

A more powerful subvector Anderson Rubin test in linear instrumental variable regression*

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Abstract

We study subvector inference in the linear instrumental variables model assuming homoskedasticity but allowing for weak instruments. The subvector Anderson and Rubin (1949) test that uses chi square critical values with degrees of freedom reduced by the number of parameters not under test, proposed by Guggenberger et al (2012), controls size but is generally conservative. We propose a conditional subvector Anderson and Rubin test that uses data-dependent critical values that adapt to the strength of identification of the parameters not under test. This test has correct size and strictly higher power than the subvector Anderson and Rubin test by Guggenberger et al (2012). We provide tables with conditional critical values so that the new test is quick and easy to use.

Keywords: Asymptotic size, linear IV regression, subvector inference, weak instruments

JEL codes: C12, C26

1 Introduction

Inference in the homoskedastic linear instrumental variables (IV) regression model with possibly weak instruments has been the subject of a growing literature.¹ Most of this literature has focused on the problem of inference on the full vector of slope coefficients of the endogenous regressors. Weak-instrument robust

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¹See e.g., Nelson and Startz (1990), Staiger and Stock (1997), Kleibergen (2002), Moreira (2003), Andrews et al. (2006, 2008) Chernozhukov et al. (2009), and Hillier (2009a,b).

inference on subvectors of slope coefficients is a harder problem, because the parameters not under test become additional nuisance parameters, and has received less attention in the literature, see e.g., Dufour and Taamouti (2005), Guggenberger et al. (2012) (henceforth GKMC), and Kleibergen (2015).

The present paper contributes to that part of the literature and focuses on the subvector Anderson and Rubin (1949) (AR) test studied by GKMC. Chernozhukov et al (2009) showed that the full vector AR test is admissible, see also Montiel-Olea (2017). GKMC proved that the use of $\chi_{k-m_W}^2$ critical values, where k is the number of instruments and m_W is the number of unrestricted slope coefficients under the null hypothesis, results in a subvector AR test with asymptotic size equal to the nominal size, thus providing a power improvement over the projection approach, see Dufour and Taamouti (2005), that uses χ_k^2 critical values.

This paper is motivated by the insight that the largest quantiles of the subvector AR test statistic, namely the quantiles of a $\chi_{k-m_W}^2$ distribution, occur under strong identification of the nuisance parameters. Therefore, there may be scope for improving the power of the subvector AR test by using data-dependent critical values that adapt to the strength of identification of the nuisance parameters. Indeed, we propose a new data-dependent critical value for the subvector AR test that is smaller than the $\chi_{k-m_W}^2$ critical value in GKMC. The new critical value depends monotonically on a statistic that measures the strength of identification of the nuisance parameters under the null (akin to a first-stage F statistic in a model with $m_W = 1$), and converges to the $\chi_{k-m_W}^2$ critical value when the conditioning statistic gets large. We prove that the new conditional subvector AR test has correct asymptotic size and strictly higher power than the test in GKMC, and therefore the subvector AR test in GKMC is inadmissible.

At least in the case $m_W = 1$, there is little scope for exploring alternative approaches, such as, e.g., Bonferroni, for using information about the strength of identification to improve the power of the subvector GKMC test. Specifically, in the case $m_W = 1$, we use the approach of Elliott et al. (2015) to obtain a point-optimal power bound for any test that only uses the subvector AR statistic and our measure of identification strength, and find that the power of the new conditional subvector AR test is very close to it.

Implementation of the new subvector test is trivial. The test statistic is the same as in GKMC and the critical values, as functions of a scalar conditioning statistic, are tabulated.

Our analysis relies on the insight that the subvector AR statistic is the likelihood ratio statistic for testing that the mean of a $k \times p$ Gaussian matrix with Kronecker covariance is of reduced rank. When the covariance matrix is known, this statistic corresponds to the minimum eigenvalue of a noncentral Wishart matrix. This enables us to draw on a large related statistical literature, see Muirhead (2009). A useful result from Perlman and Olkin (1980) establishes the monotonicity of the distribution of the subvector AR statistic with respect to the concentration parameter which measures the strength of identification when $m_W = 1$. The proposed conditional critical values are based on results given in Muirhead (1978) on approximations to the distribution of the eigenvalues of noncentral Wishart matrices.

In the normal linear IV model, we show that the finite-sample size of the conditional subvector AR test depends only on a m_W -dimensional nuisance parameter. When $m_W = 1$, it is therefore straightforward to compute the finite-sample size by simulation or numerical integration, and we prove that finite-sample size for general m_W is bounded by size in the case $m_W = 1$. The conditional subvector AR test depends on eigenvalues of quadratic forms of random matrices. We combine the method of Andrews et al. (2011) that was used in GKMC with results in Andrews and Guggenberger (2015) to show that the asymptotic size of the new test can be computed from finite-sample size when errors are Gaussian and their covariance matrix is known.

Three other related papers are Rhodes Jr (1981) that studies the exact distribution of the likelihood ratio statistic for testing the validity of overidentifying restrictions in a Gaussian simultaneous equations model; and Nielsen (1999, 2001) that study conditional tests of rank in bivariate canonical correlation analysis, which is related to the present problem when $k = 2$ and $m_W = 1$. These papers do not provide results on asymptotic size or power.

In ongoing work, Kleibergen (2015) provides power improvements over projection for the conditional likelihood ratio test for a subvector hypothesis in the linear IV model. Building on the approach of Chaudhuri and Zivot (2011), Andrews (2017) proposes a two-step Bonferroni-like method that applies more generally to nonlinear models with non-iid heteroskedastic data, and is asymptotically efficient under strong identification. Our paper focuses instead on power improvement under weak identification. Another related recent paper on subvector inference in the linear IV model is Zhu (2015), whose setup also allows for conditional heteroskedasticity and is based on the Bonferroni method. Andrews and Mikusheva (2016) develop robust subvector inference in nonlinear models. Han and McCloskey (2017) study subvector inference in nonlinear models with near singular Jacobian. Kaido et al. (2016) and Bugni et al. (2017) consider subvector inference in models defined by moment inequalities.

The analysis in this paper relies critically on the assumption of homoskedasticity. Allowing for heteroskedasticity is difficult because the number of nuisance parameters grows with k , and finite-sample distribution theory becomes intractable. When testing hypotheses on the full vector of coefficients in linear IV regression, robustness to heteroskedasticity is asymptotically costless since the heteroskedasticity-robust AR test is asymptotically equivalent to the nonrobust one under homoskedasticity, and the latter is admissible. However, in the subvector case, our paper shows that one can exploit the structure of the homoskedastic linear IV model to obtain more powerful tests, while it is not at all clear whether this is feasible under heteroskedasticity. Therefore, given the current state of the art, our results seem to indicate that there is a trade-off between efficiency and robustness to heteroskedasticity for subvector testing in the linear IV model.

The structure of the paper is as follows. Section 2 provides the finite-sample results with Gaussian errors, fixed instruments, and known covariance matrix, Section 3 gives asymptotic results, and Section 4 concludes. All proofs of the main results in the paper and tables of conditional critical values and additional numerical results are provided in the Supplemental Material (SM).

We use the following notation. For a full column rank matrix A with n rows let $P_A = A(A'A)^{-1}A'$ and $M_A = I_n - P_A$, where I_n denotes the $n \times n$ identity matrix. If A has zero columns, then we set $M_A = I_n$. The chi square distribution with k degrees of freedom and its $1 - \alpha$ -quantile are written as χ_k^2 and $\chi_{k,1-\alpha}^2$, respectively. For an $n \times n$ matrix A , $\rho(A)$ denotes the rank of A and $\kappa_i(A)$, $i = 1, \dots, n$ denote the eigenvalues of A in non-increasing order. By $\kappa_{\min}(A)$ and $\kappa_{\max}(A)$ we denote the smallest and largest eigenvalue of A , respectively. We write $0^{n \times k}$ to denote a matrix of dimensions n by k with all entries equal to zero and typically write 0^n for $0^{n \times 1}$.

2 Finite sample analysis

The model is given by the equations

$$\begin{aligned} y &= Y\beta + W\gamma + \varepsilon \\ Y &= Z\Pi_Y + V_Y \\ W &= Z\Pi_W + V_W, \end{aligned} \tag{2.1}$$

where $y \in \mathfrak{R}^n$, $Y \in \mathfrak{R}^{n \times m_Y}$, $W \in \mathfrak{R}^{n \times m_W}$, and $Z \in \mathfrak{R}^{n \times k}$. We assume that $k - m_W \geq 1$. The reduced form can be written as

$$\begin{pmatrix} y & Y & W \end{pmatrix} = Z \begin{pmatrix} \Pi_Y & \Pi_W \end{pmatrix} \begin{pmatrix} \beta & I_{m_Y} & 0^{m_Y \times m_W} \\ \gamma & 0^{m_W \times m_Y} & I_{m_W} \end{pmatrix} + \underbrace{\begin{pmatrix} v_y & V_Y & V_W \end{pmatrix}}_V, \quad (2.2)$$

where $v_y := V_Y \beta + V_W \gamma + \varepsilon$. By V_i we denote the i -th row of V written as a column vector and similarly for other matrices. Let $m := m_Y + m_W$.

Throughout this section, we make the following assumption.

Assumption A: 1. $V_i := (v_{yi}, V'_{Yi}, V'_{Wi})' \sim \text{i.i.d.} N(0^{(m+1) \times (m+1)}, \Omega)$, $i = 1, \dots, n$, where $\Omega \in \mathfrak{R}^{(m+1) \times (m+1)}$ is known and positive definite. 2. The instruments $Z \in \mathfrak{R}^{n \times k}$ are fixed and $Z'Z \in \mathfrak{R}^{k \times k}$ is positive definite.

The objective is to test the hypothesis

$$H_0 : \beta = \beta_0 \text{ against } H_1 : \beta \neq \beta_0, \quad (2.3)$$

using tests whose size, i.e. the highest null rejection probability (NRP) over the unrestricted nuisance parameters Π_Y , Π_W , and γ , equals the nominal size α . In particular, weak identification and non-identification of β and γ are allowed for.

The subvector AR statistic for testing H_0 is defined as

$$AR_n(\beta_0) := \min_{\tilde{\gamma} \in \mathfrak{R}^{m_W}} \frac{(\bar{Y}_0 - W\tilde{\gamma})' P_Z (\bar{Y}_0 - W\tilde{\gamma})}{(1, -\tilde{\gamma}') \Omega(\beta_0) (1, -\tilde{\gamma})'}, \quad (2.4)$$

where

$$\Omega(\beta_0) := \begin{pmatrix} 1 & 0^{1 \times m_W} \\ -\beta_0 & 0^{m_Y \times m_W} \\ 0^{m_W \times 1} & I_{m_W} \end{pmatrix}' \Omega \begin{pmatrix} 1 & 0^{1 \times m_W} \\ -\beta_0 & 0^{m_Y \times m_W} \\ 0^{m_W \times 1} & I_{m_W} \end{pmatrix}, \quad (2.5)$$

and

$$\bar{Y}_0 := y - Y\beta_0. \quad (2.6)$$

Denote by $\hat{\kappa}_i$ for $i = 1, \dots, p := 1 + m_W$ the roots of the following characteristic polynomial in κ

$$\left| \kappa \Omega(\beta_0) - (\bar{Y}_0, W)' P_Z (\bar{Y}_0, W) \right| = 0, \quad (2.7)$$

ordered non-increasingly. Then,

$$AR_n(\beta_0) = \hat{\kappa}_p, \quad (2.8)$$

that is, $AR_n(\beta_0)$ equals the smallest characteristic root, see, e.g. (Schmidt, 1976, chapter 4.8). The subvector AR test in GKMC rejects H_0 at significance level α if $AR_n(\beta_0) > \chi_{k-m_W, 1-\alpha}^2$, while the AR test based on projection rejects if $AR_n(\beta_0) > \chi_{k, 1-\alpha}^2$.

Under Assumption A, the subvector AR statistic equals the minimum eigenvalue of a noncentral Wishart matrix. More precisely, we show in the SM (Subsection S.1.1) that the roots $\hat{\kappa}_i$ of (2.7) for $i = 1, \dots, p$, satisfy

$$0 = |\hat{\kappa}_i I_p - \Xi' \Xi|, \quad (2.9)$$

where $\Xi \sim N(\mathcal{M}, I_{kp})$ for some nonrandom $\mathcal{M} \in \mathfrak{R}^{k \times p}$ (defined in (S-15) in the SM). Furthermore, under the

null hypothesis H_0 , $\mathcal{M} = (0^k, \Theta_W)$ for some $\Theta_W \in \mathfrak{R}^{k \times m_W}$ (defined in (S-17) in the SM) and thus $\rho(\mathcal{M}) \leq m_W$. Therefore, $\Xi' \Xi \sim \mathcal{W}_p(k, I_p, \mathcal{M}' \mathcal{M})$, where the latter denotes a non-central Wishart distribution with k degrees of freedom, covariance matrix I_p , and noncentrality matrix

$$\mathcal{M}' \mathcal{M} = \begin{pmatrix} 0 & 0^{1 \times m_W} \\ 0^{m_W \times 1} & \Theta_W' \Theta_W \end{pmatrix}. \quad (2.10)$$

The joint distribution of the eigenvalues of a noncentral Wishart matrix only depends on the eigenvalues of the noncentrality matrix $\mathcal{M}' \mathcal{M}$ (see e.g. Muirhead, 2009). Hence, the distribution of $(\hat{\kappa}_1, \dots, \hat{\kappa}_p)$ under the null only depends on the eigenvalues of $\Theta_W' \Theta_W$, which we denote by

$$\kappa_i := \kappa_i(\Theta_W' \Theta_W), \quad i = 1, \dots, m_W. \quad (2.11)$$

We can think of $\Theta_W' \Theta_W$ as the concentration matrix for the endogenous regressors W , see e.g. Stock et al. (2002). In the case when $m_W = 1$, $\Theta_W' \Theta_W$ is a scalar, and corresponds to the well-known concentration parameter (see e.g. Staiger and Stock (1997)) that measures the strength of the identification of the parameter vector γ not under test.

2.1 Motivation for conditional subvector AR test: Case $m_W = 1$

The above established that when $m_W = 1$ the distribution of $AR_n(\beta_0)$ under H_0 depends only on the single nuisance parameter κ_1 . The following result gives a useful monotonicity property of this distribution.

Theorem 1 *Suppose that Assumption A holds and $m_W = 1$. Then, under the null hypothesis $H_0 : \beta = \beta_0$, the distribution function of the subvector AR statistic in (2.4) is monotonically decreasing in the parameter κ_1 , defined in (2.11), and converges to χ_{k-1}^2 as $\kappa_1 \rightarrow \infty$.*

This result follows from (Perlman and Olkin, 1980, Theorem 3.5), who established that the eigenvalues of a $k \times p$ noncentral Wishart matrix are stochastically increasing in the nonzero eigenvalue of the noncentrality matrix when the noncentrality matrix is of rank 1.

Theorem 1 shows that the subvector AR test in GKMC is conservative for all $\kappa_1 < \infty$, because its NRP $\Pr_{\kappa_1} \left(AR_n(\beta_0) > \chi_{k-1, 1-\alpha}^2 \right)$ is monotonically increasing in κ_1 and the worst case occurs at $\kappa_1 = \infty$. Hence, it seems possible to improve the power of the subvector AR test by reducing the χ_{k-1}^2 critical value based on information about the value of κ_1 .

If κ_1 were known, which it is not, one would set the critical value equal to the $1 - \alpha$ quantile of the exact distribution of $AR_n(\beta_0)$ and obtain a similar test with higher power than the subvector AR test in GKMC. Alternatively, if there was a one-dimensional minimal sufficient statistic for κ_1 under H_0 , one could obtain a similar test by conditioning on it. Unfortunately, we are not aware of such a statistic. However, an approximation to the density of eigenvalues of noncentral Wishart matrices by Leach (1969), specialized to this case, implies that the largest eigenvalue $\hat{\kappa}_1$ is approximately sufficient for κ_1 when κ_1 is ‘‘large’’ and $\kappa_2 = 0$. Based on this approximation, (Muirhead, 1978, Section 6) provides an approximate, nuisance parameter free, conditional density of the smallest eigenvalue $\hat{\kappa}_2$ given the largest one $\hat{\kappa}_1$. This approximate density (with respect to Lebesgue measure) of $\hat{\kappa}_2$ given $\hat{\kappa}_1$ can be written as

$$f_{\hat{\kappa}_2 | \hat{\kappa}_1}^*(x_2 | \hat{\kappa}_1) = f_{\chi_{k-1}^2}(x_2) (\hat{\kappa}_1 - x_2)^{1/2} g(\hat{\kappa}_1), \quad x_2 \in [0, \hat{\kappa}_1], \quad (2.12)$$

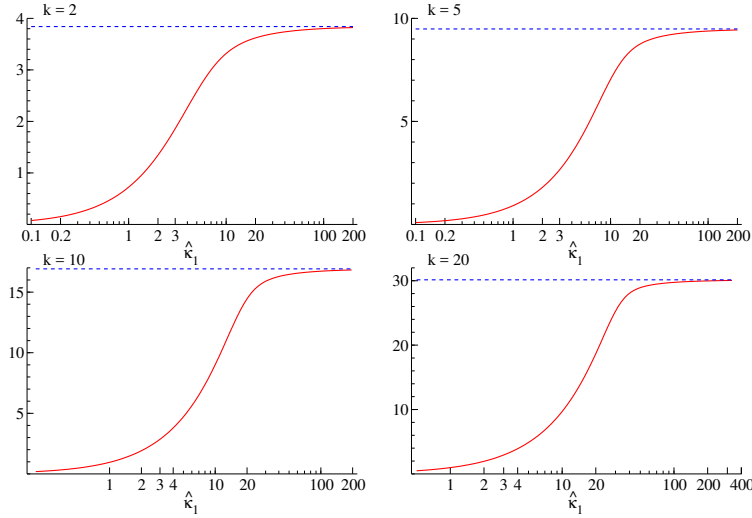


Figure 1: Conditional quantile function. The solid line plots the $1 - \alpha$ quantile of the distribution with density (2.12), for $\alpha = 5\%$. The dotted straight blue line gives the corresponding quantile of χ_{k-1}^2 .

where $f_{\chi_{k-1}^2}(\cdot)$ is the density of a χ_{k-1}^2 and $g(\hat{\kappa}_1)$ is a function that does not depend on any unknown parameters, see (S-26) in the SM.

