Parametric and Nonparametric Quantile Regression Methods for First-Price Auctions: A Signal Approach

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Work in Progress

Plan of the talk

Notations

Quantile regression and additive/interactive quantile models

Quantile and auction

Identification of linear (sieve) quantile specification

Augmented (Sieve) Quantile Regression: dimension reduction and boundary free estimation

A small simulation experiment

Extension to interdependent values

Sealed bids first-price auction

- Auctioned good, with characteristics known to the bidders and econometrician
- Bidder forms a bid which is not observed by his opponents
- Bids are sealed and collected
- Bids are opened
- Winner = largest bid
- Paid price = bid of the winner = largest bid

Notations

ℓ = auction, $\ell = 1, \dots, L$
x_ℓ = auction good covariate
I_ℓ = number of bidders
i = bidders, $i = 1, \dots, I_\ell$

Notations (cont'd): private value case

Private value $V_{i\ell}$: iid given (x_ℓ, I_ℓ)

- Common knowledge cdf $F(v|x_\ell, I_\ell)$, continuous pdf $f(v|x_\ell, I_\ell) > 0$ over its compact support
- Conditional quantile $V(\alpha|x_\ell, I_\ell)$, quantile level $\alpha \in [0, 1]$
- Private value rank $A_{i\ell} = F(V_{i\ell}|x_\ell, I_\ell)$: prob. that an opponent has a pv smaller than $V_{i\ell}$

Important property of $A_{i\ell}$: $[0, 1]$-uniform and independent of (x_ℓ, I_ℓ)

Notations (cont'd): observed bids

Observed bids $B_{i\ell}$: iid bids given (x_ℓ, I_ℓ)

- Cdf $G(b|x_\ell, I_\ell)$, pdf $g(b|x_\ell, I_\ell)$
- Conditional quantile $B(\alpha|x_\ell, I_\ell)$, $\alpha \in [0,1]$
- Bid rank $U_{i\ell} = G(B_{i\ell}|x_\ell, I_\ell)$

Why quantiles?

Fondamental Simulation Theorem: *The private value rank* $A_{i\ell}$ *is independent of* (x_ℓ, I_ℓ) *with a* $[0,1]$*-uniform distribution, and satisfies,*

$$V_{i\ell} = V\left(A_{i\ell}|x_\ell, I_\ell\right)$$

- Allow to simulate $V_{i\ell}$ in full generality

- Since, in most case,

$$\left(V_{i\ell}, B_{i\ell}\right) = \left(V\left(A_{i\ell}|x_\ell, I_\ell\right), B\left(A_{i\ell}|x_\ell, I_\ell\right)\right)$$

  counterfactuals as the expected revenue $\mathbb{E}\left[V_{i\ell} - B_{i\ell}|x_\ell\right]$ for a given mechanism

- Since $V\left(\alpha|x_\ell, I_\ell\right) = F^{-1}\left(\alpha|x_\ell, I_\ell\right)$, quantile can be estimated nonparametrically as fast as a c.d.f. and with faster rates than a p.d.f

Private value rank and Milgrom-Weber model

- In Milgrom & Weber (1982),

$$W_{i\ell} = W_i\left(\widetilde{A}_{0\ell}, \widetilde{A}_{1\ell}, \ldots, \widetilde{A}_{I_\ell \ell}, x_\ell\right), \quad \widetilde{A}_{1\ell}, \ldots, \widetilde{A}_{I\ell} \text{ bidders signals}$$

where the ith bidder knows $\widetilde{A}_{i\ell}$ but not $\widetilde{A}_{j\ell}$, and $\left(\widetilde{A}_{1\ell}, \ldots, \widetilde{A}_{I\ell}\right)$ independent of x_ℓ

- In the private value case

$$V_{i\ell} = V_i\left(\widetilde{A}_{i\ell}, x_\ell\right), \quad \widetilde{A}_{i\ell} \text{ independent r.v}$$

$\Rightarrow$ The private value rank $A_{i\ell}$ can be viewed as a standardized signal

- Issue: the general model is not identified (Laffont & Vuong, 1996)
  $\Rightarrow$ Extension of the paper: a new additive specification for the general model

An additive specification

$$
\begin{aligned}
W_{i\ell} &= W_i\left(A_{1\ell}, \ldots, A_{I_\ell \ell}; x_\ell\right) \\
&= \sum_{j=0}^{I_\ell} \pi_{ij} V_j\left(A_{j\ell}; x_\ell\right), \quad \pi_{ii} = 1
\end{aligned}
$$

- $V_i\left(A_{i\ell}, x_\ell\right) = V_i\left(A_{i\ell}, x_\ell, z_{i\ell}\right)$, $z_{i\ell}$ individual characteristic ("capacity")
- $V_i\left(A_{i\ell}, x_\ell\right)$: intrinsic private value of the good for the ith bidder
- Interactions $\Rightarrow$ the final value must aggregate the intrinsic private values ("prestige", trading after auction, etc...).

