} B_0'$$

has reduced rank $\leq K$ and its null space contains the orthogonal complement of the range of matrix B_0, for all u, $\tilde{u}$

4. HIGHER-ORDER NL MOMENT RESTRICTIONS

4.4 Semi-nonparametric identification of the linear model with contagion and (non Gaussian) common factor

Assumption A.4:

i) $\dfrac{\partial^2 h_0(u, \tilde{u}, C)}{\partial u \partial \tilde{u}'}$ *has reduced rank* $\forall (u, \tilde{u})$ *iff* $C = C_0$.

ii) *Processes* (f_t) *and* (ε_t) *are **not** Gaussian, and process* (f_t) *is **not** i.i.d.*

iii) $\exists (v^*, w^*)$ *:* $\dfrac{\partial^2 \psi_f^0(v^*, w^*)}{\partial v \partial w'}$ *has full rank.*

Proposition 5: *Under Assumption A.4, the dimension* K *of the factor space, the range of matrix* B_0*, and the contagion matrix* C_0 *are identifiable.*

Under Assumption A.4 identification is possible even if the order condition fails!

ii) Identification of ψ_ε^0 *and* ψ_f^0

Assumption A.5: *The factor process* (f_t) *is Compound Autoregressive (CaR):*

$$E_0[\exp(u' f_t)|f_{t-1}] = \exp[a_0(u)' f_{t-1} + b_0(u)]$$

Functional parameters: a_0 and marginal log-Laplace transform $\varphi_f^0 = \psi_f^0(\cdot, 0)$

Normalize (f_t) such that the upper (K, K) block of matrix B_0 is Id_K, and consider the identifiable function

$$\tau_0(u, v; h) := \log \frac{E_0[\exp(u'\tilde{y}_t + v'\tilde{y}_{t-h})]}{E_0[\exp(u'\tilde{y}_t)]E_0[\exp(v'\tilde{y}_{t-h})]},$$

where $\tilde{y}_t$ denotes the $(K, 1)$ vector process of the first K elements of $y_t - C_0 y_{t-1}$. Then:

$$\tau_0(u, v; h) = \varphi_f^0[a_0^{\circ h}(u) + v] - \varphi_f^0[a_0^{\circ h}(u)] - \varphi_f^0(v), \quad \forall u, v, \ \forall h \geq 1, \qquad (6)$$

where $a_0^{\circ h}$ denotes function a_0 compounded h times with itself

4. HIGHER-ORDER NL MOMENT RESTRICTIONS

4.4 Semi-nonparametric identification of the linear model with contagion and (non Gaussian) common factor

From (6) it follows

$$\tau_0(u, v; 2) = \tau_0[a_0(u), v; 1], \quad \forall u, v$$

and

$$\frac{\partial \tau_0}{\partial v}(u, 0; 1) = \frac{\partial \varphi_f^0}{\partial u}[a_0(u)], \quad \forall u$$

Proposition 6: *Suppose that Assumptions A.4-A.5 hold and function $a_0 : \mathcal{D}_0 \to \mathcal{R}_0$ is one-to-one, where domain $\mathcal{D}_0$ is a neighbourhood of 0 in $\mathbb{C}^K$, and $\mathcal{R}_0 \subset \mathcal{D}_0$. Then:*

- **i)** *Function $a_0(u)$ is identifiable for $u \in \mathcal{D}_0$,*
- **ii)** *Function $\varphi_f^0(u)$ is identifiable for $u \in \mathcal{R}_0$, and*
- **iii)** *Function $\psi_\varepsilon(u)$ is identifiable for $B_0' u \in \mathcal{R}_0$.*

CONCLUDING REMARKS

This paper studies identification in nonlinear panel or time series models with unobservable dynamic effects under the assumption of an exponential affine specification

Nonlinear moment restrictions are obtained from the conditional Laplace transform of the endogenous observable variables y_t, y_{t-1}, ... by integrating out the unobservable effects

Second-order nonlinear moment restrictions generically identify all model parameters, under a parametric specification for the individual effects dynamics

Third-order nonlinear moment restrictions are generically sufficient for nonparametric identification of the individual effects dynamics

CONCLUDING REMARKS

Even when the order condition fails, identification can be possible by exploiting “corner”, or reduced-rank, properties of the true parameter values

Identification may be simpler in a nonlinear framework, with non Gaussian factors and innovations, and with dynamic factors

The proposed identification strategies suggest parametric estimation with GMM [see e.g. Singleton (2001), Jiang, Knight (2002), Chacko, Viceira (2003), Carrasco, Chernov, Florens, Ghysels (2007)] and nonparametric estimation by plug-in methods