Because (2.12) is analytically available, the quantiles of the distribution whose density is given in (2.12) can be computed easily using numerical integration for fixed values of $\hat{\kappa}_1$. Figure 1 plots the $1 - \alpha$ quantile of that distribution as a function of $\hat{\kappa}_1$ for $\alpha = 5\%$ and $k = 2, 5, 10$, and 20 . It is evident that this conditional quantile function is strictly increasing in $\hat{\kappa}_1$ and asymptotes to $\chi_{k-1, 1-\alpha}^2$.² We propose to use the above conditional quantile function to obtain conditional critical values for the subvector AR statistic.

In practice, to make implementation of the test straightforward for empirical researchers, the conditional critical value function will be tabulated for different $k - 1$ and α over a grid of points $\hat{\kappa}_{1,j}$, $j = 1, \dots, J$, say, and conditional critical values for any given $\hat{\kappa}_1$ will be obtained by linear interpolation.³ Specifically, let $q_{1-\alpha,j}(k-1)$ denote the $1 - \alpha$ quantile of the distribution whose density is given by (2.12) with $\hat{\kappa}_1$ replaced by $\hat{\kappa}_{1,j}$. The end point of the grid $\hat{\kappa}_{1,J}$ should be chosen high enough so that $q_{1-\alpha,J}(k-1) \approx \chi_{k-1, 1-\alpha}^2$. For any realization of $\hat{\kappa}_1 \leq \hat{\kappa}_{1,J}$,⁴ find j such that $\hat{\kappa}_1 \in [\hat{\kappa}_{1,j-1}, \hat{\kappa}_{1,j}]$ with $\hat{\kappa}_{1,0} = 0$ and $q_{1-\alpha,0}(k-1) = 0$, and let

$$c_{1-\alpha}(\hat{\kappa}_1, k-1) := \frac{\hat{\kappa}_{1,j} - \hat{\kappa}_1}{\hat{\kappa}_{1,j} - \hat{\kappa}_{1,j-1}} q_{1-\alpha,j-1}(k-1) + \frac{\hat{\kappa}_1 - \hat{\kappa}_{1,j-1}}{\hat{\kappa}_{1,j} - \hat{\kappa}_{1,j-1}} q_{1-\alpha,j}(k-1). \quad (2.13)$$

Table 1 gives conditional critical values at significance level 5% for a fine grid for the conditioning statistic $\hat{\kappa}_1$ for the case $k - 1 = 4$. To mitigate any slight over-rejection induced by interpolation, the reported critical values have been rounded up to one decimal.

We will see that by using $c_{1-\alpha}(\hat{\kappa}_1, k-1)$ as a critical value for the subvector AR test, one obtains a close to similar test, except for small values of κ_1 . Note that $\hat{\kappa}_1$, the largest root of the characteristic polynomial in (2.7) is comparable to the first-stage F statistic in the case $m_W = 1$ for the hypothesis that

²The monotonicity statement is made based on numerical integration without an analytical proof. An analytical proof of the limiting result is given in Section S.1.2 in the SM.

³For general m_W , discussed in the next subsection, the role of $k - 1$ is played by $k - m_W$.

⁴When $\hat{\kappa}_1 > \hat{\kappa}_{1,J}$, we can define $c_{1-\alpha}(\hat{\kappa}_1, k-1)$ using nonlinear interpolation between $\hat{\kappa}_{1,J}$ and ∞ , i.e., $c_{1-\alpha}(\hat{\kappa}_1, k-1) := (1 - F(\hat{\kappa}_1 - \hat{\kappa}_{1,J})) q_{1-\alpha,J}(k-1) + F(\hat{\kappa}_1 - \hat{\kappa}_{1,J}) \chi_{k-1, 1-\alpha}^2$, where F is some distribution function.

$\alpha = 5\%, \quad k - 1 = 4$																	
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv		
1.2	1.1	2.1	1.9	3.2	2.9	4.5	3.9	5.9	4.9	7.4	5.9	9.4	6.9	12.5	7.9	20.9	8.9
1.3	1.2	2.3	2.1	3.5	3.1	4.7	4.1	6.2	5.1	7.8	6.1	9.9	7.1	13.4	8.1	26.5	9.1
1.4	1.3	2.5	2.3	3.7	3.3	5.0	4.3	6.5	5.3	8.2	6.3	10.5	7.3	14.5	8.3	39.9	9.3
1.6	1.5	2.7	2.5	4.0	3.5	5.3	4.5	6.8	5.5	8.6	6.5	11.1	7.5	15.9	8.5	57.4	9.4
1.8	1.7	3.0	2.7	4.2	3.7	5.6	4.7	7.1	5.7	9.0	6.7	11.7	7.7	17.9	8.7	1000	9.48

Table 1: $1 - \alpha$ quantile of the conditional distribution with density given in (2.12), $cv = c_{1-\alpha}(\hat{\kappa}_1, k - 1)$ at different values of the conditioning variable $\hat{\kappa}_1$. Computed by numerical integration.

$\Pi_W = 0^{k \times m_W}$ (γ is unidentified) under the null hypothesis $H_0 : \beta = \beta_0$. So given α , $c_{1-\alpha}(\hat{\kappa}_1, k - 1)$ is a data-dependent critical value that depends only on the integer $k - 1$ (the number of IVs minus the number of untested parameters), and the nonnegative scalar $\hat{\kappa}_1$ which is a measure of the strength of identification of the unrestricted coefficient γ .

2.2 Definition of the conditional subvector AR test for general m_W

We will now define the conditional subvector AR test for the general case when $m_W \geq 1$. The conditional subvector AR test rejects H_0 at nominal size α if

$$AR_n(\beta_0) > c_{1-\alpha}(\hat{\kappa}_1, k - m_W), \quad (2.14)$$

where $c_{1-\alpha}(\cdot, \cdot)$ has been defined in (2.13) for any argument consisting of a vector with first component in $\mathfrak{R}_+ \cup \{\infty\}$ and second component in \mathbb{N} . Tables of critical values for significance levels $\alpha = 10\%$, 5% , and 1% , and degrees of freedom $k - m_W = 1$ to 20 are provided in Section S.3 of the SM. Since $AR_n(\beta_0) = \hat{\kappa}_p$, the associated test function can be written as

$$\varphi_c(\hat{\kappa}) := \mathbf{1}[\hat{\kappa}_p > c_{1-\alpha}(\hat{\kappa}_1, k - m_W)], \quad (2.15)$$

where $\mathbf{1}[\cdot]$ is the indicator function, $\hat{\kappa} := (\hat{\kappa}_1, \hat{\kappa}_p)$ and the subscript c abbreviates ‘‘conditional’’.

The subvector AR test in GKMC that uses $\chi_{k-m_W}^2$ critical value has test function

$$\varphi_{GKMC}(\hat{\kappa}) := \mathbf{1}[\hat{\kappa}_p > c_{1-\alpha}(\infty, k - m_W)]. \quad (2.16)$$

Since $c_{1-\alpha}(x, \cdot) < c_{1-\alpha}(\infty, \cdot)$ for all $x < \infty$, it follows that $E[\varphi_c(\hat{\kappa})] > E[\varphi_{GKMC}(\hat{\kappa})]$, i.e., the conditional subvector AR test φ_c has strictly higher power than the (unconditional) subvector AR test φ_{GKMC} in GKMC.

2.3 Finite sample size of φ_c when $m_W = 1$

As long as the conditional critical values $c_{1-\alpha}(\hat{\kappa}_1, k - m_W)$ guarantee size control for the new test φ_c , the actual quality of the approximation (2.12) to the true *conditional* density is not of major concern to us, and the main purpose of (2.12) was to give us a simple analytical expression to generate data-dependent critical values.

We next compute the size of the conditional subvector AR test, and because we don’t have available an analytical expression of the NRP, we need to do that numerically. This can be done easily because the

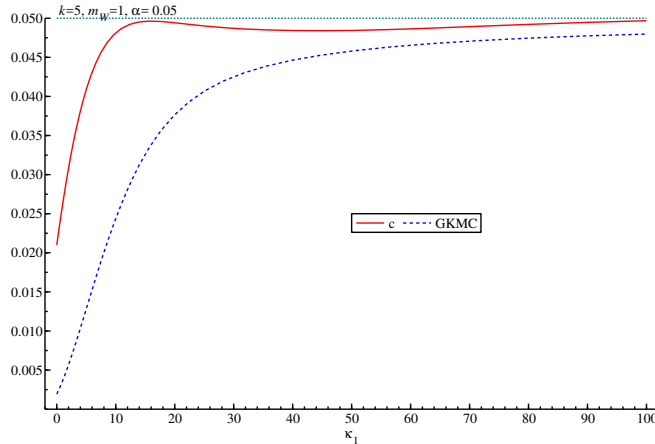


Figure 2: Null rejection probabilities of conditional (2.15) (red solid) and GKMC subvector AR (2.16) (blue dotted) tests at nominal size 5% as a function of the nuisance parameter κ_{m_W} . The number of instruments is $k = 5$ and the number of nuisance parameters is $m_W = 1$. Computed by numerical integration of the exact density (2.12).

nuisance parameter κ_1 is one-dimensional, and the density of the data is analytically available, so the NRP of the test can be estimated accurately by Monte Carlo simulation or numerical integration. Using (low-dimensional) simulations to calculate the (asymptotic) size of a testing procedure has been used in several recent papers, see e.g. Elliott et al. (2015).

Figure 2 plots the NRPs of both φ_c and the subvector AR test φ_{GKMC} of GKMC in (2.16) at $\alpha = 5\%$ as a function of κ_1 for $k = 5$ and $m_W = 1$. The conditional test φ_c is evaluated using the critical values reported in Table 1 with interpolation.⁵

We notice that the size of the conditional subvector AR test φ_c is controlled, because the NRPs never exceed the nominal size no matter the value of κ_1 . The NRPs of the subvector AR test φ_{GKMC} are monotonically increasing in κ_1 in accordance with Theorem 1. Therefore the proposed conditional test φ_c strictly dominates the unconditional test φ_{GKMC} . The results for other significance levels and other values of k are the same, and they are reported in Table S.21 of the SM. We summarize this finding in the following theorem.

Theorem 2 *Under Assumption A, the finite-sample size of the conditional subvector AR test φ_c defined in (2.15) is equal to its nominal size α .*

Comment. To reiterate, the proof of Theorem 2 for given $k - m_W$ and nominal size α is a combination of an analytical step that shows that the null rejection probability of the new test depends on only a scalar parameter and of a numerical step where it is shown by numerical integration and Monte Carlo simulation that none of the NRPs exceeds the nominal size. We performed these simulations for $k - m_W = 1, \dots, 20$ and $\alpha = 10\%, 5\%$, and 1% using 1 million Monte Carlo replications, and in no case did we find size distortion.

⁵E.g. if $\hat{\kappa}_1 = 2.4$ which is an element of $[2.3, 2.5]$, then from Table 1 the critical value employed would be 2.2. To produce Figure 2 we use a grid of 42 points for κ_1 , evenly spaced in log-scale between 0 and 100. In this figure, the NRPs were computed by numerical integration using the Quadpack in Ox, see Doornik (2001). The densities were evaluated using the algorithm of Koev and Edelman (2006) for the computation of hypergeometric functions of two matrix arguments. The NRPs are essentially the same when estimated by Monte Carlo integration with 1 million replications, see Section S.2 in the SM.

2.4 Power analysis when $m_W = 1$

One main advantage of the conditional subvector AR test (2.14) is its computational simplicity. For general m_W there are other approaches one might consider based on the information in the eigenvalues $(\hat{\kappa}_1, \dots, \hat{\kappa}_{m_W})$ that, at the expense of potentially much higher computational cost, might yield higher power than the conditional subvector AR test. For example, one could apply the critical value function approach of Moreira et al. (2016) to derive conditional critical values. One could condition on the largest m_W eigenvalues rather than just the largest one.

To assess the scope for power improvements over the subvector AR test in GKMC, we consider the case $m_W = 1$ and compute power bounds of all tests that depend on the statistic $(\hat{\kappa}_1, \hat{\kappa}_2)$. These are point-optimal bounds based on the least favorable distribution for the nuisance parameter κ_1 under the null that $\kappa_2 = 0$, see the SM S.2.3 for details. We consider both the approximately least favorable distribution (ALFD) method of Elliott et al. (2015) and the one-point least favorable distribution of (Andrews et al., 2008, section 4.2), but report here only the ALFD bound for brevity and because it is very similar to the Andrews et al. (2008) upper bound. The results based on the Andrews et al. (2008) method are discussed in Section S.4.2 of the SM.

Recall from (2.11) that under $H_0 : \beta = \beta_0$ in (2.3), the joint distribution of $(\hat{\kappa}_1, \dots, \hat{\kappa}_p)$ only depends on the vector of eigenvalues $(\kappa_1, \dots, \kappa_{m_W})$ of $\Theta'_W \Theta_W$, where $\Theta_W \in \mathfrak{R}^{k \times m_W}$ appears in the noncentrality matrix $\mathcal{M} = (0^k, \Theta_W)$ of $\Xi \sim N(\mathcal{M}, I_{kp})$. It follows easily from (S-17) in the SM that if Π_W ranges through all matrices in $\mathfrak{R}^{k \times m_W}$, then $(\kappa_1, \dots, \kappa_{m_W})'$ ranges through all vectors in $[0, \infty)^{m_W}$.

Define $A := E(Z'(y - Y\beta_0, W)) \in \mathfrak{R}^{k \times p}$ and consider the null hypothesis

$$H'_0 : \rho(A) \leq m_W \text{ versus } H'_1 : \rho(A) = p. \quad (2.17)$$

Clearly, whenever H_0 holds H'_0 holds too, but the reverse is not true; by (S-18) in the SM, H'_0 holds iff Π_W is rank deficient or $\Pi_Y(\beta - \beta_0) \in \text{span}(\Pi_W)$. It is shown in the SM (Case 2 in Subsection S.1.1) that under H'_0 the joint distribution of $(\hat{\kappa}_1, \dots, \hat{\kappa}_p)$ is the same as the one of the vector of eigenvalues of a Wishart matrix $\mathcal{W}_p(k, I_p, \mathcal{M}'\mathcal{M})$ with rank deficient noncentrality matrix and therefore depends only on the vector of the largest m_W eigenvalues $(\kappa_1, \dots, \kappa_{m_W})' \in \mathfrak{R}^{m_W}$ of $\mathcal{M}'\mathcal{M}$. The important implication of that result is that any test $\varphi(\hat{\kappa}_1, \dots, \hat{\kappa}_p) \in [0, 1]$ for some measurable function φ that has size bounded by α under H_0 also has size (in the parameters $(\beta, \gamma, \Pi_Y, \Pi_W)$) bounded by α under H'_0 . In particular, no test $\varphi(\hat{\kappa}_1, \dots, \hat{\kappa}_p)$ that controls size under H_0 has power exceeding size under alternatives $H'_0 \setminus H_0$.