Differs from Somaini (2014), $W_i = W_i\left(A_{1\ell}, \ldots, A_{I_\ell \ell}; x_\ell, z_{i\ell}\right)$

Dimension reduction issues

Many covariate available in auction datasets:

- Athey, Levin & Seira (2011) or Li & Zheng (2009, 2012): 5 to 15 covariates for 1,000 observations
- Haile & Tamer (2003), Aradillas-López, Ghandi & Quint (2013): 5-6 covariates for few thousands observations

Not many dimension reduction methods for first-price auction

- Paarsch & Hong (2006): implement p.d.f. estimation as in G., Perrigne & Vuong (2000) using an index assuming $V_{i\ell} = g\left(x_\ell' \beta\right) + \varepsilon_{i\ell}$. A quantile approach as in Chaudhuri, Doksum & Samarov (1997) would be less restrictive
- Haile, Hong & Shum (2003), Rezende (2008): $V_{i\ell} = x_\ell' \beta + v_{i\ell}$ implies $B_{i\ell} = x_\ell' \beta + b_{i\ell}$ where the p.d.f of $v_{i\ell}$ can be estimated from the ones of $b_{i\ell}$ as in G., Perrigne & Vuong (2000)

Additive quantile specification

Various lower dimensional models have been proposed to restrict the general quantile specification

$$V_{i\ell} = V\left(A_{i\ell}|x_\ell, I_\ell\right)$$

- Quantile regression (Koenker & Bassett, 1978);

$$V_{i\ell} = x_{i\ell}'\beta_1\left(A_{i\ell}|I_\ell\right) + \beta_0\left(A_{i\ell}|I_\ell\right) = X_{i\ell}'\beta\left(A_{i\ell}|I_\ell\right)$$

  Nests Haile, Hong & Shum (2003), Rezende (2008) ($\beta_1\left(A_{i\ell}|I_\ell\right) = \beta_1$) and allows for interactions between signal and covariates

- Additive specification (Horowitz & Lee, 2005): for $x_\ell = \left[x_{1\ell}, \ldots, x_{d\ell}\right]$,

$$V_{i\ell} = V_1\left(A_{i\ell}|x_{1\ell}, I_\ell\right) + \cdots + V_d\left(A_{i\ell}|x_{1d}, I_\ell\right)$$

- Additive interactive specification (Andrews & Whang (1990) for regression)

$$V_{i\ell} = \sum_{k=1}^{D} \sum_{j_1<\cdots<j_k} V_{j_1,\ldots,j_D}\left(A_{i\ell} | x_{j_1\ell}, \ldots, x_{j_k\ell}, I_\ell\right)$$

$\Rightarrow$ A wide class of models ranging from parametric to nonparametric

The general linear quantile specification of the paper

All previous specifications can be nested in the linear sieve specification with D interactions ($0 \leq D \leq \dim x$)

$$V_{i\ell} = \sum_{k=0}^{\infty} P_k\left(x_\ell\right)\gamma_k\left(A_{i\ell}|I_\ell\right), \quad P_k\left(x_\ell\right) = P_k\left(x_{j_1(k)\ell}, \ldots, x_{j_D(k)\ell}\right),$$

and where $\gamma_k\left(\alpha|I\right) = \left\langle V\left(\alpha|x, I\right), P_k\left(x\right)\right\rangle_x$ for orthonormal sieve

$\Rightarrow$ An infinite dimensional version of Koenker & Bassett (1978) quantile regression

Other econometric issues

Econometric issues with G., Perrigne & Vuong (2000) two step kernel density estimation method

- Boundary bias for the upper and lower tails distribution (Hickman & Hubbard, 2014)
- Lack of clearcut bandwidth choice (Henderson, List, Millmet, Parmeter & Price, 2012)

The proposed new quantile methodology is helpful regarding these issues

Quantile and auction in the econometric literature

- Haile, Hong & Shum (2003): Quantile, dimension reduction using a regression model. See also Rezende (2008)
- Marmer & Shneyerov (2012): avoids estimation of private values
- G. & Sabbah (2012), Fan, Li & Pesendorfer (2013,WP): LP quantile estimation
- Menzel & Morganti (2013): order statistic (sample quantile) approach for second-price auction
- Gimenes (2013, WP): QR for ascending auction

Rest of the talk

- Quantile identification
- A key property: Stability of linear quantile specification
- Augmented (Sieve) Quantile regression
- Interdependent value extension

Quantile identification: a preliminary lemma

Lemma *Suppose that the values W_i are such,*

$$W_i = W_i\left(A_0, A_1, \ldots, A_I, x, I\right), \; i = 1, \ldots, I,$$

where $\left(A_0, A_1, \ldots, A_I\right)$ is independent of (x, I), each A_i are $[0,1]$ uniform, and that each bidder plays a strictly increasing strategy,

$$B_i = s_i\left(A_i | x, I\right), \; s_i\left(\cdot | x, I\right) \; \uparrow \; \text{for all} \; (x, I).$$

- $\Rightarrow$ No equilibrium assumption. Increasing strategy assumption strong enough to identify A_i and $s_i\left(\cdot|\cdot,\cdot\right)$ in a constructive way
- Bayesian Nash Equilibrium generates increasing strategies (Reny & Zhamir, 2004)

Lemma cont'd: Signal identification

(i) The signal A_i, $i \geq 1$, *can be recovered from the observed bids with,*

$$A_i = G_i\left(B_i | x, I\right),$$

where $G_i\left(\cdot | x, I\right)$ *is the conditional c.d.f of* B_i;

- $\Rightarrow$ the joint distribution of $\left(A_1, \ldots, A_I\right)$ is identified
- The signal A_i can be estimated (known identity or $G_i\left(B_i | x, I\right) = G\left(B_i | x, I\right)$)

Lemma cont'd: strategy identification

$$A_i = G_i\left(B_i|x, I\right) \Rightarrow B_i = B_i\left(A_i|x, I\right)$$

(ii) the strategy $s_i\left(\cdot|x, I\right)$ *is identified by the conditional bid quantile function,*

$$s_i\left(A|x, I\right) = B_i\left(A|x, I\right), \text{ for any } A \in [0, 1];$$

- Contrasts with strategies depending upon the private value for symmetric IPV.

Lemma cont'd: Probability of Winning

(iii) under symmetric IPV, that is if A_i *independent,* $V_i = V(A_i, x, I)$ *and* $B_i = B(A_i|x, I)$ *for all* $i = 2, \ldots, I$*, the probability that a bid* $B(A|x, I)$ *wins is* A^{I-1}.

- Under asymmetry or interdependent value, the probability that a bid $B_1(A|z, I)$ is also identified since it is

$$\mathbb{P}\left(B_1(A|x, I) > \max_{i=2,\ldots,I} B_i(A_i|x, I)\,|A_1, x, I\right)$$

which depends upon the identified $B_i(\cdot|x, I)$ and the identified joint distribution of $(A_1, \ldots, A_I)'$. But no close form expression in general

Quantile under symmetric IPV and Bayesian Nash Equilibrium

$\Rightarrow$ Under symmetric IPV and BNE, $B(\cdot|x, I)$ is the optimal strategy

This identifies $V(\alpha|x, I)$ in a simple linear way under risk neutrality

The risk neutral expected utily of a bid $B(a|x, I)$ given first bidder signal $A_1 = A$ is

$$(V_1 - B(a|x, I))\, a^{I-1} = (V(A|x, I) - B(a|x, I))\, a^{I-1}$$

Since the optimal bid is $B(A|x,I)$,

$$(V(A|x,I) - B(a|x,I))\, a^{I-1} \leq (V(A|x,I) - B(A|x,I))\, A^{I-1}$$

for all $a \in [0,1]$.

Hence, for all $A \in (0,1)$,

$$\frac{\partial}{\partial a}\left[(V(A|x,I) - B(a|x,I))\, a^{I-1}\right]\Big|_{a=A} = 0$$
$$\Leftrightarrow -B^{(1)}(A|x,I)\, A^{I-1} + (I-1)(V(A|x,I) - B(A|x,I))\, A^{I-2} = 0$$

Rearranging gives the differential equation,

$$V(A|x,I) = B(A|x,I) + \frac{A \times B^{(1)}(A|x,I)}{I-1}, \quad B(0|x,I) = V(0|x,I)$$

which is the quantile version of the identification method in G., Perrigne & Vuong (2000)

$$V_{i\ell} = B_{i\ell} + \frac{1}{I_\ell - 1}\frac{G(B_{i\ell}|x_\ell, I_\ell)}{g(B_{i\ell}|x_\ell, I_\ell)}$$

Suggests to estimate $V(\alpha|x, I)$ using

$$\widehat{V}(\alpha|x, I) = \widehat{B}(\alpha|x, I) + \frac{\alpha \widehat{B}^{(1)}(\alpha|x, I)}{I-1}$$

as in G. & Sabbah (2012) or Fan et al. (2013).

However,

- Not so many good estimators of $B^{(1)}(\alpha|x, I)$ in the literature
- It may be fruitful to solve the linear differential equation before estimating

A key lemma: (i) stability of linear specification

(i) The conditional quantile function of optimal bids is given by the linear operator,

$$B\left(\alpha|x,I\right)=\frac{I-1}{\alpha^{I-1}}\int_0^\alpha t^{I-2}V\left(t|x,I\right)dt.$$

linear specification for $V\left(\cdot|x,I\right)$

$\Rightarrow$

linear specification for $B\left(\cdot|x,I\right)$

as noted in Haile et al. (2003) and Rezende (2008) for the particular case of regression.

Example: if, for some $\gamma_k(\alpha|I) = \langle V(\alpha|\cdot, I), P_k(\cdot)\rangle$

$$V(\alpha|x, I) = \sum_{k=0}^{\infty} P_k(x)\,\gamma_k(\alpha|I),$$

then

$$B(\alpha|x, I) = \sum_{k=0}^{\infty} P_k(x)\,\beta_k(\alpha|I)$$

with

$$\beta_k(\alpha|I) = \frac{I-1}{\alpha^{I-1}} \int_0^{\alpha} t^{I-2}\gamma_k(t|I)\,dt.$$

A key lemma (ii): identification

(ii) The conditional private values quantile function can be recovered from the bid one,

$$V(\alpha|x,I) = B(\alpha|x,I) + \frac{\alpha}{I-1} B^{(1)}(\alpha|x,I).$$

Example (Cont'd): since

$$B(\alpha|x,I)=\sum_{k=0}^{\infty}P_k(x)\,\beta_k(\alpha|I)$$

$$\text{with } \beta_k(\alpha|I)=\frac{I-1}{\alpha^{I-1}}\int_0^{\alpha}t^{I-2}\gamma_k(t|I)\,dt,$$

then

$$V(\alpha|x,I)=\sum_{k=0}^{\infty}P_k(x)\,\gamma_k(\alpha|I)$$

$$\text{with } \gamma_k(\alpha|I)=\beta_k(\alpha|I)+\frac{\alpha}{I-1}\beta_k^{(1)}(\alpha|I).$$