Now assume $m_W = 1$. We compute the power of the conditional and unconditional subvector tests φ_c and φ_{GKMC} at the 5% level for $k = 5$ and the associated power bound for a grid of values of the parameters $\kappa_1 \geq \kappa_2 > 0$ under the alternative, see Section S.2.3 in the SM for details. The power curves are computed using 100,000 Monte Carlo replications without importance sampling (results for other k are similar and given in the SM). The left panel of Figure 3 plots the difference between the power function of the conditional test φ_c and the power bound across all alternatives. Except at alternatives very close to the null, and when κ_1 is very close to κ_2 (so the nuisance parameter is weakly identified), the power of the conditional subvector test φ_c is essentially on the power bound. The fact that the power of φ_c for small κ_1 is somewhat below the power bound can be explained by the fact that the test is not exactly similar, so its rejection probability can fall below α for some alternatives. The right panel of Figure 3 plots the power curves for alternatives with $\kappa_1 = \kappa_2$, which seem to be the least favorable to the conditional test. The power of the conditional test is mostly on the power bound, while the subvector test φ_{GKMC} is well below the bound. Two-dimensional

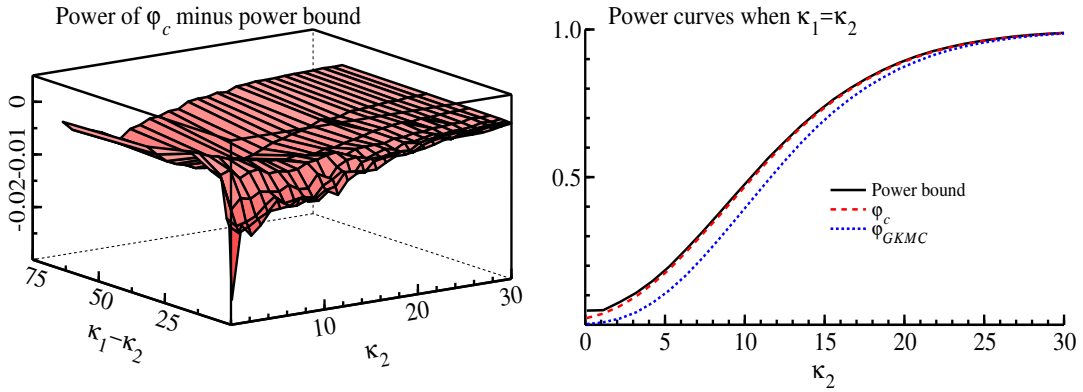


Figure 3: Power of conditional (2.15) and GKMC (2.16) subvector AR tests, φ_c and φ_{GKMC} , and point optimal power envelope computed using the ALFD method of Elliott et al. (2015). The number of instruments is $k = 5$ and the number of nuisance parameters is $m_W = 1$. The left panel plots the power of φ_c minus the power bound across all alternatives. The right panel plots the power curves for both tests and the power bound when $\kappa_1 = \kappa_2$.

plots for other values of $\kappa_1 - \kappa_2$ are provided in the SM. As $\kappa_1 - \kappa_2$ gets larger, the power of φ_{GKMC} gets closer to the power envelope, as expected.

2.5 Size of φ_c when $m_W > 1$ - inadmissibility of φ_{GKMC}

We cannot extend the monotonicity result of Theorem 1 to the general case $m_W > 1$. This is because the distribution of the subvector AR statistic depends on all the m_W eigenvalues of $\mathcal{M}'\mathcal{M}$ in (2.10), and the method of the proof of Theorem 1 only works for the case that $\rho(\mathcal{M}'\mathcal{M}) = 1$.⁶ However, the following result suffices to establish correct finite-sample size of the proposed conditional subvector AR test (2.15) and the inadmissibility of the subvector test φ_{GKMC} in (2.16) in the general case.

Theorem 3 *Suppose that Assumption A holds with $m_W > 1$. Denote by $\tilde{\Xi}_{11} \in \mathbb{R}^{k-m_W+1 \times 2}$ the upper left submatrix of $\tilde{\Xi} := \Xi O \in \mathbb{R}^{k \times p}$, where Ξ and the random orthogonal matrix $O \in \mathbb{R}^{p \times p}$ are defined below (2.9) and in (S-4) of the SM, respectively. Then, under the null hypothesis $H_0 : \beta = \beta_0$*

$$\tilde{\Xi}'_{11} \tilde{\Xi}_{11} | O \sim \mathcal{W}_2 \left(k - m_W + 1, I_2, \tilde{\mathcal{M}}'_{11} \tilde{\mathcal{M}}_{11} \right),$$

where $\tilde{\mathcal{M}}_{11}$ is defined in (S-7) in the SM and satisfies $\rho(\tilde{\mathcal{M}}'_{11} \tilde{\mathcal{M}}_{11}) \leq 1$.

Note that

$$AR_n(\beta_0) = \kappa_{\min}(\Xi'\Xi) = \kappa_{\min}(\tilde{\Xi}'\tilde{\Xi}) \leq \kappa_{\min}(\tilde{\Xi}'_{11}\tilde{\Xi}_{11}) \leq \kappa_{\max}(\tilde{\Xi}'_{11}\tilde{\Xi}_{11}) \leq \kappa_{\max}(\tilde{\Xi}'\tilde{\Xi}) = \kappa_{\max}(\Xi'\Xi), \quad (2.18)$$

where the first and third inequalities hold by the inclusion principle, see (Lütkepohl, 1996, p. 73) and the second and last equalities hold because O is orthogonal. Therefore,

$$P(AR_n(\beta_0) > c_{1-\alpha}(\kappa_{\max}(\Xi'\Xi), k - m_W)) \leq P(\kappa_{\min}(\tilde{\Xi}'_{11}\tilde{\Xi}_{11}) > c_{1-\alpha}(\kappa_{\max}(\tilde{\Xi}'_{11}\tilde{\Xi}_{11}), k - m_W)) \leq \alpha, \quad (2.19)$$

⁶See (Perلمان and Olkin, 1980, p. 1337) for some more discussion of the difficulties involved in extending the result to the general case.

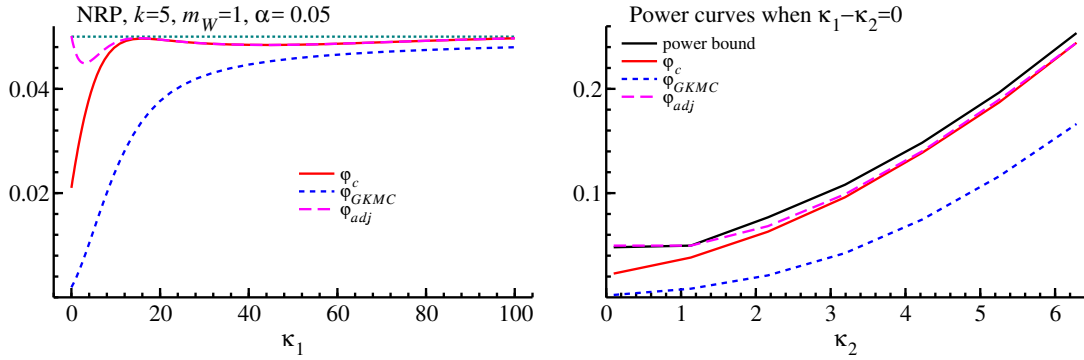


Figure 4: Left panel: NRP of (2.15), GKMC (2.16) and adjusted subvector AR tests, φ_c , φ_{GKMC} and φ_{adj} . Right panel: comparison of power curves when $\kappa_1 = \kappa_2$ to point optimal power envelope computed using the ALFD method of Elliott et al. (2015).

where the first inequality follows from (2.18). The second inequality follows from Theorem 2 for the case $m_W = 1$ and from Theorem 3 by conditioning on O , where the role of k is now played by $k - m_W + 1$. Hence, the conditional subvector AR test has correct size for any m_W . Because $c_{1-\alpha}(\kappa_{\max}(\Xi'\Xi), k - m_W) < \chi_{k-m_W, 1-\alpha}^2$, it follows that the subvector AR test φ_{GKMC} given in (2.16) is inadmissible.

2.6 Refinement

Figure 2 shows that the NRPs of test φ_c for nominal size 5% is considerably below 5% for small values of κ_1 , which causes a loss of power for some alternatives that are close to H_0 , see Figure 3. However, we can reduce the under-rejection by adjusting the conditional critical values to bring the test closer to similarity.⁷ For the case $k = 5$, $m_W = 1$, and $\alpha = 5\%$, let φ_{adj} be the test that uses the critical values in Table 1 where the smallest 8 critical values are divided by 5 (e.g., the critical value for $\hat{\kappa}_1 = 2.5$ becomes 0.46). Figure 4 shows that φ_{adj} still has size 5%, that it is much closer to similarity than φ_c , and does not suffer from any loss of power relative to the power bound near H_0 . This approach can be applied to all other values of α and k , but needs to be adjusted for each case.

3 Asymptotics

In this section, Assumption A is replaced by

Assumption B: The random vectors $(\varepsilon_i, Z'_i, V'_{Y,i}, V'_{W,i})$ for $i = 1, \dots, n$ in (2.1) are i.i.d. with distribution F .

Therefore, the instruments are random, the reduced form errors are not necessarily normally distributed, and the matrix $\Omega = E_F V_i V'_i$ is unknown. We define the parameter space \mathcal{F} for $(\gamma, \Pi_W, \Pi_Y, F)$ under the null hypothesis $H_0 : \beta = \beta_0$ exactly as in GKMC.⁸ Namely, for $U_i = (\varepsilon_i, V'_{W,i})'$ let

⁷We thank Ulrich Müller for this suggestion.

⁸Regarding the notation $(\gamma, \Pi_W, \Pi_Y, F)$ and elsewhere, note that we allow as components of a vector column vectors, matrices (of different dimensions), and distributions.

$$\begin{aligned}
\mathcal{F} = \{ & (\gamma, \Pi_W, \Pi_Y, F) : \gamma \in \mathfrak{R}^{m_W}, \Pi_W \in \mathfrak{R}^{k \times m_W}, \Pi_Y \in \mathfrak{R}^{k \times m_Y}, \\
& E_F(\|T_i\|^{2+\delta}) \leq B, \text{ for } T_i \in \{Z_i \varepsilon_i, \text{vec}(Z_i V'_{W,i}), V_{W,i} \varepsilon_i, \varepsilon_i, V_{W,i}, Z_i\}, \\
& E_F(Z_i V'_i) = 0^{k \times (m+1)}, E_F(\text{vec}(Z_i U'_i) (\text{vec}(Z_i U'_i))') = (E_F(U_i U'_i) \otimes E_F(Z_i Z'_i)), \\
& \kappa_{\min}(A) \geq \delta \text{ for } A \in \{E_F(Z_i Z'_i), E_F(U_i U'_i)\} \}
\end{aligned} \tag{3.1}$$

for some $\delta > 0$ and $B < \infty$, where “ \otimes ” denotes the Kronecker product of two matrices and $\text{vec}(\cdot)$ the column vectorization of a matrix. Note that the factorization of the covariance matrix into a Kronecker product in line three of (3.1) is a slightly weaker assumption than conditional homoskedasticity.

Rather than controlling the *finite-sample* size the objective is to demonstrate that the new conditional subvector AR test has *asymptotic* size, that is the limit of the finite-sample size with respect to \mathcal{F} , equal to the nominal size.

We next define the test statistic and the critical value for the case here where Ω is unknown. With some abuse of notation (by using the same symbol for another object than above), the subvector AR statistic $AR_n(\beta_0)$ is defined as the smallest root $\hat{\kappa}_{pn}$ of the roots $\hat{\kappa}_{in}$, $i = 1, \dots, p$ (ordered nonincreasingly) of the characteristic polynomial

$$\left| \hat{\kappa} I_p - \hat{U}_n (\bar{Y}_0, W)' P_Z (\bar{Y}_0, W) \hat{U}_n \right| = 0, \tag{3.2}$$

where

$$\hat{U}_n := ((n - k)^{-1} (\bar{Y}_0, W)' M_Z (\bar{Y}_0, W))^{-1/2} \tag{3.3}$$

and \hat{U}_n^{-2} is a consistent estimator (under certain drifting sequences from the parameter space \mathcal{F}) for $\Omega(\beta_0)$ in (2.5), see Lemma 1 in the SM for details. The conditional subvector AR test rejects H_0 at nominal size α if

$$AR_n(\beta_0) > c_{1-\alpha}(\hat{\kappa}_{1n}, k - m_W), \tag{3.4}$$

where $c_{1-\alpha}(\cdot, \cdot)$ has been defined in (2.13) and $\hat{\kappa}_{1n}$ is the largest root of (3.2).

Theorem 4 *Under Assumption B, the conditional subvector AR test in (3.4) implemented at nominal size $\alpha \in (0, 1)$ has asymptotic size equal to α for the parameter space \mathcal{F} defined in (3.1).*

Comments. 1. The proof of Theorem 4 is given in Section S.1.3 in the SM. It relies on showing that the limiting NRP is smaller or equal to α along all relevant drifting sequences of parameters from \mathcal{F} . This is done by showing that the limiting NRPs equal finite-sample NRPs under Assumption A. Therefore the same comment applies to Theorem 4 as the comment below Theorem 2. The analysis is substantially more complicated here than in GKMC, in part because the critical values are also random.

2. Theorem 4 remains true if the conditional critical value $c_{1-\alpha}(\hat{\kappa}_{1n}, k - m_W)$ of the subvector AR test is replaced by any other critical value, $\tilde{c}_{1-\alpha}(\hat{\kappa}_{1n}, k - m_W)$ say, where $\tilde{c}_{1-\alpha}(\cdot, k - m_W)$ is a continuous non-decreasing function such that the corresponding test under Assumption A has finite-sample size equal to α . In particular, besides the critical values obtained from Table 1 by interpolation also the critical values suggested in Section 2.6 could be used.

4 Conclusion

We show that the subvector AR test of GKMC is inadmissible by developing a new conditional subvector AR test that has correct size and uses data-dependent critical values that are always smaller than the $\chi_{k-m_W}^2$ critical values in GKMC. The critical values are increasing in a conditioning statistic that relates to the strength of identification of the parameters not under test. Our proposed test has considerably higher power under weak identification than the GKMC procedure.

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Supplemental Material

The Supplemental Material contains proofs of the results in the main text and additional graphs and figures.

S.1 Proofs and derivations

Proof of Theorem 1: The monotonicity follows from (Perlman and Olkin, 1980, Theorem 3.5). The proof relies on the following result, available in (Muirhead, 2009, Theorem 10.3.8), which states that a 2×2 non-central Wishart matrix with noncentrality matrix of rank 1 can be expressed as $T'T$, where

$$T = \begin{pmatrix} t_{11} & t_{12} \\ 0 & t_{22} \end{pmatrix},$$

$t_{11}^2 \sim \chi_k^2(\kappa_1)$ (non-central χ^2 with noncentrality parameter κ_1), $t_{22}^2 \sim \chi_{k-1}^2$, $t_{12} \sim N(0, 1)$, and t_{11}, t_{12}, t_{22} are mutually independent. The minimum eigenvalue of $T'T$, $\hat{\kappa}_{\min}$, is given by

$$\hat{\kappa}_{\min} = \frac{t_{11}^2 + t_{12}^2 + t_{22}^2 - \sqrt{(t_{11}^2 + t_{12}^2 + t_{22}^2)^2 - 4t_{11}^2 t_{22}^2}}{2}.$$

It is straightforward to show that $\hat{\kappa}_{\min} \leq t_{22}^2$, which establishes the upper bound in the distribution of $\hat{\kappa}_{\min}$ in GKMC. It is also straightforward to establish that $\hat{\kappa}_{\min}$ is monotonically increasing in t_{11}^2 , and since t_{11}^2 is stochastically increasing in κ_1 (see, e.g., (Johnson and Kotz, 1970, ch. 28)), then $\hat{\kappa}_{\min}$ is stochastically increasing in κ_1 , as shown formally in (Perlman and Olkin, 1980, Theorem 3.5). Finally, $\hat{\kappa}_{\min} - t_{22}^2 \xrightarrow{P} 0$ as $\kappa_1 \rightarrow \infty$ (because $t_{11}^2 \xrightarrow{P} \infty$), and therefore, $\hat{\kappa}_{\min} \xrightarrow{d} \chi_{k-1}^2$, as required. \square

Proof of Theorem 3: Consider the $k \times (m_W + 1)$ Gaussian matrix $\Xi \sim N(\mathcal{M}, I_{k(m_W+1)})$, with \mathcal{M} nonstochastic and $\rho(\mathcal{M}) \leq m_W$. Partition Ξ as

$$\Xi = \begin{pmatrix} \Xi_{11} & \Xi_{12} \\ \Xi_{21} & \Xi_{22} \end{pmatrix}, \quad (\text{S-1})$$

where Ξ_{11} is $(k - m_W + 1) \times 2$, Ξ_{12} is $(k - m_W + 1) \times (m_W - 1)$, Ξ_{21} is $(m_W - 1) \times 2$, and Ξ_{22} is $(m_W - 1) \times (m_W - 1)$. Partition \mathcal{M} conformably with Ξ . Let μ_i , $i = 1, \dots, m_W$, denote the possibly nonzero singular values of \mathcal{M} (the order doesn't matter for the arguments below). Wlog, we can set

$$\mathcal{M} = \begin{pmatrix} \mathcal{M}_{11} & 0^{k-m_W+1 \times m_W-1} \\ 0^{m_W-1 \times 2} & \mathcal{M}_{22} \end{pmatrix}, \quad (\text{S-2})$$

where

$$\mathcal{M}_{11} := \begin{pmatrix} 0^{k-m_W \times 1} & 0^{k-m_W \times 1} \\ 0 & \mu_{m_W} \end{pmatrix}, \text{ and } \mathcal{M}_{22} := \text{diag}(\mu_1, \dots, \mu_{m_W-1}). \quad (\text{S-3})$$