Estimation methology

1. Postulate a quantile regression specification for the private values or set $X = (P_1(x), \ldots, P_{k_L}(x))$,

$$\Rightarrow V(\alpha|x,I) = X'\gamma(\alpha|I) + bias_V \text{ (no bias for QR)}$$

2. By the stability property

$$B(\alpha|x,I) = X'\beta(\alpha|I) + bias_B \text{ (no bias for QR)}$$

3. Given an estimation of $\widehat{\beta}(\alpha|I)$ and $\widehat{\beta}^{(1)}(\alpha|I)$, set

$$\widehat{\gamma}(\alpha|I) = \widehat{\beta}(\alpha|I) + \frac{\alpha\widehat{\beta}^{(1)}(\alpha|I)}{I-1}, \quad \widehat{V}(\alpha|x,I) = X'\widehat{\gamma}(\alpha|I)$$

$\Rightarrow$ Needs new techniques to find good estimation of $\beta^{(1)}(\alpha|I)$, an issue mostly ignored in the literature.

Standard quantile regression

Check function $\rho_\alpha(t) = t(\alpha - \mathbb{I}(t \leq 0))$

$$\beta(\alpha|I) = \arg\min_\beta \mathbb{E}\left[\mathbb{I}(I_\ell = I)\,\rho_\alpha\left(B_{i\ell} - X_\ell'\beta\right)\right]$$

$$\Rightarrow \widehat{\beta}(\alpha|I) = \arg\min_\beta \frac{1}{L}\sum_{\ell=1}^{L} \mathbb{I}(I_\ell = I)\,\rho_\alpha\left(B_{i\ell} - X_\ell'\beta\right)$$

- Does not give an estimator of $\beta^{(1)}(\alpha|I)$
- Difficult to define for $\alpha = 0$ or $\alpha = 1$

Augmented quantile regression

- Allow small variation of α in the check function $\rho_\alpha(t)$
- Expand $\beta(\alpha + ht)$ to estimate $\beta^{(1)}(\alpha|I)$ by local polynomial smoothing

For $a = \alpha + ht$, $h > 0$ bandwidth, and $\beta(\cdot|I)$ $s+2$ differentiable,

$$
\begin{aligned}
&X'\beta(a|I) \\
&\quad = X'\left\{\beta(\alpha|I) + (a-\alpha)\beta^{(1)}(\alpha|I) + \cdots + \frac{(a-\alpha)^{s+1}}{(s+1)!}\beta^{(s+1)}(\alpha|I)\right\} \\
&\qquad + O\left((a-\alpha)^{s+2}\right) \\
&\quad = X(a-\alpha)'\boldsymbol{\beta}(\alpha|I) + O\left((a-\alpha)^{s+2}\right),\ X(t) = \begin{bmatrix} 1 \\ \vdots \\ \frac{t^{s+1}}{(s+1)!} \end{bmatrix} \otimes X
\end{aligned}
$$

where $\boldsymbol{\beta}(\alpha|I) = \left[\beta(\alpha|I)', \beta^{(1)}(\alpha|I)', \ldots, \beta^{(s+1)}(\alpha|I)'\right]'$.

Objective function of the augmented quantile regression

$K(\cdot)$ kernel, h bandwidth $\Rightarrow$ objective function $\widehat{\mathcal{R}}(\boldsymbol{\beta};\alpha,I)$ is

$$\frac{1}{LIh}\sum_{\ell=1}^{L}\mathbb{I}(I_\ell=I)\sum_{i=1}^{I_\ell}\int_0^1\rho_a\left(B_{i\ell}-X_\ell(a-\alpha)'\boldsymbol{\beta}\right)K\left(\frac{a-\alpha}{h}\right)da$$

$$=\frac{1}{LI}\sum_{\ell=1}^{L}\mathbb{I}(I_\ell=I)\sum_{i=1}^{I_\ell}\int_{-\frac{\alpha}{h}}^{\frac{1-\alpha}{h}}\rho_{\alpha+ht}\left(B_{i\ell}-X_\ell(ht)'\boldsymbol{\beta}\right)K(t)\,dt.$$

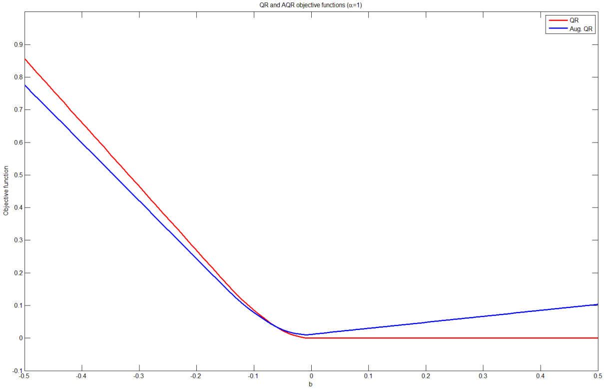
The augmented quantile regression estimator is

$$\widehat{\boldsymbol{\beta}}(\alpha|I) = \arg\min_{\boldsymbol{\beta}} \widehat{\mathcal{R}}(\boldsymbol{\beta}; \alpha, I), \quad \widehat{\boldsymbol{\beta}}(\alpha|I) = \begin{bmatrix} \widehat{\beta}^{(0)}(\alpha|I) \\ \widehat{\beta}^{(1)}(\alpha|I) \\ \vdots \\ \widehat{\beta}^{(s+1)}(\alpha|I) \end{bmatrix}$$

$$\widehat{V}(\alpha|x, I) = X' \left[\widehat{\beta}^{(0)}(\alpha|I) + \frac{\alpha}{I-1} \widehat{\beta}^{(1)}(\alpha|I) \right]$$

Boundary behavior

Smoothing gives a convex AQR function for $\alpha = 0, 1 \Rightarrow \widehat{\boldsymbol{\beta}}\,(0|I)$ and $\widehat{\boldsymbol{\beta}}\,(1|I)$ are well defined



Assumptions (QR case)

1. X in a compact set, $-\infty < X'\gamma\,(0|I) < X'\gamma\,(1|I) < \infty$, $\sup_\alpha X'\gamma^{(1)}\,(\alpha|I) < \infty$, $\inf_\alpha X'\gamma^{(1)}\,(\alpha|I) > 0$

   $\Rightarrow$ boundary bias for kernel estimation

2. $\alpha \in [0,1] \mapsto \gamma\,(\alpha|I)$ $(s+1)$th continuously differentiable $\Rightarrow$ $\beta\,(\alpha|I)$ $(s+2)$ cont. diff. except at $\alpha = 0$.

Theoretical results for quantile regression models

Theorem *Suppose the private value quantile regression specification is correct. Then if* $h \to 0$ *with* $\log^3 L / \left(Lh^2\right) = O\left(1\right)$

$$\sup_{(\alpha,x)\in[0,1]\times\mathcal{X}} \left|\widehat{V}\left(\alpha|x,I\right) - V\left(\alpha|x,I\right)\right| = O_{\mathbb{P}}\left(\left(\frac{\log L}{LIh}\right)^{1/2} + h^{s+1}\right).$$

It also holds that

$$\sup_{(\alpha,x)\in[0,1]\times\mathcal{X}} \left|\widehat{B}\left(\alpha|x,I\right) - B\left(\alpha|x,I\right)\right| = O_{\mathbb{P}}\left(\left(\frac{1}{LI}\right)^{1/2} + h^{s+2}\right).$$

Uniform consistency rate for private values

$$\sup_{(\alpha,x)\in[0,1]\times\mathcal{X}} \left| \widehat{V}\left(\alpha|x,I\right) - V\left(\alpha|x,I\right) \right| = O_{\mathbb{P}}\left(\left(\frac{\log L}{LIh} \right)^{1/2} + h^{s+1} \right)$$

- Rate given by $\widehat{\beta}^{(1)}\left(\alpha|I\right)$. No boundary bias at $\alpha = 0$ or 1.

- Optimal rate $= \left(\frac{\log L}{LI}\right)^{\frac{s+1}{2(s+1)+1}}$ = minimax optimal rate of G., Perrigne & Vuong (2000) with no covariate and for all $s > 0$. Achieved when
$$h \asymp \left(\frac{\log L}{LI}\right)^{\frac{1}{2(s+1)+1}}.$$

- CLT + MSE decomposition allowing for plug in bandwidth choice

Private value estimation

$$\widehat{A}_{i\ell} = \arg\min_{\alpha\in[0,1]} \left| B_{i\ell} - \widehat{B}\left(\alpha | x_\ell, I_\ell\right) \right|,$$
$$\widehat{V}_{i\ell} = \widehat{V}\left(\widehat{A}_{i\ell} | x_\ell, I_\ell\right).$$

Lemma *It holds that,*

$$\max_{\ell=1,\dots,L} \max_{i=1,\dots,I_\ell} \left| \widehat{V}_{i\ell} - V_{i\ell} \right| = O_{\mathbb{P}}\left(\left(\frac{\log L}{LIh} \right)^{1/2} + h^{s+1} \right).$$

$$\Rightarrow O_{\mathbb{P}}\left(\frac{\log L}{L} \right)^{\frac{s+1}{2(s+1)+1}} \text{ for optimal bandwidth choice}$$

Holds for all private values due to the absence of boundary bias

Sieve interactive specification

With D interactions and localized sieve as wavelets of cardinal B splines and under suitable bandwdith $(K = h^{-D})$ and smoothness assumptions,

$$\sup_{(\alpha,x)\in[0,1]\times\mathcal{X}} \left| \widehat{V}(\alpha|x,I) - V(\alpha|x,I) \right| = O_{\mathbb{P}}\left(\left(\frac{\log L}{LIh^{D+1}} \right)^{1/2} + h^{s+1} \right)$$

under conditions which imposes $s > \frac{3}{2}(D-1)$ for the optimal $h = (L/\ln L)^{-1/(2s+D+3)}$

IMSE, MSE expansions and CLT

Simulation example

$L = 50$ and $I = 2$

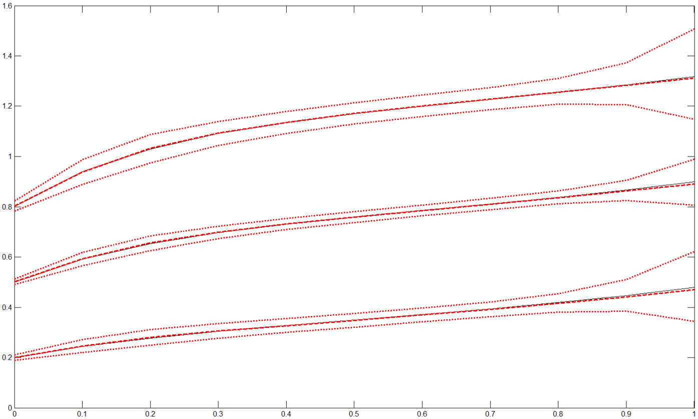
Second-order LP ($s + 1 = 2$), Epachnikov kernel, data-driven $\widehat{h}$ computed from a regression model with truncated exponential error