Let

$$O := \begin{pmatrix} (I_2 + \Xi'_{21} \Xi_{22}^{-1'} \Xi_{22}^{-1} \Xi_{21})^{-1/2} & \Xi'_{21} \Xi_{22}^{-1'} (I_{m_W-1} + \Xi_{22}^{-1} \Xi_{21} \Xi'_{21} \Xi_{22}^{-1'})^{-1/2} \\ -\Xi_{22}^{-1} \Xi_{21} (I_2 + \Xi'_{21} \Xi_{22}^{-1'} \Xi_{22}^{-1} \Xi_{21})^{-1/2} & (I_{m_W-1} + \Xi_{22}^{-1} \Xi_{21} \Xi'_{21} \Xi_{22}^{-1'})^{-1/2} \end{pmatrix} \quad (\text{S-4})$$

and

$$\tilde{\Xi} := \Xi O = \begin{pmatrix} \tilde{\Xi}_{11} & \tilde{\Xi}_{12} \\ 0 & \tilde{\Xi}_{22} \end{pmatrix}, \quad (\text{S-5})$$

where

$$\tilde{\Xi}_{11} := (\Xi_{11} - \Xi_{12}\Xi_{22}^{-1}\Xi_{21}) (I_2 + \Xi'_{21}\Xi_{22}^{-1}\Xi_{22}^{-1}\Xi_{21})^{-1/2}. \quad (\text{S-6})$$

Moreover, since Ξ_{21} and Ξ_{22} are independent of Ξ_{11} and Ξ_{12} , and $O'O = I_{m_W+1}$, conditional on O , $\tilde{\Xi}_{11} \in \mathfrak{R}^{(k-m_W+1) \times 2}$ is Gaussian with covariance matrix $I_{2(k-m_W+1)}$ and mean

$$\begin{aligned} \tilde{\mathcal{M}}_{11} &:= (\mathcal{M}_{11} - \mathcal{M}_{12}\Xi_{22}^{-1}\Xi_{21}) (I_2 + \Xi'_{21}\Xi_{22}^{-1}\Xi_{22}^{-1}\Xi_{21})^{-1/2} \\ &= \mathcal{M}_{11} (I_2 + \Xi'_{21}\Xi_{22}^{-1}\Xi_{22}^{-1}\Xi_{21})^{-1/2}. \end{aligned} \quad (\text{S-7})$$

Since $\rho(\mathcal{M}_{11}) \leq 1$ by (S-3), the same holds for $\rho(\tilde{\mathcal{M}}_{11})$. Hence, conditional on O , $\tilde{\Xi}'_{11}\tilde{\Xi}_{11} \sim \mathcal{W}_2(k-m_W+1, I_2, \tilde{\mathcal{M}}'_{11}\tilde{\mathcal{M}}_{11})$ with $\rho(\tilde{\mathcal{M}}'_{11}\tilde{\mathcal{M}}_{11}) \leq 1$. \square

S.1.1 Joint distribution of the vector of eigenvalues of eigenproblem (2.7)

We study the joint distribution of the vector of eigenvalues $(\hat{\kappa}_1, \dots, \hat{\kappa}_{m_W})$ of the eigenproblem that defines the subvector statistic $AR_n(\beta_0)$ when the hypothesized β_0 does not necessarily equal the true slope parameter β . Recall the model (2.1) and the eigenproblem of the subvector AR statistic (2.7). Pre/post-multiplying (2.7) by

$$\begin{vmatrix} 1 & 0 \\ -\gamma & I_{m_W} \end{vmatrix} \text{ yields } 0 = |\kappa\Sigma - (u, W)' P_Z(u, W)| \quad (\text{S-8})$$

an equivalent eigenproblem, where

$$u := y - Y\beta_0 - W\gamma = \varepsilon + Y(\beta - \beta_0), \quad \Sigma := \begin{pmatrix} \sigma_{uu} & \Sigma_{uV_W} \\ \Sigma'_{uV_W} & \Sigma_{V_W V_W} \end{pmatrix}, \quad (\text{S-9})$$

and σ_{uu} and $\Sigma'_{uV_W} \in \mathfrak{R}^{m_W}$ denote the variance of u and the covariance between u and V_W , respectively. Note that u does not equal the structural error ε in (2.1) unless $\beta = \beta_0$. Note that for

$$C := \begin{pmatrix} \sigma_{uu}^{-1/2} & 0 \\ -\Sigma_{V_W V_W \cdot u}^{-1/2} \Sigma'_{uV_W} \sigma_{uu}^{-1} & \Sigma_{V_W V_W \cdot u}^{-1/2} \end{pmatrix} \text{ with } \Sigma_{V_W V_W \cdot u} := \Sigma_{V_W V_W} - \Sigma'_{uV_W} \Sigma_{uV_W} \sigma_{uu}^{-1} \in \mathfrak{R}^{m_W \times m_W}, \quad (\text{S-10})$$

$C\Sigma C' = I_p$ holds. Therefore, pre and postmultiplying (S-8) by $|C|$ leads to

$$0 = \left| \kappa I_p - \left(u/\sigma_{uu}^{1/2}, \left(W - u \frac{\Sigma_{uV_W}}{\sigma_{uu}} \right) \Sigma_{V_W V_W \cdot u}^{-1/2} \right)' P_Z \left(u/\sigma_{uu}^{1/2}, \left(W - u \frac{\Sigma_{uV_W}}{\sigma_{uu}} \right) \Sigma_{V_W V_W \cdot u}^{-1/2} \right) \right| \quad (\text{S-11})$$

or

$$0 = \left| \kappa I_{1+m_W} - \begin{pmatrix} \xi'_u \xi_u & \xi'_u \xi_{W \cdot u} \\ \xi'_{W \cdot u} \xi_u & \xi'_{W \cdot u} \xi_{W \cdot u} \end{pmatrix} \right|, \quad (\text{S-12})$$

where

$$\xi_u := (Z'Z)^{-1/2} Z'u/\sigma_{uu}^{1/2} \in \mathfrak{R}^k \text{ and } \xi_{W \cdot u} := (Z'Z)^{-1/2} Z' \left(W - u \frac{\Sigma_{uV_W}}{\sigma_{uu}} \right) \Sigma_{V_W V_W \cdot u}^{-1/2} \in \mathfrak{R}^{k \times m_W}. \quad (\text{S-13})$$

Now,

$$\begin{aligned}
E(\xi_u) &= E(Z'Z)^{-1/2} Z'Y(\beta - \beta_0) / \sigma_{uu}^{1/2} \\
&= (Z'Z)^{1/2} \Pi_Y(\beta - \beta_0) / \sigma_{uu}^{1/2} \text{ and} \\
E(\xi_{W.u}) &= (Z'Z)^{1/2} \left(\Pi_W - \Pi_Y(\beta - \beta_0) \frac{\Sigma_u V_W}{\sigma_{uu}} \right) \Sigma_{V_W V_W.u}^{-1/2}.
\end{aligned} \tag{S-14}$$

Hence,

$$\begin{aligned}
\Xi &:= [\xi_u, \xi_{W.u}] \sim N(\mathcal{M}, I_{kp}) \text{ and } \Xi' \Xi \sim \mathcal{W}_p(k, I_p, \mathcal{M}' \mathcal{M}), \text{ where} \\
\mathcal{M} &:= (Z'Z)^{1/2} \left[\Pi_Y(\beta - \beta_0) / \sigma_{uu}^{1/2}, \left(\Pi_W - \Pi_Y(\beta - \beta_0) \frac{\Sigma_u V_W}{\sigma_{uu}} \right) \Sigma_{V_W V_W.u}^{-1/2} \right].
\end{aligned} \tag{S-15}$$

Case 1) Assume that H_0 in (2.3) holds. In that case $u = \varepsilon$ and we write

$$\Sigma = \begin{pmatrix} \sigma_{\varepsilon\varepsilon} & \Sigma_{\varepsilon V_W} \\ \Sigma'_{\varepsilon V_W} & \Sigma_{V_W V_W} \end{pmatrix} \tag{S-16}$$

and $\Sigma_{V_W V_W.\varepsilon} := \Sigma_{V_W V_W} - \Sigma'_{\varepsilon V_W} \Sigma_{\varepsilon V_W} \sigma_{\varepsilon\varepsilon}^{-1}$. Defining

$$\Theta_W := (Z'Z)^{1/2} \Pi_W \Sigma_{V_W V_W.\varepsilon}^{-1/2} \in \mathfrak{R}^{k \times m_W}, \tag{S-17}$$

it follows that $\mathcal{M} = (0^k, \Theta_W)$.

Case 2) Assume instead that H'_0 in (2.17) holds. Note that

$$A = Z'Z [\Pi_Y(\beta - \beta_0) + \Pi_W \gamma, \Pi_W] \tag{S-18}$$

and therefore for \mathcal{M} defined in (S-15)

$$\mathcal{M} = (Z'Z)^{-1/2} AT \text{ for } T := \begin{pmatrix} 1/\sigma_{uu}^{1/2} & -\frac{\Sigma_u V_W}{\sigma_{uu}^{1/2}} \Sigma_{V_W V_W.u}^{-1/2} \\ -\gamma/\sigma_{uu}^{1/2} & (I_{m_W} + \gamma \frac{\Sigma_u V_W}{\sigma_{uu}^{1/2}}) \Sigma_{V_W V_W.u}^{-1/2} \end{pmatrix}. \tag{S-19}$$

Because $(Z'Z)^{-1/2}$ and T are both of full rank it follows that $\rho(\mathcal{M}) = \rho(A)$.⁹

⁹To see the former, note that T is of full rank iff

$$\tilde{T} := \begin{pmatrix} 1 & -c' \\ -\gamma & \Sigma_{V_W V_W.u}^{-1/2} + \gamma c' \end{pmatrix}$$

is of full rank, where $c' := \Sigma_u V_W \Sigma_{V_W V_W.u}^{-1/2} \sigma_{uu}^{-1/2}$. But whenever $\tilde{T}(a_1, a_2)' = 0^p$ it follows that $a_1 - c'a_2 = 0$ and $-\gamma a_1 + \Sigma_{V_W V_W.u}^{-1/2} a_2 + \gamma c' a_2 = 0^{m_W}$. Inserting the former into the latter equality yields $\Sigma_{V_W V_W.u}^{-1/2} a_2 = 0^{m_W}$ and thus $a_2 = 0^{m_W}$. The latter implies $a_1 = 0$. Finally, $(Z'Z)^{-1/2}$ is of full rank by Assumption A 2.

S.1.2 The approximate conditional distribution

This section replicates the analysis in Muirhead (1978, Section 6). As a special case of (James, 1964, eq. (68)), the joint density of the eigenvalues $\hat{\kappa}_1$ and $\hat{\kappa}_2$ of $\Xi' \Xi \sim \mathcal{W}_2(k, I_2, \mathcal{M}'\mathcal{M})$ can be written as

$$f_{\hat{\kappa}_1, \hat{\kappa}_2}(x_1, x_2; \kappa_1, \kappa_2) = \frac{\pi^2}{2^k \Gamma_2(k/2) \Gamma_2(1)} \exp\left(-\frac{1}{2}(x_1 + x_2)\right) x_1^{\frac{k-3}{2}} x_2^{\frac{k-3}{2}} (x_1 - x_2) \quad (\text{S-20})$$

$$\times \exp\left(-\frac{1}{2}(\kappa_1 + \kappa_2)\right) {}_0F_1^{(2)}\left(\frac{1}{2}k; \frac{1}{4} \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}, \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}\right)$$

for $x_1 \geq x_2 \geq 0$, where $\Gamma_m(a) := \pi^{m(m-1)/4} \prod_{i=1}^m \Gamma(a - \frac{1}{2}(i-1))$ and ${}_0F_1^{(2)}$ is the hypergeometric function of two matrix arguments. Thus, $\Gamma_2(a) := \pi^{1/2} \Gamma(a) \Gamma(a - \frac{1}{2})$, $\Gamma_2(1) := \pi^{1/2} \Gamma(1) \Gamma(\frac{1}{2}) = \pi$ and $\Gamma_2(k/2) = \pi^{1/2} \Gamma(k/2) \Gamma(\frac{k-1}{2})$. So, the joint density (S-20) can also be written as

$$\frac{\pi^{1/2}}{2^k \Gamma(k/2) \Gamma(\frac{k-1}{2})} \exp\left(-\frac{1}{2}(x_1 + x_2)\right) x_1^{\frac{k-3}{2}} x_2^{\frac{k-3}{2}} (x_1 - x_2) \quad (\text{S-21})$$

$$\times \exp\left(-\frac{1}{2}(\kappa_1 + \kappa_2)\right) {}_0F_1^{(2)}\left(\frac{1}{2}k; \frac{1}{4} \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}, \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}\right).$$

Under the assumption that $\kappa_1 > \kappa_2 = 0$, where κ_1 is large, Leach (1969) has shown that

$${}_0F_1^{(2)}\left(\frac{1}{2}k; \frac{1}{4} \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}, \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}\right) \sim \frac{2^{\frac{k-2}{2}}}{\pi} \Gamma(k/2) \exp\left((x_1 \kappa_1)^{\frac{1}{2}}\right) \quad (\text{S-22})$$

$$\times (\kappa_1 x_1)^{\frac{2-k}{4}} (\kappa_1 (x_1 - x_2))^{-\frac{1}{2}}.$$

Substituting equation (S-22) into equation (S-21) gives an asymptotic representation for the density function of $\hat{\kappa}_1$ and $\hat{\kappa}_2$ under the assumption that κ_1 is large,

$$\frac{\pi^{-1/2}}{2^{\frac{k+2}{2}} \Gamma(\frac{k-1}{2})} \exp\left(-\frac{1}{2}\kappa_1\right) \kappa_1^{-\frac{k}{4}} \hat{\kappa}_1^{\frac{k-4}{4}} \exp\left[-\frac{1}{2}x_1 + (x_1 \kappa_1)^{\frac{1}{2}}\right] \quad (\text{S-23})$$

$$\times \exp\left(-\frac{1}{2}x_2\right) x_2^{\frac{k-3}{2}} (x_1 - x_2)^{\frac{1}{2}}.$$

This is a special case of Muirhead (1978, (6.5)) with his k, m , and n corresponding to $1, p = 2$, and k , respectively, and using $\kappa_2 = 0$. Integrating the second line of (S-23) w.r.t. x_2 yields

$$\int_0^{\hat{\kappa}_1} \exp\left(-\frac{1}{2}x_2\right) x_2^{\frac{k-3}{2}} (x_1 - x_2)^{\frac{1}{2}} dx_2 \quad (\text{S-24})$$

$$= \frac{\pi^{\frac{1}{2}}}{2} x_1^{k/2} \frac{\Gamma(\frac{k-1}{2})}{\Gamma(\frac{k+2}{2})} {}_1F_1\left(\frac{k-1}{2}, \frac{k+2}{2}; -\frac{x_1}{2}\right),$$

where ${}_1F_1(a, c; z)$ is the confluent hypergeometric function. Combined with (S-23), the approximate conditional distribution of $\hat{\kappa}_2$ given $\hat{\kappa}_1$ is

$$f_{\hat{\kappa}_2 | \hat{\kappa}_1}^*(x_2 | \hat{\kappa}_1) = \frac{\Gamma(\frac{k+2}{2})}{\Gamma(\frac{k-1}{2})} \frac{2 \exp(-\frac{1}{2}x_2) x_2^{\frac{k-3}{2}} (\hat{\kappa}_1 - x_2)^{\frac{1}{2}}}{\hat{\kappa}_1^{\frac{k}{2}} \sqrt{\pi} {}_1F_1\left(\frac{k-1}{2}, \frac{k+2}{2}; -\frac{\hat{\kappa}_1}{2}\right)}. \quad (\text{S-25})$$

The last equation reduces to (2.12) if we use the definition of the density of χ_{k-1}^2 , $f_{\chi_{k-1}^2}(x_2) = \frac{1}{2^{\frac{k-1}{2}} \Gamma(\frac{k-1}{2})} x_2^{\frac{k-3}{2}} e^{-\frac{x_2}{2}}$. Hence, the integrating constant $g(\hat{\kappa}_1)$ in the approximate conditional density (2.12) is given by

$$g(\hat{\kappa}_1) = \frac{\Gamma\left(\frac{k+2}{2}\right) 2^{\frac{k+1}{2}}}{\hat{\kappa}_1^{\frac{k}{2}} \sqrt{\pi} {}_1F_1\left(\frac{k-1}{2}, \frac{k+2}{2}; -\frac{\hat{\kappa}_1}{2}\right)}. \quad (\text{S-26})$$

The result that $c_{1-\alpha}(\infty, k-1) = \chi_{k-1, 1-\alpha}^2$ follows from the fact that $\lim_{\hat{\kappa}_1 \rightarrow \infty} f_{\hat{\kappa}_2 | \hat{\kappa}_1}^*(\cdot | \hat{\kappa}_1) = f_{\chi_{k-1}^2}(\cdot)$. This can be proven using the property that ${}_1F_1(a, c; -z) z^a \rightarrow \Gamma(c) / \Gamma(c-a)$ as $z \rightarrow \infty$ (Olver, 1997, p. 257, eq. 10.08). It follows that $\frac{2^{\frac{k+1}{2}} (x_1 - x_2)^{1/2} \Gamma(\frac{k+2}{2})}{x_1^{\frac{k}{2}} \sqrt{\pi} {}_1F_1(\frac{k-1}{2}, \frac{k+2}{2}; -\frac{x_1}{2})} \rightarrow \frac{2^{\frac{k+1}{2}} \Gamma(\frac{k+2}{2} - \frac{k-1}{2})}{\sqrt{\pi} 2^{\frac{k-1}{2}}} = \frac{2\Gamma(\frac{3}{2})}{\sqrt{\pi}} = 1$ as $x_1 \rightarrow \infty$.

S.1.3 Proof of Theorem 4

Uniformity Reparametrization To prove that the new subvector AR test has asymptotic size bounded by the nominal size α we use a general result in Andrews, Cheng, and Guggenberger (2011, ACG from now on). To describe it, consider a sequence of arbitrary tests $\{\varphi_n : n \geq 1\}$ of a certain null hypothesis and denote by $RP_n(\lambda)$ the null rejection probability of φ_n when the DGP is pinned down by the parameter vector $\lambda \in \Lambda$, where Λ denotes the parameter space of λ . By definition, the asymptotic size of φ_n is defined as

$$AsySz = \lim_{n \rightarrow \infty} \sup_{\lambda \in \Lambda} RP_n(\lambda). \quad (\text{S-27})$$

Let $\{h_n(\lambda) : n \geq 1\}$ be a sequence of functions on Λ , where $h_n(\lambda) = (h_{n,1}(\lambda), \dots, h_{n,J}(\lambda))'$ with $h_{n,1}(\lambda) \in \mathfrak{R} \forall j = 1, \dots, J$. Define

$$H = \{h \in (\mathfrak{R} \cup \{\pm\infty\})^J : h_{w_n}(\lambda_{w_n}) \rightarrow h \text{ for some subsequence } \{w_n\} \text{ of } \{n\} \text{ and some sequence } \{\lambda_{w_n} \in \Lambda : n \geq 1\}\} \quad (\text{S-28})$$

Assumption B in ACG: For any subsequence $\{w_n\}$ of $\{n\}$ and any sequence $\{\lambda_{w_n} \in \Lambda : n \geq 1\}$ for which $h_{w_n}(\lambda_{w_n}) \rightarrow h \in H$, $RP_{w_n}(\lambda_{w_n}) \rightarrow [RP^-(h), RP^+(h)]$ for some $RP^-(h), RP^+(h) \in (0, 1)$.¹⁰

The assumption states, in particular, that along certain drifting sequences of parameters λ_{w_n} indexed by a localization parameter h the NRP of the test cannot asymptotically exceed a certain threshold $RP^+(h)$ indexed by h .

Proposition 1 (ACG, Theorem 2.1(a) and Theorem 2.2) *Suppose Assumption B in ACG holds. Then, $\inf_{h \in H} RP^-(h) \leq AsySz \leq \sup_{h \in H} RP^+(h)$.*

We next verify Assumption B in ACG for the subvector AR test and establish that $\sup_{h \in H} RP^+(h) = \alpha$ when the test is implemented at nominal size α . To do so, we use Andrews and Guggenberger (2015, AG from now on), namely Proposition 12.5 in AG, to derive the joint limiting distribution of the eigenvalues $\hat{\kappa}_{in}$, $i = 1, \dots, p$ in (3.2). We reparameterize the null distribution F to a vector λ . The vector λ is chosen such that for a subvector of λ convergence of a drifting subsequence of the subvector (after suitable renormalization) yields convergence in distribution of the test statistic and the critical value. For given F define

$$Q_F := (E_F Z_i Z_i')^{1/2} \text{ and } U_F := \Omega(\beta_0)^{-1/2}. \quad (\text{S-29})$$

¹⁰By definition, the notation $x_n \rightarrow [x_{1,\infty}, x_{2,\infty}]$ means that $x_{1,\infty} \leq \liminf_{n \rightarrow \infty} x_n \leq \limsup_{n \rightarrow \infty} x_n \leq x_{2,\infty}$.

Let

$$B_F \text{ denote a } p \times p \text{ orthogonal matrix of eigenvectors of } U'_F(\Pi_W\gamma, \Pi_W)'Q'_F Q_F(\Pi_W\gamma, \Pi_W)U_F \quad (\text{S-30})$$

ordered so that the p corresponding eigenvalues $(\eta_{1F}, \dots, \eta_{pF})$ are nonincreasing. Let

$$C_F \text{ denote a } k \times k \text{ orthogonal matrix of eigenvectors of } Q_F(\Pi_W\gamma, \Pi_W)U_F U'_F(\Pi_W\gamma, \Pi_W)'Q'_F. \quad (\text{S-31})$$

The corresponding k eigenvalues are $(\eta_{1F}, \dots, \eta_{pF}, 0, \dots, 0)$. Let

$$(\tau_{1F}, \dots, \tau_{pF}) \text{ denote the singular values of } Q_F(\Pi_W\gamma, \Pi_W)U_F \in \mathfrak{R}^{k \times p}, \quad (\text{S-32})$$

which are nonnegative, ordered so that τ_{jF} is nonincreasing. (Some of these singular values may be zero.) As is well-known, the squares of the p singular values of a $k \times p$ matrix A equal the p largest eigenvalues of $A'A$ and AA' . In consequence, $\eta_{jF} = \tau_{jF}^2$ for $j = 1, \dots, p$. In addition, $\eta_{jF} = 0$ for $j = p+1, \dots, k$.

Define the elements of λ to be¹²

$$\begin{aligned} \lambda_{1,F} &:= (\tau_{1F}, \dots, \tau_{pF})' \in \mathfrak{R}^p, \\ \lambda_{2,F} &:= B_F \in \mathfrak{R}^{p \times p}, \\ \lambda_{3,F} &:= C_F \in \mathfrak{R}^{k \times k}, \\ \lambda_{4,F} &:= E_F Z_i Z'_i \in \mathfrak{R}^{k \times k}, \\ \lambda_{5,F} &:= (\lambda_{5,1,F}, \dots, \lambda_{5,3,F}) = E_F(V_{W,i} V'_{W,i}, \varepsilon_i^2, V_{W,i} \varepsilon_i), \\ \lambda_{6,F} &:= (\lambda_{6,1F}, \dots, \lambda_{6,p-1F})' := \left(\frac{\tau_{2F}}{\tau_{1F}}, \dots, \frac{\tau_{pF}}{\tau_{p-1F}} \right)' \in [0, 1]^{p-1}, \text{ where } 0/0 := 0, \\ \lambda_{7,F} &:= Q_F \in \mathfrak{R}^{k \times k}, \\ \lambda_{8,F} &:= U_F = \Omega(\beta_0)^{-1/2} \in \mathfrak{R}^{p \times p}, \\ \lambda_{9,F} &:= \gamma \in \mathfrak{R}^{m_W}, \\ \lambda_{10,F} &:= F, \text{ and} \\ \lambda &:= \lambda_F := (\lambda_{1,F}, \dots, \lambda_{10,F}). \end{aligned} \quad (\text{S-33})$$

The parameter space Λ for λ and the function $h_n(\lambda)$ (that appears in Assumption B in ACG) are defined by

$$\begin{aligned} \Lambda &:= \{\lambda : \lambda = (\lambda_{1,F}, \dots, \lambda_{10,F}) \text{ for some } F \in \mathcal{F}\}, \\ h_n(\lambda) &:= (n^{1/2} \lambda_{1,F}, \lambda_{2,F}, \lambda_{3,F}, \lambda_{4,F}, \dots, \lambda_{8,F}, \lambda_{9,F}). \end{aligned} \quad (\text{S-34})$$

We define λ and $h_n(\lambda)$ as in (S-33) and (S-34) because, as shown below, the asymptotic distributions of the test statistic and conditional critical values under a sequence $\{F_n : n \geq 1\}$ for which $h_n(\lambda_{F_n}) \rightarrow h$ depend on $\lim n^{1/2} \lambda_{1,F_n}$ and $\lim \lambda_{m,F_n}$ for $m = 2, \dots, 9$. Note that we can view $h \in (\mathfrak{R} \cup \{\pm\infty\})^J$ (for an

¹¹The matrices B_F and C_F are not uniquely defined. We let B_F denote one choice of the matrix of eigenvectors of $U'_F(\Pi_W\gamma, \Pi_W)'Q'_F Q_F(\Pi_W\gamma, \Pi_W)U_F$ and analogously for C_F .

Note that the role of $E_F G_i$ in AG, Section 12, is played by $(\Pi_W\gamma, \Pi_W) \in \mathfrak{R}^{k \times p}$ and the role of W_F is played by Q_F .

¹²For simplicity, as above, when writing $\lambda = (\lambda_{1,F}, \dots, \lambda_{10,F})$ or $\lambda_{5,F} = (\lambda_{5,1,F}, \dots, \lambda_{5,3,F})$ (and likewise in similar expressions) we allow the elements to be scalars, vectors, matrices, and distributions.

appropriately chosen finite $J \in \mathbb{N}$).

For notational convenience, for any subsequence $\{w_n : n \geq 1\}$,

$$\{\lambda_{w_n, h} : n \geq 1\} \text{ denotes a sequence } \{\lambda_{w_n} \in \Lambda : n \geq 1\} \text{ for which } h_{w_n}(\lambda_{w_n}) \rightarrow h. \quad (\text{S-35})$$

It follows that the set H defined in (S-28) is given as the set of all $h \in (\mathfrak{R} \cup \{\pm\infty\})^J$ such that there exists $\{\lambda_{w_n, h} : n \geq 1\}$ for some subsequence $\{w_n : n \geq 1\}$.

We decompose h analogously to the decomposition of the first nine components of λ : $h = (h_1, \dots, h_9)$, where $\lambda_{m, F}$ and h_m have the same dimensions for $m = 1, \dots, 9$. We further decompose the vector h_1 as $h_1 = (h_{1,1}, \dots, h_{1,p})'$, where the elements of h_1 could equal ∞ . Again, by definition, under a sequence $\{\lambda_{n, h} : n \geq 1\}$, we have

$$n^{1/2} \tau_{jF_n} \rightarrow h_{1,j} \geq 0 \quad \forall j = 1, \dots, p, \quad \lambda_{m, F_n} \rightarrow h_m \quad \forall m = 2, \dots, 9. \quad (\text{S-36})$$

Note that $h_{1,p} = \tau_{pF_n} = 0$ because $\rho(\Pi_W \gamma, \Pi_W) < p$. By Lyapunov-type WLLNs and CLTs, using the moment restrictions imposed in (3.1), we have under $\lambda_{n, h}$

$$\begin{aligned} & \left(\begin{array}{c} (E_F(\varepsilon_i^2))^{-1/2} (n^{-1} Z' Z)^{-1/2} n^{-1/2} Z' \varepsilon \\ \left((E_F(V_{W,i} V_{W,i}'))^{-1/2} \otimes (n^{-1} Z' Z)^{-1/2} \right) \text{vec} (n^{-1/2} Z' V_W) \end{array} \right) \xrightarrow{d} \\ & \left(\begin{array}{c} \xi_{\varepsilon, h} \\ \xi_{V_W, h} \end{array} \right) \sim N \left(0^{pk \times 1}, \left(\begin{array}{cc} 1 & h_{5,2}^{-1/2} h'_{5,3} h_{5,1}^{-1/2} \\ h_{5,2}^{-1/2} h_{5,1}^{-1/2} h_{5,3} & I_{m_W} \end{array} \right) \otimes I_k \right), \\ & n^{-1} (V_W' V_W, \varepsilon' \varepsilon, V_W' \varepsilon) \xrightarrow{p} h_5, \lambda_{4,F}^{-1} (n^{-1} Z' Z) \xrightarrow{p} I_k, \quad n^{-1} Z' [\varepsilon : V_W] \xrightarrow{p} 0^{k \times p}, \end{aligned} \quad (\text{S-37})$$

where the random vector $(\xi_{\varepsilon, h}, \xi_{V_W, h}')'$ is defined here.

Asymptotic Distributions Let $q = q_h \in \{0, \dots, p-1\}$ be such that

$$h_{1,j} = \infty \text{ for } 1 \leq j \leq q_h \text{ and } h_{1,j} < \infty \text{ for } q_h + 1 \leq j \leq p, \quad (\text{S-38})$$

where $h_{1,j} := \lim n^{1/2} \tau_{jF_n} \geq 0$ for $j = 1, \dots, p$ by (S-36) and the distributions $\{F_n : n \geq 1\}$ correspond to $\{\lambda_{n, h} : n \geq 1\}$ defined in (S-35). This value q exists because $\{h_{1,j} : j \leq p\}$ are nonincreasing in j (since $\{\tau_{jF} : j \leq p\}$ are nonincreasing in j , as defined in (S-32)). Note that q is the number of singular values of $Q_{F_n}(\Pi_{W_n} \gamma_n, \Pi_{W_n}) U_{F_n} \in \mathfrak{R}^{k \times p}$ that diverge to infinity when multiplied by $n^{1/2}$. Note again that $q < p$ because $\rho(\Pi_{W_n} \gamma_n, \Pi_{W_n}) < p$.

An analogue to Lemma 12.4 in AG is given by the following statement. Define

$$\widehat{D}_n := (Z' Z)^{-1} Z' (\overline{Y}_0, W) \quad \text{and} \quad \widehat{Q}_n := (n^{-1} Z' Z)^{1/2}. \quad (\text{S-39})$$

Lemma 1 Under all sequences $\{\lambda_{n, h} : n \geq 1\}$ with $\lambda_{n, h} \in \Lambda$, $n^{1/2}(\widehat{D}_n - (\Pi_{W_n} \gamma_n, \Pi_{W_n})) \rightarrow_d \overline{D}_h$, where

$$\overline{D}_h \sim (h_4^{-1} \text{vec}_{k, m_W}^{-1} ((h_{5,1}^{1/2} \otimes h_4^{1/2}) \xi_{V_W, h}) h_9 + h_{5,2}^{1/2} h_4^{-1/2} \xi_{\varepsilon, h}, h_4^{-1} \text{vec}_{k, m_W}^{-1} ((h_{5,1}^{1/2} \otimes h_4^{1/2}) \xi_{V_W, h})),$$

$\widehat{U}_n^{-2} - \Omega(\beta_0) \rightarrow_p 0^{p \times p}$, and $\widehat{Q}_n - Q_{F_n} \rightarrow_p 0^{k \times k}$, where $\text{vec}_{k, m_W}^{-1}(\cdot)$ denotes the inverse vec operation that transforms a km_W vector into a $k \times m_W$ matrix and \widehat{U}_n is defined in (3.3).

Proof of Lemma 1: We have

$$\begin{aligned}
& n^{1/2}(\widehat{D}_n - (\Pi_{W_n}\gamma_n, \Pi_{W_n})) \\
&= n^{1/2}((Z'Z)^{-1}Z'(y - Y\beta_0, W) - (\Pi_{W_n}\gamma_n, \Pi_{W_n})) \\
&= n^{1/2}((Z'Z)^{-1}Z'(Z\Pi_{W_n}\gamma_n + V_W\gamma_n + \varepsilon, Z\Pi_{W_n} + V_W) - (\Pi_{W_n}\gamma_n, \Pi_{W_n})) \\
&= (n^{-1}Z'Z)^{-1/2}[(n^{-1}Z'Z)^{-1/2}n^{-1/2}Z'(V_W\gamma_n + \varepsilon, V_W)] \rightarrow_d \overline{D}_h,
\end{aligned} \tag{S-40}$$

where the first equality uses the definition of \widehat{D}_n in (S-39), the second equality uses the formulas in (2.1), and the convergence results holds by the (triangular array) CLT and WLLN in (S-37). Also,

$$\begin{aligned}
& (n-k)^{-1}(\overline{Y}_0, W)' M_Z(\overline{Y}_0, W) \\
&= (n-k)^{-1}(V_W\gamma_n + \varepsilon, V_W)' M_Z(V_W\gamma_n + \varepsilon, V_W) \\
&= (n-k)^{-1}(V_W\gamma_n + \varepsilon, V_W)'(V_W\gamma_n + \varepsilon, V_W) + o_p(1),
\end{aligned} \tag{S-41}$$

where the first equality uses the formulas in (2.1) and the fact that $M_Z Z = 0^{n \times k}$ and the second equality follows directly from (S-37). Because $\Omega(\beta_0) = E(V'_{W,i}\gamma + \varepsilon_i, V'_{W,i})'(V'_{W,i}\gamma + \varepsilon_i, V'_{W,i})$ an application of WLLNs as in (S-37) yields the desired convergence result. Likewise, an application of a WLLN using the uniform moment conditions on Z_i in \mathcal{F} in (3.1) and the continuous mapping theorem immediately imply the desired result $\widehat{Q}_n - Q_{F_n} \rightarrow_p 0^{k \times k}$. \square