10,000 replications

$$V(\alpha|x) = \gamma_0(\alpha) + x_1 + \gamma_2(\alpha) x_2,$$
$$\gamma_0(\alpha) = -0.1 \times \log\left(1 - \frac{\alpha}{e}\right),$$
$$\gamma_2(\alpha) = 1 - \exp(-\alpha).$$

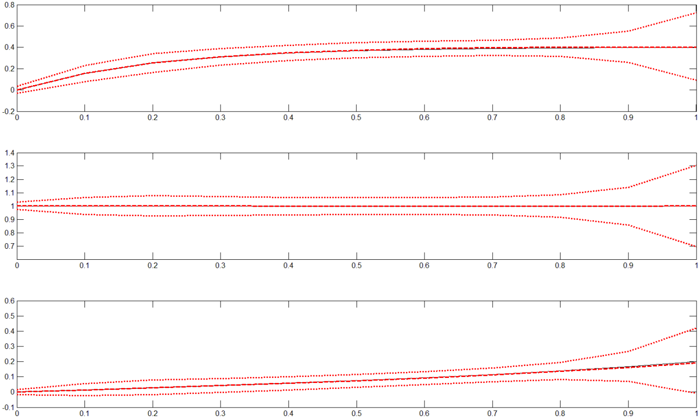
The covariates x_1 and x_2 are two independent uniform variables. x_2 inactive for small α

Quantiles



$$x_1 = x_2 \in \{0.2, 0.5.0.8\}$$

Slope coefficients



$$\gamma_0(\alpha) = -0.1 \times \log\left(1 - \frac{\alpha}{e}\right),\ \gamma_1(\alpha) = 1 \text{ and}$$
$$\gamma_2(\alpha) = 1 - \exp(-\alpha)$$

Extension to additive interdependent value

- I bidders with known identity from now on
- z_i: characteristic of ith bidder observed by all ("capacity" variable as distance to the project, labor force, cash flow,...) $z = (1, z_1, \dots, z_I)'$ full-rank

The general additive specification

$$W_i\left(A;x,z\right)=W_0\left(x,z\right)+V_0\left(A_0;x\right)+\sum_{j=1}^{I}\pi_{ij}V_j\left(A_j;x,z_j\right),$$

$\pi_{ii}=1,\pi_{ij}\geq 0$

$V_j\left(\cdot;x,z_j\right)\uparrow,\ V_j\left(0;x,z_j\right)=0$, and

$$V_j\left(A_j;x,z_j\right)=v_j\left(A_j;z_j\right)+\int_0^{z_j}\frac{\partial V_j\left(A_j;x,t\right)}{\partial z_j}dt$$

$\Rightarrow V_j\left(A_j;x,z_j\right)\neq V_{1j}\left(A_j;x\right)+V_{2j}\left(A_j;x,z_j\right)$

$\Rightarrow$ force an interaction between A_j and z_j

A simple interdependent value specification

$$W_i = \gamma_0 (A_0) + \sum_{j=1}^{I} \pi_{ij} z_j \gamma_j (A_j)$$

- A_j: jth bidder private signal with a $U_{[0,1]}$ distribution
- $z_j \gamma_j (A_j)$: jth bidder "private" component of the ith bidder value W_i, $i = 1, \ldots, I$

  Weighted by π_{ij} in W_i
- $\gamma_0 (A_0)$: common component of the values W_i, $i = 1, \ldots, I$
  A_0: $U_{[0,1]}$ distribution
  Not identified without a completness assumption

Parameter of interest: slope coefficients $\gamma_1 (\cdot), \ldots, \gamma_I (\cdot)$

Assumption

1. *The signals $A_0, A_1, \ldots, A_I$ are affiliated with a conditional c.d.f which is bounded away from 0 over $[0,1]^{I+1}$. The signals are independent of z*
2. *The slope cofficients $\gamma_j(\cdot)$ are strictly increasing with $\gamma_j(0) = 0$ and*
$$\pi_{ii} = 1,\ \pi_{ij} \geq 0$$
3. *Each bidder plays a best-response strictly increasing and differentiable strategy $s_i(A_i; z)$ (Reny and Zamir, 2004)*

$$\Rightarrow s_i(A_i; z) = B_i(A_i; z)$$

Expected profit and best response condition

- Expected profit of a bid $B_i\left(a|z\right)$ given $A_i = \alpha$

$$\mathbb{E}\left[\left(W_i - B_i\left(a|z\right)\right)\mathbb{I}\left\{B_i\left(a|z\right) \geq \max_{j\neq i} B_j\right\}|A_i = \alpha, z\right]$$
$$= \overline{W}_i\left(a|\alpha, z\right) - B_i\left(a|z\right)\omega_i\left(a|\alpha, z\right)$$

where

$$\omega_i\left(a|\alpha, z\right) = \mathbb{E}\left[\mathbb{I}\left\{B_i\left(a|z\right) \geq \max_{j\neq i} B_j\right\}|A_i = \alpha, z\right]$$
$$= \mathbb{P}\left(B_i\left(a|z\right) \text{ wins}|A_i = \alpha, z\right)$$
$$\overline{W}_i\left(a|\alpha, z\right) = \mathbb{E}\left[W_i\mathbb{I}\left\{B_i\left(a|z\right) \geq \max_{j\neq i} B_j\right\}|A_i = \alpha, z\right]$$