Note that the matrix $n\widehat{U}_n\widehat{D}'_n\widehat{Q}_n\widehat{Q}'_n\widehat{D}_n\widehat{U}_n$ equals the matrix $\widehat{U}_n(\overline{Y}_0, W)' P_Z(\overline{Y}_0, W)\widehat{U}_n$ that appears in (3.2). Thus, $\widehat{\kappa}_{in}$ for $i = 1, \dots, p$ equals the i th eigenvalue of $n\widehat{U}_n\widehat{D}'_n\widehat{Q}'_n\widehat{Q}_n\widehat{D}_n\widehat{U}_n$, ordered nonincreasingly, and $\widehat{\kappa}_{pn}$ is the subvector AR test statistic. To describe the limiting distribution of $(\widehat{\kappa}_{1n}, \dots, \widehat{\kappa}_{pn})$ we need additional notation, namely:

$$\begin{aligned}
& h_2 = (h_{2,q}, h_{2,p-q}), \quad h_3 = (h_{3,q}, h_{3,k-q}), \\
& h_{1,p-q}^\diamond := \begin{bmatrix} 0^{q \times (p-q)} \\ \text{Diag}\{h_{1,q+1}, \dots, h_{1,p-1}, 0\} \\ 0^{(k-p) \times (p-q)} \end{bmatrix} \in \mathfrak{R}^{k \times (p-q)}, \\
& \overline{\Delta}_h := (\overline{\Delta}_{h,q}, \overline{\Delta}_{h,p-q}) \in \mathfrak{R}^{k \times p}, \quad \overline{\Delta}_{h,q} := h_{3,q} \in \mathfrak{R}^{k \times q}, \\
& \overline{\Delta}_{h,p-q} := h_3 h_{1,p-q}^\diamond + h_4^{1/2} \overline{D}_h h_8 h_{2,p-q} \in \mathfrak{R}^{k \times (p-q)},
\end{aligned} \tag{S-42}$$

where $h_{2,q} \in \mathfrak{R}^{p \times q}$, $h_{2,p-q} \in \mathfrak{R}^{p \times (p-q)}$, $h_{3,q} \in \mathfrak{R}^{k \times q}$, $h_{3,k-q} \in \mathfrak{R}^{k \times (k-q)}$, $\overline{\Delta}_{h,q} \in \mathfrak{R}^{k \times q}$, and $\overline{\Delta}_{h,p-q} \in \mathfrak{R}^{k \times (p-q)}$.¹³ Note that because $\Omega(\beta_0) = E_F(V'_{W,i}\gamma + \varepsilon_i, V'_{W,i})'(V'_{W,i}\gamma + \varepsilon_i, V'_{W,i})$ and $\lambda_{8,F} = \Omega(\beta_0)^{-1/2}$ we have

$$h_8 = \begin{pmatrix} h_{5,2} + 2h'_9 h_{5,3} + h'_9 h_{5,1} h_9 & h'_9 h_{5,1} + h'_{5,3} \\ h_{5,1} h_9 + h_{5,3} & h_{5,1} \end{pmatrix}^{-1/2} \in \mathfrak{R}^{p \times p}. \tag{S-43}$$

Let $T_n := B_{F_n} S_n$ and $S_n := \text{Diag}\{(n^{1/2}\tau_{1F_n})^{-1}, \dots, (n^{1/2}\tau_{qF_n})^{-1}, 1, \dots, 1\} \in \mathfrak{R}^{p \times p}$. The same proof as the one of Lemma 12.4 in AG shows that $n^{1/2}Q_{F_n}\widehat{D}_n U_{F_n} T_n \rightarrow_d \overline{\Delta}_h$ under all sequences $\{\lambda_{n,h} : n \geq 1\}$ with $\lambda_{n,h} \in \Lambda$. The following proposition is an analogue to Proposition 12.5 in AG.

Proposition 2 *Under all sequences $\{\lambda_{n,h} : n \geq 1\}$ with $\lambda_{n,h} \in \Lambda$,*

¹³There is some abuse of notation here. E.g., $h_{2,q}$ and $h_{2,p-q}$ denote different matrices even if $p-q$ equals q .

- (a) $\widehat{\kappa}_{jn} \rightarrow_p \infty$ for all $j \leq q$,
- (b) the (ordered) vector of the smallest $p - q$ eigenvalues of $n\widehat{U}'_n \widehat{D}'_n \widehat{Q}_n \widehat{Q}_n \widehat{D}_n \widehat{U}_n$, i.e., $(\widehat{\kappa}_{(q+1)n}, \dots, \widehat{\kappa}_{pn})'$, converges in distribution to the (ordered) $p - q$ vector of the eigenvalues of $\overline{\Delta}'_{h,p-q} h_{3,k-q} h'_{3,k-q} \times \overline{\Delta}_{h,p-q} \in \mathfrak{R}^{(p-q) \times (p-q)}$,
- (c) the convergence in parts (a) and (b) holds jointly with the convergence in Lemma 1, and
- (d) under all subsequences $\{w_n\}$ and all sequences $\{\lambda_{w_n,h} : n \geq 1\}$ with $\lambda_{w_n,h} \in \Lambda$, the results in parts (a)-(c) hold with n replaced with w_n .

Comments. 1. The proof of Proposition 2 follows directly from Proposition 12.5 in AG. Note that Assumption WU in AG is fulfilled with the roles of W_{2F} , W_F , U_{2F} , and U_F in AG played here by Q_F , Q_F , $U_F = \Omega(\beta_0)^{-1/2}$, and U_F while the roles of W_1 and U_1 in AG are played by the identity function. The roles of \widehat{W}_{2n} and \widehat{W}_n in AG are both played by \widehat{Q}_n and those of both \widehat{U}_{2n} and \widehat{U}_n by \widehat{U}_n . Lemma 1 shows consistency $\widehat{W}_{2n} - W_{2F_n} \rightarrow_p 0^{k \times k}$ and $\widehat{U}_{2n} - U_{2F_n} \rightarrow_p 0^{p \times p}$ under sequences $\{\lambda_{n,h} : n \geq 1\}$ with $\lambda_{n,h} \in \Lambda$ and trivially the functions W_1 and U_1 are continuous in our case. Note that by the restrictions in \mathcal{F} in (3.1) the requirements in the parameter space \mathcal{F}_{WU} in AG, namely “ $\kappa_{\min}(Q_F)$ and $\kappa_{\min}(U_F)$ are uniformly bounded away from zero and $\|Q_F\|$ and $\|U_F\|$ are uniformly bounded away from infinity”, are fulfilled. To see this for $U_F = \Omega(\beta_0)^{-1/2}$, simply use $\kappa_{\min}(E_F((\varepsilon_i, V'_{W,i})'(\varepsilon_i, V'_{W,i}))) \geq \delta$ in \mathcal{F} in (3.1) and the fact that under the null

$$\Omega(\beta_0) = \begin{pmatrix} 1 & \gamma' \\ 0^{m_W \times 1} & I_{m_W} \end{pmatrix} E(\varepsilon_i, V'_{W,i})'(\varepsilon_i, V'_{W,i}) \begin{pmatrix} 1 & \gamma' \\ 0^{m_W \times 1} & I_{m_W} \end{pmatrix}' \in \mathfrak{R}^{p \times p}. \quad (\text{S-44})$$

2. Proposition 2 yields the desired joint limiting distribution of the p eigenvalues in (3.2). Using repeatedly the general formula $(C' \otimes A) \text{vec}(B) = \text{vec}(ABC)$ for three conformable matrices A, B, C , we have

$$\begin{aligned} & \text{vec}(h_4^{1/2} \overline{D}_h h_8) \\ &= \text{vec}(((h_4^{-1/2} \text{vec}_{k,m_W}^{-1}((h_{5,1}^{1/2} \otimes h_4^{1/2}) \xi_{V_{W,h}}) h_9 + h_{5,2}^{1/2} \xi_{\varepsilon,h}, h_4^{-1/2} \text{vec}_{k,m_W}^{-1}((h_{5,1}^{1/2} \otimes h_4^{1/2}) \xi_{V_{W,h}}))) h_8) \\ &= (h_8' \otimes I_k) \left(\begin{pmatrix} h_{5,2}^{1/2} & h_9' h_{5,1}^{1/2} \\ 0^{m_W \times 1} & h_{5,1}^{1/2} \end{pmatrix} \otimes I_k \right) \begin{pmatrix} \xi_{\varepsilon,h} \\ \xi_{V_{W,h}} \end{pmatrix} \\ &\sim \text{vec}(v_1, \dots, v_p), \end{aligned} \quad (\text{S-45})$$

where, by definition, v_j , $j = 1, \dots, p$ are i.i.d. normal k -vectors with zero mean and covariance matrix I_k , and the distributional statement follows by straightforward calculations using the second line in (S-37). Therefore, by Lemma 1, the definition of $\overline{\Delta}_{h,p-q}$ in (S-42), and by noting that

$$h'_{3,k-q} h_3 h_{1,p-q}^\diamond = \begin{pmatrix} \text{Diag}\{h_{1,q+1}, \dots, h_{1,p-1}, 0\} \\ 0^{(k-p) \times (p-q)} \end{pmatrix} \quad (\text{S-46})$$

we obtain

$$\begin{aligned} h'_{3,k-q} \overline{\Delta}_{h,p-q} &= \begin{pmatrix} \text{Diag}\{h_{1,q+1}, \dots, h_{1,p-1}, 0\} \\ 0^{(k-p) \times (p-q)} \end{pmatrix} + h'_{3,k-q} (v_1, \dots, v_p) h_{2,p-q} \\ &\sim \begin{pmatrix} \text{Diag}\{h_{1,q+1}, \dots, h_{1,p-1}, 0\} \\ 0^{(k-p) \times (p-q)} \end{pmatrix} + (w_1, \dots, w_{p-q}), \end{aligned} \quad (\text{S-47})$$

where, by definition, w_j , $j = 1, \dots, p - q$ are i.i.d. normal $(k - q)$ -vectors with zero mean and covariance matrix I_{k-q} . The distributional equivalence in the second line holds because $(v_1, \dots, v_p)h_{2,p-q} \sim (\tilde{v}_1, \dots, \tilde{v}_{p-q})$, where \tilde{v}_j , $j = 1, \dots, p - q$ are i.i.d. $N(0^k, I_k)$ as $h_{2,p-q}$ has orthogonal columns of length 1. Analogously, $h'_{3,k-q}(\tilde{v}_1, \dots, \tilde{v}_{p-q}) \sim (w_1, \dots, w_{p-q})$ because $h_{3,k-q}$ has orthogonal columns of length 1.

E.g. when $q = p - 1 = m_W$ (which could be called the "strong IV" case), we obtain from (S-47) $h'_{3,k-q}\bar{\Delta}_{h,p-q} = w_1 \in \mathfrak{R}^{k-m_W}$. Therefore $\bar{\Delta}'_{h,p-q}h_{3,k-q}h'_{3,k-q}\bar{\Delta}_{h,p-q} \sim \chi_{k-m_W}^2$ and thus by part (b) of Proposition 2 the limiting distribution of the subvector AR statistic is $\chi_{k-m_W}^2$ in that case, while all the larger roots in (3.2) converge in probability to infinity by part (a).

Proof of Theorem 4. By construction, for $\alpha \in (0, 1)$, $c_{1-\alpha}(z, k - m_W)$ is an increasing continuous function in z on $(0, \infty)$, where $c_{1-\alpha}(z, k - m_W)$ is defined in (2.13) with $\hat{\kappa}_1$ replaced by z . Furthermore, $c_{1-\alpha}(z, k - m_W) \rightarrow \chi_{k-m_W, 1-\alpha}^2$ as $z \rightarrow \infty$. Thus, defining $c_{1-\alpha}(\infty, k - m_W) := \chi_{k-m_W, 1-\alpha}^2$, we can view $c_{1-\alpha}(z, k - m_W)$ as a continuous function in z on $(0, \infty]$. Finally, for $\alpha \in (0, 1)$ we have $P(\hat{\kappa}_p = c_{1-\alpha}(\hat{\kappa}_1, k - m_W)) = 0$ whenever $\hat{\kappa}_p$ and $\hat{\kappa}_1$ are the smallest and largest eigenvalues of the Wishart matrix $\Xi'\Xi \sim \mathcal{W}_p(k, I_p, \mathcal{M}'\mathcal{M})$ and any choice of eigenvalues $(\kappa_1, \dots, \kappa_{m_W}, 0)$ of $\mathcal{M}'\mathcal{M} \in \mathfrak{R}^{p \times p}$.

According to Proposition 1 in order to show that $AsySz \leq \alpha$ it is sufficient to establish that $RP^+(h) \leq \alpha$ for all $h \in H$, where $RP^+(h)$ appears in Assumption B in ACG. We therefore need to establish that for every drifting sequence $\{\lambda_{w_n, h} \in \Lambda : n \geq 1\}$ the null rejection probability of the conditional subvector AR test $RP_{w_n}(\lambda_{w_n, h})$ satisfies $RP_{w_n}(\lambda_{w_n, h}) \rightarrow [RP^-(h), RP^+(h)]$ for some $RP^+(h) \leq \alpha$. We also show that under strong IV sequences the limiting rejection probability equals α which then implies that the asymptotic size equals α . For notational simplicity we write n instead of w_n .

By the discussion below Proposition 2 when $q = p - 1 = m_W$, the strong IV case, $AR_n(\beta_0) \rightarrow_d \chi_{k-m_W}^2$ under $\{\lambda_{n, h} \in \Lambda : n \geq 1\}$ while the largest root $\hat{\kappa}_{1n}$ goes off to infinity in probability. Thus, by the definition of convergence in distribution and the features of $c_{1-\alpha}(z, k - m_W)$ described above

$$RP_n(\lambda_{n, h}) = P_{F_n}(AR_n(\beta_0) > c_{1-\alpha}(\hat{\kappa}_{1n}, k - m_W)) \rightarrow RP^+(h) = P(\chi_{k-m_W}^2 > \chi_{k-m_W, 1-\alpha}^2) = \alpha. \quad (\text{S-48})$$

When $0 < q < m_W$, then, just like above, the largest root $\hat{\kappa}_{1n}$ goes off to infinity in probability and $c_{1-\alpha}(\hat{\kappa}_{1n}, k - m_W) \rightarrow_p \chi_{k-m_W, 1-\alpha}^2$. By Proposition 2(b) the limiting distribution of $\hat{\kappa}_{pn} = AR_n(\beta_0)$ in (3.2) equals the distribution of the smallest eigenvalue, $\kappa(p - q)$ say, of $\bar{\Delta}'_{h,p-q}h_{3,k-q}h'_{3,k-q}\bar{\Delta}_{h,p-q} \in \mathfrak{R}^{p-q \times p-q}$, where $h'_{3,k-q}\bar{\Delta}_{h,p-q} = (\tilde{w}_1, \dots, \tilde{w}_{p-q})$, where $\tilde{w}_j \in \mathfrak{R}^{k-q}$ for $j = 1, \dots, p - q$ are independent $N(m_j, I_{k-q})$ with $m_j = (0^{j-1}, h_{1,q+j}, 0^{k-q-j})'$ for $j < p - q$ and $m_{p-q} = 0^{k-q}$, respectively. Therefore,

$$RP_n(\lambda_{n, h}) = P_{F_n}(AR_n(\beta_0) > c_{1-\alpha}(\hat{\kappa}_{1n}, k - m_W)) \rightarrow RP^+(h) = P(\kappa(p - q) > \chi_{k-m_W, 1-\alpha}^2), \quad (\text{S-49})$$

where the convergence holds by the features of $c_{1-\alpha}(z, k - m_W)$ described above. Consider a finite-sample scenario as in (2.9) in Section 2 with the roles of k, p, Ξ and \mathcal{M} played by $k - q, p - q, h'_{3,k-q}\bar{\Delta}_{h,p-q}$, and (m_1, \dots, m_{p-q}) , respectively. From the discussion below Theorem 3 we know that $P(\kappa(p - q) > c_{1-\alpha}(\kappa(1), k - m_W)) \leq \alpha$ for any choice of $\kappa(1) \geq 0$, where $\kappa(1)$ denotes the largest eigenvalue of $\bar{\Delta}'_{h,p-q}h_{3,k-q}h'_{3,k-q}\bar{\Delta}_{h,p-q}$. But given that $c_{1-\alpha}(\kappa(1), k - m_W)$ is increasing in $\kappa(1)$ and converges to $\chi_{k-m_W, 1-\alpha}^2$ as $\kappa(1) \rightarrow \infty$, it must also hold that $P(\kappa(p - q) > \chi_{k-m_W, 1-\alpha}^2) \leq \alpha$.

By Proposition 2(b) when $q = 0$, the limiting distribution of the two roots $(\hat{\kappa}_{1n}, AR_n(\beta_0))$ in (3.2) equals the distribution of the largest and smallest eigenvalues, $\kappa(1)$ and $\kappa(p)$ say, of $\bar{\Delta}'_{h,p}h_{3,k}h'_{3,k}\bar{\Delta}_{h,p} \in \mathfrak{R}^{p \times p}$, where $h'_{3,k}\bar{\Delta}_{h,p} = (\tilde{w}_1, \dots, \tilde{w}_p)$, where $\tilde{w}_j \in \mathfrak{R}^k$ for $j = 1, \dots, p$ are independent $N(m_j, I_k)$ with $m_j =$

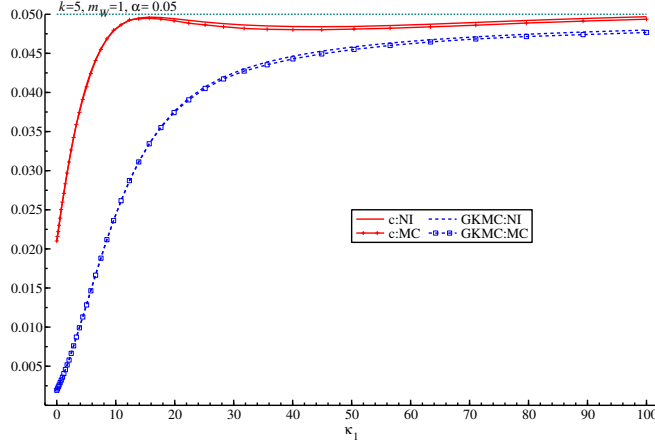


Figure S.1: Comparison of estimates of NRP obtained by numerical integration (NI) and Monte Carlo simulation (MC) with 1 million draws. NRP of 5% level conditional (2.15) (red solid) and GKMC subvector AR (blue dotted) tests as a function of the nuisance parameter κ_{m_W} . The number of instruments is $k = 5$ and the number of nuisance parameters is $m_W = 1$.

$(0^{j-1}, h_{1,j}, 0^{k-j})'$ for $j < p$ and $m_p = 0^k$, respectively. Consider a finite-sample scenario as in (2.9) in Section 2 with the roles of Ξ and \mathcal{M} played by $h'_{3,k} \bar{\Delta}_{h,p}$ and (m_1, \dots, m_p) , respectively. From the discussion below Theorem 3 we know that $P(\kappa(p) > c_{1-\alpha}(\kappa(1), k - m_W)) \leq \alpha$. Therefore,

$$RP_n(\lambda_{n,h}) = P_{F_n}(AR_n(\beta_0) > c_{1-\alpha}(\hat{\kappa}_{1n}, k - m_W)) \rightarrow RP^+(h) = P(\kappa(p) > c_{1-\alpha}(\kappa(1), k - m_W)) \leq \alpha, \quad (\text{S-50})$$

where the convergence holds again from the features of $c_{1-\alpha}(z, k - m_W)$ described above. \square

S.2 Computational details

S.2.1 Computation of the hypergeometric function

The function ${}_0F_1^{(2)}$ of two matrix arguments, which appears in the kernel of the density (S-20), involves an infinite series of Jack functions that converge very slowly and it is notoriously hard to compute accurately. We use the recently developed algorithm of Koev and Edelman (2006) which is efficient and fast. The algorithm approximates ${}_0F_1^{(2)}$ using a finite sum of terms M terms, so we need to choose M large enough for an accurate approximation. By extensive experimentation with different values of M up to 500, we found that $M = 200$ seems to be sufficiently large for all the cases we considered, because the results are unchanged when M is increased further. Hence, we used $M = 200$ in all calculations.

S.2.2 Size calculations

The computation of the NRP in Section 2 was conducted using numerical integration of the exact density (S-21). Their accuracy depends in part on the accuracy of the computation of ${}_0F_1^{(2)}$. To assess that, we compare in Figure S.1 the NRP computed using Monte Carlo integration with 1 million replications to the one reported in Figure 2. The results are essentially identical to 3 decimals.

Further results on the size of conditional subvector AR test are given in Section S.4.1.

S.2.3 Power bounds

In this section, we explain how we compute bounds to the power of the rank testing problem in Section 2.4 using the methods of (Andrews et al., 2008, Section 4.2) and Elliott et al. (2015) (henceforth AMS and EMW respectively). The testing problem is

$$H_0 : \kappa_2 = 0, \kappa_1 \geq 0 \text{ versus } H_1 : \kappa_2 > 0, \kappa_1 \geq \kappa_2,$$

where κ_1, κ_2 are the eigenvalues of the noncentrality parameter $\kappa_i(\mathcal{M}'\mathcal{M})$ of the 2×2 noncentral Wishart matrix $\Xi'\Xi \sim \mathcal{W}_2(k, I_2, \mathcal{M}'\mathcal{M})$, and $\hat{\kappa}_i = \kappa_i(\Xi'\Xi)$. The joint density of the eigenvalues $f_{\hat{\kappa}_1, \hat{\kappa}_2}(x_1, x_2; \kappa_1, \kappa_2)$ is given in (S-21).

All simulations in this section are performed using importance sampling. The parameter space for κ_1 under H_0 is discretized into $N_{\kappa_1} = 42$ points, in the same way as for the size calculations before, i.e., $\kappa_1 \in \{\kappa_{1,1}, \dots, \kappa_{1, N_{\kappa_1}}\}$, where $\kappa_{1,j}$ are equally spaced in log-scale between 0 and 100. We will compute point-optimal power bounds over a grid of point alternatives. Let F denote a distribution over H_1 , so that $H_{1,F} \in H_1$ is a point alternative, and let g denote the density of the data under $H_{1,F}$. For the power envelope, we consider one-point distributions F , whose support varies over the range $\kappa_2 \in [0.1, \bar{\kappa}_2(k)]$, discretized into 30 equally spaced points, and $\kappa_1 - \kappa_2 \in \{0, 1, 2, 4, 8, 16, 32, 64\}$. We do not consider greater values of $\kappa_1 - \kappa_2$ because the power curves of κ_2 are already indistinguishable at $\kappa_1 - \kappa_2 = 64$. The upper bound of κ_2 under H_1 , $\bar{\kappa}_2(k)$, is chosen to be about just high enough for the power of the conditional subvector AR test φ_c to be above 0.99, and it necessarily varies with k (larger values are needed for higher k). With some experimentation, we picked $\bar{\kappa}_2(2) = 25$, $\bar{\kappa}_2(5) = 30$, $\bar{\kappa}_2(10) = 38$, $\bar{\kappa}_2(20) = 46$. We index the density of the data under the alternative by $r = 1, \dots, N_r = 30 \times 8$, so that $g_r(\cdot) = f_{\hat{\kappa}_1, \hat{\kappa}_2}(\cdot; \kappa_{1,r}, \kappa_{2,r})$.

Let $\mathbf{x}_{i,j} \in \mathbb{R}^2$ denote a draw from $\mathcal{W}_2(k, I_2, \text{diag}(\kappa_{1,j}, 0))$. We draw N_0 simulations from each of N_{κ_1} data generating processes (DGPs). We abbreviate by $f_l(\mathbf{x}_{i,j})$ the joint density (S-21) at parameter l evaluated at the i th draw $\mathbf{x}_{i,j}$ from DGP j , i.e.,

$$\begin{aligned} f_l(\mathbf{x}_{i,j}) &= f_{\hat{\kappa}_1, \hat{\kappa}_2}(x_{1,i,j}, x_{2,i,j}; \kappa_{1,l}, 0), \\ x_{s,i,j} &= \kappa_s(X_i), \quad s = 1, 2 \quad X_i \sim \mathcal{W}_2(k, I_2, \text{diag}(\kappa_{1,j}, 0)), \\ l, j &= 1, \dots, N_{\kappa_1}, \quad i = 1, \dots, N_0. \end{aligned}$$

The rejection probability of any test $\varphi(\mathbf{x})$, under DGP j , $RP_j(\varphi)$, is computed by Monte Carlo integration with importance sampling using the formula

$$\widehat{RP}_j(\varphi) = \frac{1}{N_{\kappa_1} N_0} \sum_{l=1}^{N_{\kappa_1}} \sum_{i=1}^{N_0} \frac{f_j(\mathbf{x}_{i,l})}{\bar{f}(\mathbf{x}_{i,l})} \varphi(\mathbf{x}_{i,l}), \quad (\text{S-1})$$

where $\bar{f}(\cdot) = N_{\kappa_1}^{-1} \sum_{j=1}^{N_{\kappa_1}} f_j(\cdot)$.

Let Λ denote a distribution over the space of the nuisance parameter κ_1 , i.e., a distribution over H_0 . A point null hypothesis $H_{0,\Lambda} \in H_0$ is defined by the distribution $\int f_{\kappa_1} d\Lambda$, and is approximated here by $\sum_{l=1}^{N_{\kappa_1}} f_l(\cdot) w_{l,\Lambda}$, where $w_{l,\Lambda}$, $l = 1, \dots, N_{\kappa_1}$ are the weights over the (discretized) support of Λ . A least favorable distribution Λ^{LF} for testing H_0 against a particular point alternative $H_{1,F}$ (if it exists) is such that the α -level Neyman-Pearson test of $H_{0,\Lambda^{LF}}$ against $H_{1,F}$ has size α under H_0 .

The least favorable distribution Λ^{LF} is not known in this application. As shown in (Elliott et al., 2015,

Lemma 1), any Neyman-Pearson test φ_Λ of size α under $H_{0,\Lambda}$ will provide an upper bound on the power of tests of H_0 . But the power bound may be quite conservative in the sense that it could be far above the least upper bound. The procedures in AMS and EMW are designed to produce bounds that are close to the least upper bound obtained using Λ^{LF} . AMS consider one-point distributions Λ , and provide upper and lower bounds on the power envelope. The upper bound is obtained by looking for the (one-point) distribution Λ^* that gives the smallest size under H_0 , i.e., $\max_{\kappa_1} E_{\kappa_1}(\varphi_{\Lambda^*}) \leq \max_{\kappa_1} E_{\kappa_1}(\varphi_\Lambda)$ for all one-point distributions Λ , where $E_{\kappa_1}(\cdot)$ is expectation w.r.t. the Null distribution indexed by κ_1 . When the size of φ_{Λ^*} exceeds α this bound may be too high. We will report here only the upper bound of AMS, because it is close to, and often indistinguishable from, the bound obtained by the ALFD method of EMW.

S.2.3.1 AMS bound

The AMS algorithm for the upper bound on power, with a slight modification to do importance sampling, is as follows.

1. For each j , $j = 1, \dots, N_{\kappa_1}$, generate N_0 draws $\mathbf{x}_{i,j}$, $i = 1, \dots, N_0$ with density f_j . The draws must be independent across i and j .
2. Compute and store the importance sampling weights $\omega_{l,i,j} = f_l(\mathbf{x}_{i,j}) / \bar{f}(\mathbf{x}_{i,j})$, $l, j = 1, \dots, N_{\kappa_1}$, $i = 1, \dots, N_0$.
3. Set $r = 1$.
4. Compute $LR_l(\mathbf{x}_{i,j}) = g_r(\mathbf{x}_{i,j}) / f_l(\mathbf{x}_{i,j})$, $l, j = 1, \dots, N_{\kappa_1}$, $i = 1, \dots, N_0$.
5. Computation of cvs under H_0 : For each $l = 1, \dots, N_{\kappa_1}$, find \varkappa_l by solving $\widehat{RP}_l(\varphi_l) = \alpha$, where $\varphi_l := \mathbf{1}[LR_l > \varkappa_l]$ is the LR test of f_l against g_r with critical value \varkappa_l , and $\widehat{RP}_l(\varphi_l)$ is the Monte Carlo estimate (S-1) with the weights $\omega_{l,i,j}$ computed in step 2.
6. Computation of size of each test: For each $l, j = 1, \dots, N_{\kappa_1}$, compute $\widehat{RP}_j(\varphi_l)$, and obtain $D_l = \max_{j \in \{1, \dots, N_{\kappa_1}\}} [\widehat{RP}_j(\varphi_l) - \alpha]$.
7. Find test with size closest to α : $l^* = \arg \min_{l \in \{1, \dots, N_{\kappa_1}\}} D_l$.
8. Compute the AMS upper bound $\bar{\pi}_r = N_1^{-1} \sum_{i=1}^{N_1} \varphi_{l^*}(\mathbf{x}_i)$, where \mathbf{x}_i are i.i.d. draws of $(\hat{\kappa}_1, \hat{\kappa}_2)$ with density g_r .
9. If $r < N_r$, set $r = r + 1$ and go to step 4.

All the reported results are based on $N_0 = 10000$ and $N_1 = 100000$. (we can use a smaller number of simulations under H_0 for a similar level of precision due to importance sampling.)

Because the size of the test φ_{l^*} can exceed α , the AMS upper bound $\bar{\pi}$ may be higher than the least upper bound. To gauge this, Figure S.2 plots (Monte Carlo estimates of) the size of φ_{l^*} across the different alternatives r . Note that for most alternatives the size of the test φ_{l^*} is close to α , so $\bar{\pi}_r$ could be close to the least upper bound in those cases. However, for alternatives close to H_0 the size of φ_{l^*} deviates substantially from α , and the AMS upper bound $\bar{\pi}_r$ may be higher than the least upper bound. These are precisely the cases in which the conditional subvector AR test has the highest deviations from the power bound.

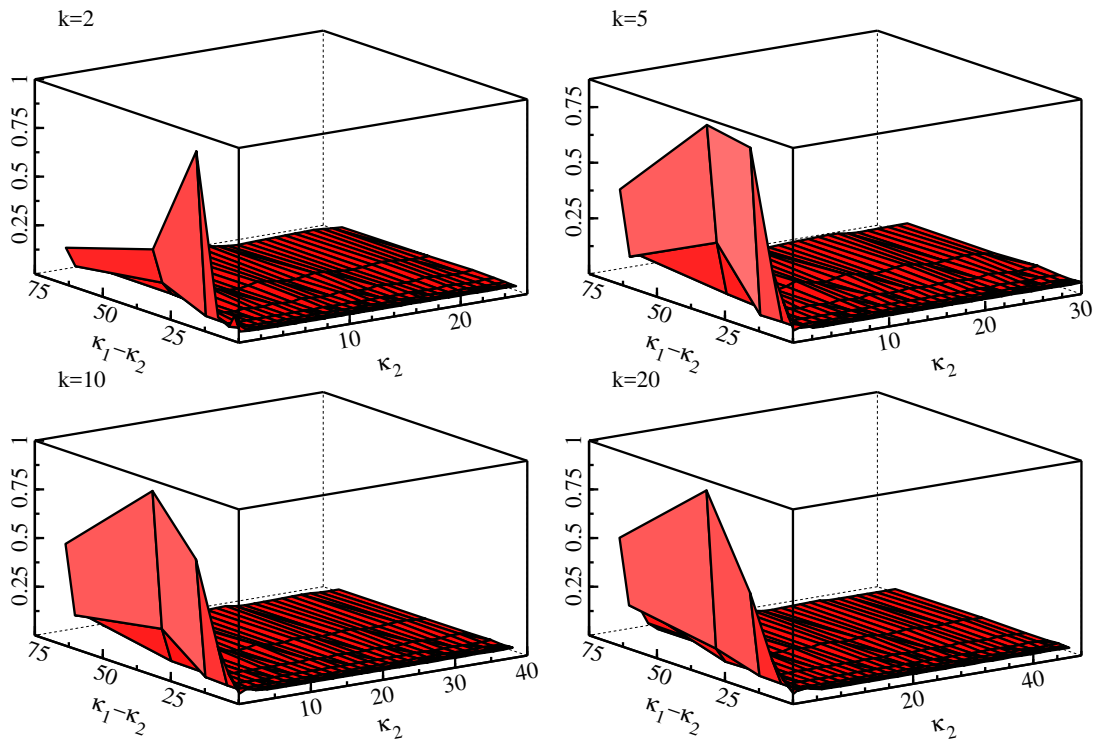


Figure S.2: Size of the AMS NP tests (step 6 of the algorithm in Section S.2.3.1) for each point under H_1 , $\kappa_2 \in [0.1, \bar{\kappa}_2(k)]$, $\bar{\kappa}_2(2) = 25$, $\bar{\kappa}_2(5) = 30$, $\bar{\kappa}_2(10) = 38$, $\bar{\kappa}_2(20) = 46$, discretized into 30 equally spaced points, and $\kappa_1 - \kappa_2 \in \{0, 1, 2, 4, 8, 16, 32, 64\}$. Calculated over 42 points of the nuisance parameter under H_0 , using 10000 Monte Carlo replications with importance sampling.

S.2.3.2 EMW bound

The PO power bound reported in Figure 3 is based on the ALFD approach of EMW. The ALFD is designed to produce tests that are at most ε away from the true (unknown) power envelope. We apply the algorithm with a slight modification to allow ε to vary across alternatives – for some alternatives we may be able to get closer to the least upper bound than for others.

We use the following modified version of the algorithm in (Elliott et al., 2015, Appendix A.2) without switching, assuming the parameter space for the nuisance parameter is compact. The modification relates to steps 6 to 8 of the original algorithm and cannot underestimate the true power bound.