- Identification issue: $\overline{W}_i\left(a|\alpha, z\right) \neq W_i$

$$
\begin{aligned}
\overline{W}_i\left(a|\alpha, z\right) = \mathbb{E}&\left[\gamma_0\left(A_0\right)\mathbb{I}\left\{B_i\left(a|z\right) \geq \max_{j\neq i} B_j\right\}|A_i = \alpha, z\right] \\
&+ \sum_{j=1}^{I} \pi_{ij} z_j \mathbb{E}\left[\gamma_j\left(A_j\right)\mathbb{I}\left\{B_i\left(a|z\right) \geq \max_{j\neq i} B_j\right\}|A_i = \alpha, z\right]
\end{aligned}
$$

$$
\Rightarrow \overline{W}_i\left(a|\alpha, z\right) = \overline{\gamma}_{i0}\left(a|\alpha, z\right) + \sum_{j=1}^{I} \pi_{ij} z_j \overline{\gamma}_{ij}\left(a|\alpha, z\right)
$$

with

$$
\begin{aligned}
\overline{\gamma}_{ij}\left(a|\alpha, z\right) &= \mathbb{E}\left[\gamma_j\left(A_j\right)\mathbb{I}\left\{B_i\left(a|z\right) \geq \max_{j\neq i} B_j\right\}|A_i = \alpha, z\right] \\
\overline{\gamma}_{ii}\left(a|\alpha, z\right) &= \gamma_i\left(\alpha\right)\mathbb{P}\left(B_i\left(a|z\right) \geq \max_{j\neq i} B_j|A_i = \alpha, z\right) \\
&= \gamma_i\left(\alpha\right)\omega_i\left(a|\alpha, z\right)
\end{aligned}
$$

Best response condition

$$\alpha = \arg\max_{\alpha} \left\{ \overline{W}_i \left(a | \alpha, z \right) - B_i \left(a | z \right) \omega_i \left(a | \alpha, z \right) \right\}$$

$$\text{FOC} \Rightarrow \frac{\partial \overline{W}_i}{\partial a} \left(\alpha | \alpha, z \right) = B_i \left(\alpha | z \right) \frac{\partial \omega_i}{\partial a} \left(\alpha | \alpha, z \right) + B_i^{(1)} \left(\alpha | z \right) \omega_i \left(\alpha | \alpha, z \right)$$

$\Leftrightarrow$

$$\mathbf{W}_i \left(\alpha ; z \right) = B_i \left(\alpha | z \right) + \Omega_i \left(\alpha ; z \right) B_i^{(1)} \left(\alpha | z \right) \text{ with I.C. } B_i \left(0 | z \right) = 0$$

where

$$\Omega_i \left(\alpha ; z \right) = \frac{1}{\frac{\partial \omega_i}{\partial a} \left(\alpha | \alpha, z \right)} \omega_i \left(\alpha | \alpha, z \right)$$

$$\mathbf{W}_i \left(\alpha ; z \right) = \frac{1}{\frac{\partial \omega_i}{\partial a} \left(\alpha | \alpha, z \right)} \frac{\partial \overline{W}_i}{\partial a} \left(\alpha | \alpha, z \right)$$

Comparison with IPV

- **Symmetric IPV case**:

$$\text{PV Quantile } (\alpha|I) = B(\alpha) + \frac{\alpha}{I-1} B^{(1)}(\alpha)$$

- **Interdependent value case:**

$$\mathbf{W}_i(\alpha; z) = B_i(\alpha|z) + \Omega_i(\alpha; z) B_i^{(1)}(\alpha|z)$$

$$\text{where } \Omega_i(\alpha; z) = \frac{1}{\frac{\partial \omega_i}{\partial a}(\alpha|\alpha, z)} \omega_i(\alpha|\alpha, z)$$

is identified as

$$\omega_i(a|\alpha, z) = \mathbb{P}\left(B_i(a|z) \geq \max_{j \neq i} B_j | A_i = \alpha, z \right).$$

$$\Rightarrow \mathbf{W}_i(\alpha; z) = \frac{1}{\frac{\partial \omega_i}{\partial a}(\alpha|\alpha, z)} \frac{\partial \overline{W}_i}{\partial a}(\alpha|\alpha, z) \text{ is identified}$$

but again $\mathbf{W}_i(\alpha; z) \neq W_i$

Stability property for additive interdependent values (1)

$$\overline{W}_i\left(a|\alpha,z\right) = \overline{\gamma}_{i0}\left(a|\alpha,z\right) + \sum_{j=1}^{I} \pi_{ij} z_j \overline{\gamma}_{ij}\left(a|\alpha,z\right)$$

$$\text{with } \overline{\gamma}_{ii}\left(a|\alpha,z\right) = \gamma_i\left(\alpha\right)\omega_i\left(a|\alpha,z\right),$$

$$\mathbf{W}_i\left(\alpha;z\right) = \frac{1}{\frac{\partial \omega_i}{\partial a}\left(\alpha|\alpha,z\right)} \frac{\partial \overline{W}_i}{\partial a}\left(\alpha|\alpha,z\right)$$

$$\Rightarrow \mathbf{W}_i\left(\alpha;z\right) = \gamma_{i0}\left(\alpha;z\right) + \sum_{j=1}^{I} \pi_{ij} z_j \gamma_{ij}\left(\alpha;z\right) \text{ with}$$

$$\gamma_{ii}\left(\alpha;z\right) = \gamma_i\left(\alpha\right) \ \textbf{(invariance of } \gamma_i\left(\alpha\right)\textbf{)}$$

$$\gamma_{ij}\left(\alpha;z\right) = \frac{1}{\frac{\partial \omega_i}{\partial a}\left(\alpha|\alpha,z\right)} \frac{\partial \overline{\gamma}_{ij}}{\partial a}\left(\alpha|\alpha,z\right)$$

Stability property (2) and identification

$$\mathbf{W}_i(\alpha; z) = \gamma_{i0}(\alpha; z) + \sum_{j=1}^{I} \pi_{ij} z_j \gamma_{ij}(\alpha; z) \text{ with } \gamma_{ii}(\alpha; z) = \gamma_i(\alpha)$$

and solving the differential equation in $B_i(\alpha|xz)$

$$\mathbf{W}_i(\alpha; z) = B_i(\alpha|z) + \Omega_i(\alpha; z) B_i^{(1)}(\alpha|z), \quad B_i(0|z) = 0$$

gives the “random coefficient” quantile regression model

$$B_i(\alpha|z) = \beta_{i0}(\alpha; z) + \sum_{j=1}^{I} z_j \beta_{ij}(\alpha; z) \text{ with}$$

$$\pi_{ij}\gamma_{ij}(\alpha; z) = \beta_{ij}(\alpha; z) + \Omega_i(\alpha; z) \beta_{ij}^{(1)}(\alpha; z) \text{ (with } \pi_{i0} = 1\text{)}$$

$$\gamma_i(\alpha) = \beta_{ii}(\alpha; z) + \Omega_i(\alpha; z) \beta_{ii}^{(1)}(\alpha; z)$$

- $\Rightarrow \gamma_i(\alpha)$ is identified for all $i \geq 1$

- $\Rightarrow \pi_{ij}\gamma_{ij}(\alpha; z)$ is identified for all $j \geq 0$

But $\gamma_{ij}(\alpha; z)$ is also identified for all $j \geq 1$, since

$$\gamma_{ij}(\alpha; z) = \frac{1}{\frac{\partial \omega_i}{\partial a}(\alpha|\alpha, z)} \frac{\partial \overline{\gamma}_{ij}}{\partial a}(\alpha|\alpha, z) \text{ where}$$

$$\overline{\gamma}_{ij}(a|\alpha, z) = \mathbb{E}\left[\gamma_j(A_j)\,\mathbb{I}\left\{B_i(a|z) \geq \max_{k \neq i} B_k\right\}|A_i = \alpha, z\right]$$

$$\omega_i(\alpha|\alpha, z) = \mathbb{P}\left(B_i(a|z) \geq \max_{k \neq i} B_k|A_i = \alpha, z\right)$$

- $\Rightarrow \pi_{ij}$ is identified

Estimation method

- **Localised Augmented Quantile**: For each i, estimate $\beta_{ij}(\alpha; z)$ and $\beta_{ij}^{(1)}(\alpha; z)$ from the "random coefficient" quantile regressions

$$B_i(\alpha|z) = \beta_{i0}(\alpha; z) + \sum_{j=1}^{I} z_j \beta_{ij}(\alpha; z)$$

using a kernel weighted AQR with weights $K\left(\frac{z_\ell - z}{h}\right)$

- **Estimate**

$$\Omega_i(\alpha; z) = \frac{1}{\frac{\partial \omega_i}{\partial a}(\alpha|\alpha, z)} \omega_i(\alpha|\alpha, z) \text{ where}$$

$$\omega_i(\alpha|\alpha, z) = \mathbb{P}\left(B_i(a|z) \geq \max_{j \neq i} B_j | A_i = \alpha, z \right)$$

from $\widehat{B}_i(\alpha|z)$ and $\widehat{A}_{i\ell}$

- **Structural parameters:** compute

$$\widehat{\gamma}_{ii}(\alpha; z) = \widehat{\beta}_{ii}(\alpha; z) + \widehat{\Omega}_i(\alpha; z)\, \widehat{\beta}_{ii}^{(1)}(\alpha; z)$$

and average to improve convergence rate

$$\widehat{\gamma}_i(\alpha) = \frac{1}{L} \sum_{\ell=1}^{L} \widehat{\gamma}_{ii}(\alpha; z_\ell)$$

$\widehat{\pi}_{ij}$ from $\widehat{\gamma}_i(\alpha)$ and $\widehat{\gamma}_{ij}(\alpha; z_\ell)$

Final remarks

- Flexible quantile regression specifications including nonparametric components which can be estimated with fast rate
- A class of additive interactive specification ranging from quantile regression to the general nonparametric quantile model. Can be tested from the data
- Address the curse of dimensionality
- No boundary bias. Allows estimation of all private values
- Bandwidth choice
- Additive interdependent value specification
- Statistical extension: dimension reduction for conditional p.d.f