1. For each j , $j = 1, \dots, N_{\kappa_1}$, generate N_0 draws $\mathbf{x}_{i,j}$, $i = 1, \dots, N_0$ with density f_j . The draws must be independent across i and j .
2. Compute and store $f_l(\mathbf{x}_{i,j})$ and $\bar{f}(\mathbf{x}_{i,j})$, $l, j = 1, \dots, N_{\kappa_1}$, $i = 1, \dots, N_0$.
3. Set $r = 1$.
4. Set $\mu^{(0)} = (-2, \dots, -2) \in \mathfrak{R}^{N_{\kappa_1}}$.
5. Compute $\mu^{(s+1)}$ from $\mu^{(s)}$ via $\mu_j^{(s+1)} = \mu_j^{(s)} + \omega \left(\widehat{RP}_j(\varphi^{(s)}) - \alpha \right)$ and $\omega = 2$, where $\varphi^{(s)} := \mathbf{1}[g_r > \sum_{i=1}^{N_{\kappa_1}} \exp(\mu_i^{(s)}) f_i]$ and $\widehat{RP}_l(\varphi^{(s)})$ is the Monte Carlo estimate (S-1) with weights computed in step 2, and repeat this step $O = 600$ times. Denote the resulting element in the simplex by $\hat{\Lambda}^* = (\hat{\lambda}_1, \dots, \hat{\lambda}_{N_{\kappa_1}})$, where $\hat{\lambda}_j = \exp(\mu_j^{(O)}) / \sum_{i=1}^{N_{\kappa_1}} \exp(\mu_i^{(O)})$.
6. Compute the number \varkappa^* such that the test $\varphi_{\hat{\Lambda}^*} := \mathbf{1}[g_r > \varkappa^* \sum_{i=1}^{N_{\kappa_1}} \hat{\lambda}_i^* f_i]$ is exactly of (Monte Carlo) level α when \mathbf{x} is drawn with density $\sum_{i=1}^{N_{\kappa_1}} \hat{\lambda}_i^* f_i$, that is, solve $\sum_{j=1}^{N_{\kappa_1}} \hat{\lambda}_j^* \left(\widehat{RP}_j(\varphi_{\hat{\Lambda}^*}) - \alpha \right) = 0$.
7. Compute the estimate of the power bound $\bar{\pi}_r = N_1^{-1} \sum_{i=1}^{N_1} \varphi_{\hat{\Lambda}^*}(\mathbf{x}_i)$, where \mathbf{x}_i are i.i.d. draws of $(\hat{\kappa}_1, \hat{\kappa}_2)$ with density g_r .
8. Compute the number \varkappa such that the test $\tilde{\varphi}_{\hat{\Lambda}^*} = \mathbf{1}[g_r > \varkappa \sum_{i=1}^{N_{\kappa_1}} \hat{\lambda}_i^* f_i]$ is exactly of (Monte Carlo) level α when \mathbf{x} is drawn with density f_i , $i = 1, \dots, N_{\kappa_1}$, that is, solve $\max_{j \in \{1, \dots, N_{\kappa_1}\}} \left(\widehat{RP}_j(\tilde{\varphi}_{\hat{\Lambda}^*}) - \alpha \right) = 0$.
9. Compute another estimate of the power bound $\tilde{\pi}_r = N_1^{-1} \sum_{i=1}^{N_1} \tilde{\varphi}_{\hat{\Lambda}^*}(\mathbf{x}_i)$, where \mathbf{x}_i are the i.i.d. draws in step 7, and $\varepsilon_r = \bar{\pi}_r - \tilde{\pi}_r$.
10. If $r < N_r$, set $r = r + 1$ and go to step 4.

All the reported results are based on $N_0 = 10000$ and $N_1 = 100000$. (we can use a smaller number of simulations under H_0 for a similar level of precision due to importance sampling.)

Up to step 7, the algorithm is identical to EMW (Appendix A.2.1). The difference is in step 7, which replaces steps 6 to 8 of the original algorithm. The number ε_r is an estimate of the maximum distance of the power bound $\tilde{\pi}_r$ from the unknown least upper bound. $\tilde{\pi}_r$ is the PO power bound used in Figure 3. Figure S.3 plots ε_r across all alternatives. In most cases ε_r is equal to zero to 3 decimals, indicating that the ALFD upper bound is essentially least favorable. The only exceptions are for a handful of alternatives very close to the null. Hence, the ALFD upper bound is arguably a good approximation of the PO power envelope.

The bound $\bar{\pi}_r$ (which is obtained from step 5 in the original EMW algorithm) can also serve as an upper bound on the power, similar to the AMS bound in the previous section. The only difference is the use of a

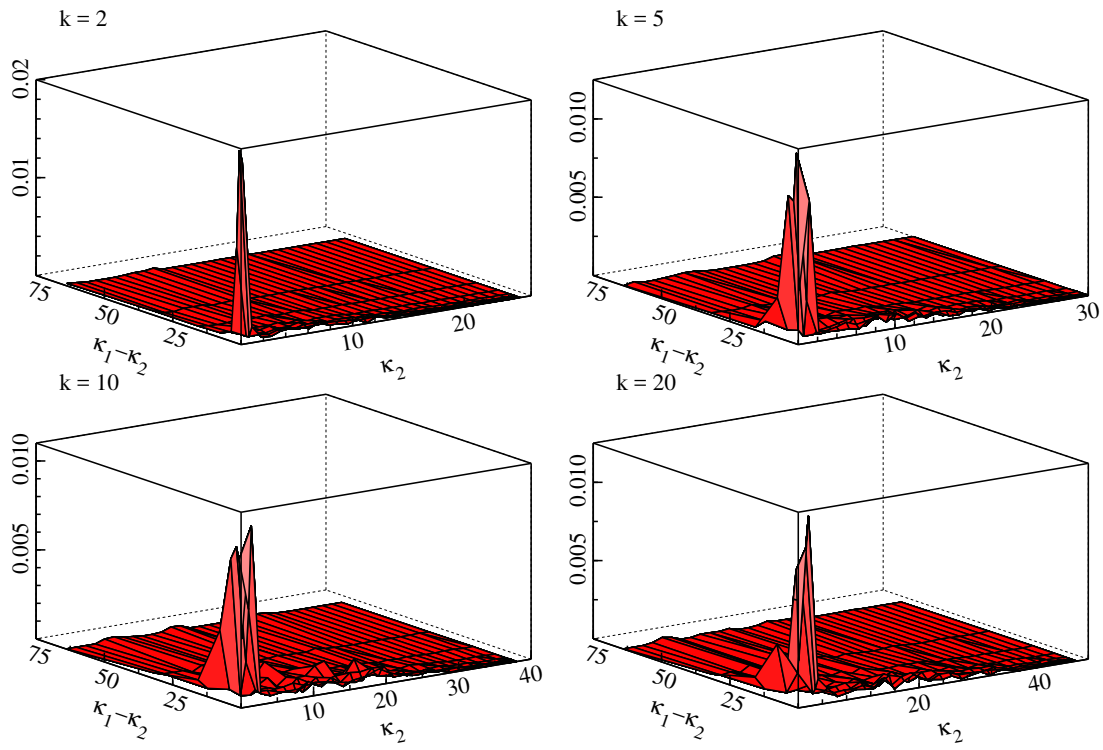


Figure S.3: Estimates of the maximum distance of the ALFD power bound from the least favorable bound, $\varepsilon = \bar{\pi} - \tilde{\pi}$ (step 9 of Algorithm in Section S.2.3.2), for each point under H_1 , $\kappa_2 \in [0.1, \bar{\kappa}_2(k)]$, $\bar{\kappa}_2(2) = 25$, $\bar{\kappa}_2(5) = 30$, $\bar{\kappa}_2(10) = 38$, $\bar{\kappa}_2(20) = 46$, discretized into 30 equally spaced points, and $\kappa_1 - \kappa_2 \in \{0, 1, 2, 4, 8, 16, 32, 64\}$. Calculated over 42 points of the nuisance parameter under H_0 , using 10000 Monte Carlo replications with importance sampling.

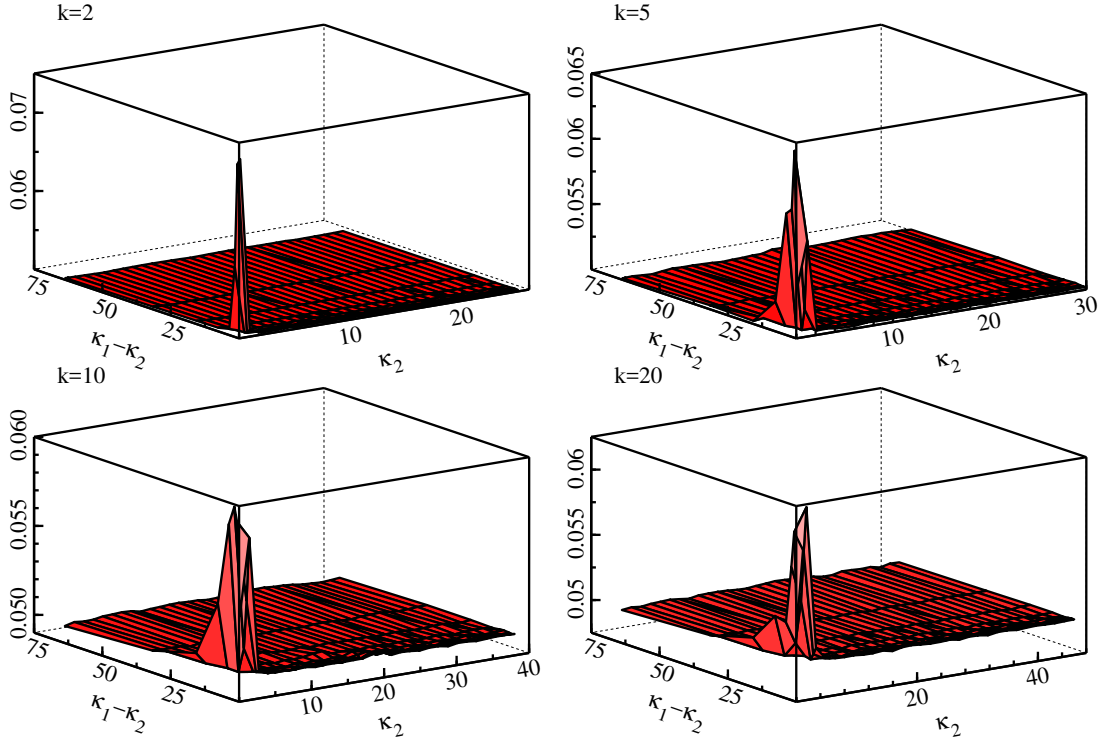


Figure S.4: Size of the tests $\varphi_{\hat{\Lambda}^*}$ (step 6 of the algorithm in Section S.2.3.2) for each point under H_1 , $\kappa_2 \in [0.1, \bar{\kappa}_2(k)]$, $\bar{\kappa}_2(2) = 25$, $\bar{\kappa}_2(5) = 30$, $\bar{\kappa}_2(10) = 38$, $\bar{\kappa}_2(20) = 46$, discretized into 30 equally spaced points, and $\kappa_1 - \kappa_2 \in \{0, 1, 2, 4, 8, 16, 32, 64\}$. Calculated over 42 points of the nuisance parameter under H_0 , using 10000 Monte Carlo replications with importance sampling.

distribution Λ^* with full support on the discretized H_0 , as opposed to a one-point candidate least favorable distribution in AMS. But as for AMS, $\bar{\pi}_r$ can be far from the least upper bound if $\varphi_{\hat{\Lambda}^*}$ is oversized under H_0 . To gauge this, Figure S.4 plots (Monte Carlo estimates of) the size of $\varphi_{\hat{\Lambda}^*}$ across the different alternatives r . The figure is directly comparable to Figure S.2 for the AMS algorithm in the previous subsection. Compared to AMS, the EMW procedure has size much closer to α across most (but not all) alternatives.

Let $\bar{\pi}_r^{AMS}$ and $\bar{\pi}_r^{EMW}$ denote the power bounds obtained from the AMS and EMW algorithms, respectively. Since they are both upper bounds to the true PO power envelope, so is their minimum, $\bar{\pi}_r^{\min} = \min(\bar{\pi}_r^{AMS}, \bar{\pi}_r^{EMW})$. We can therefore use $\bar{\pi}_r^{\min}$ as a possibly tighter upper bound on the power envelope.

S.3 Tables of critical values

10%, 5% and 1% conditional critical values $c_{1-\alpha}(\hat{\kappa}_1, k - m_W)$ were computed by numerically integrating the density (2.12) at different values of the conditioning variable $\hat{\kappa}_1$ for the cases $k - m_W = 1, \dots, 20$. The results are reported in Tables S.1 to S.20.

$k - m_W = 1$													
$\alpha = 10\%$													
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
0.3	0.2	0.9	0.6	1.7	1.0	2.7	1.4	4.0	1.8	6.4	2.2	15.2	2.6
0.4	0.3	1.1	0.7	1.9	1.1	3.0	1.5	4.5	1.9	7.4	2.3	27.6	2.7
0.5	0.4	1.3	0.8	2.1	1.2	3.3	1.6	5.0	2.0	8.8	2.4	1000	2.703
0.7	0.5	1.5	0.9	2.4	1.3	3.6	1.7	5.6	2.1	11.0	2.5	∞	2.706
$\alpha = 5\%$													
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
0.5	0.4	1.3	1.0	2.3	1.6	3.6	2.2	5.5	2.8	9.8	3.4	∞	3.841
0.6	0.5	1.5	1.1	2.5	1.7	3.9	2.3	6.0	2.9	11.4	3.5		
0.7	0.6	1.6	1.2	2.7	1.8	4.1	2.4	6.5	3.0	13.9	3.6		
0.9	0.7	1.8	1.3	2.9	1.9	4.4	2.5	7.0	3.1	18.5	3.7		
1.0	0.8	2.0	1.4	3.1	2.0	4.8	2.6	7.8	3.2	29.7	3.8		
1.2	0.9	2.1	1.5	3.4	2.1	5.1	2.7	8.6	3.3	1000	3.838		
$\alpha = 1\%$													
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
1.0	0.9	2.0	1.8	3.2	2.7	4.5	3.6	6.2	4.5	9.0	5.4	19.3	6.3
1.1	1.0	2.1	1.9	3.3	2.8	4.7	3.7	6.5	4.6	9.5	5.5	23.8	6.4
1.2	1.1	2.2	2.0	3.4	2.9	4.8	3.8	6.7	4.7	10.0	5.6	32.2	6.5
1.3	1.2	2.4	2.1	3.6	3.0	5.0	3.9	7.0	4.8	10.6	5.7	53.1	6.6
1.4	1.3	2.5	2.2	3.7	3.1	5.2	4.0	7.2	4.9	11.3	5.8	1000	6.628
1.5	1.4	2.6	2.3	3.9	3.2	5.4	4.1	7.5	5.0	12.2	5.9	∞	6.635
1.6	1.5	2.8	2.4	4.0	3.3	5.6	4.2	7.9	5.1	13.3	6.0		
1.8	1.6	2.9	2.5	4.2	3.4	5.8	4.3	8.2	5.2	14.7	6.1		
1.9	1.7	3.0	2.6	4.3	3.5	6.0	4.4	8.6	5.3	16.6	6.2		

Table S.1: $1 - \alpha$ quantile of the conditional distribution, with density given in (2.12), $cv=c_{1-\alpha}(\hat{\kappa}_1, k - m_W)$ at different values of the conditioning variable $\hat{\kappa}_1$. Computed by numerical integration.

$k - m_W = 2$													
$\alpha = 10\%$													
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
0.5	0.4	1.4	1.1	2.5	1.8	3.9	2.5	5.7	3.2	9.2	3.9	47.2	4.6
0.6	0.5	1.6	1.2	2.7	1.9	4.1	2.6	6.1	3.3	10.1	4.0	1000	4.601
0.7	0.6	1.7	1.3	2.9	2.0	4.3	2.7	6.5	3.4	11.2	4.1	∞	4.605
0.8	0.7	1.9	1.4	3.1	2.1	4.6	2.8	6.9	3.5	12.7	4.2		
1.0	0.8	2.0	1.5	3.3	2.2	4.8	2.9	7.3	3.6	15.0	4.3		
1.1	0.9	2.2	1.6	3.5	2.3	5.1	3.0	7.9	3.7	18.6	4.4		
1.3	1.0	2.4	1.7	3.7	2.4	5.4	3.1	8.5	3.8	25.9	4.5		
$\alpha = 5\%$													
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
0.7	0.6	1.6	1.4	2.7	2.2	4.0	3.0	5.5	3.8	7.8	4.6	13.0	5.4
0.8	0.7	1.8	1.5	2.9	2.3	4.2	3.1	5.8	3.9	8.2	4.7	14.5	5.5
0.9	0.8	1.9	1.6	3.0	2.4	4.3	3.2	6.0	4.0	8.6	4.8	16.5	5.6
1.0	0.9	2.0	1.7	3.2	2.5	4.5	3.3	6.3	4.1	9.1	4.9	19.5	5.7
1.1	1.0	2.2	1.8	3.3	2.6	4.7	3.4	6.5	4.2	9.7	5.0	24.7	5.8
1.3	1.1	2.3	1.9	3.5	2.7	4.9	3.5	6.8	4.3	10.3	5.1	35.4	5.9
1.4	1.2	2.4	2.0	3.6	2.8	5.1	3.6	7.1	4.4	11.0	5.2	1000	5.985
1.5	1.3	2.6	2.1	3.8	2.9	5.3	3.7	7.5	4.5	11.9	5.3	∞	5.991
$\alpha = 1\%$													
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
1.6	1.5	2.9	2.7	4.6	4.1	6.5	5.5	9.2	6.9	15.3	8.3	∞	9.210
1.7	1.6	3.1	2.9	4.8	4.3	6.9	5.7	9.7	7.1	17.5	8.5		
1.8	1.7	3.3	3.1	5.1	4.5	7.2	5.9	10.3	7.3	21.1	8.7		
2.0	1.9	3.6	3.3	5.4	4.7	7.5	6.1	11.0	7.5	28.3	8.9		
2.2	2.1	3.8	3.5	5.6	4.9	7.9	6.3	11.7	7.7	49.5	9.1		
2.4	2.3	4.1	3.7	5.9	5.1	8.3	6.5	12.6	7.9	89.0	9.2		
2.7	2.5	4.3	3.9	6.2	5.3	8.7	6.7	13.8	8.1	1000	9.201		

Table S.2: $1 - \alpha$ quantile of the conditional distribution, with density given in (2.12), $cv = c_{1-\alpha}(\hat{\kappa}_1, k - m_W)$ at different values of the conditioning variable $\hat{\kappa}_1$. Computed by numerical integration.

$k - m_W = 3$													
$\alpha = 10\%$													
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
0.6	0.5	1.7	1.4	2.9	2.3	4.4	3.2	6.2	4.1	9.1	5.0	18.8	5.9
0.7	0.6	1.8	1.5	3.1	2.4	4.6	3.3	6.5	4.2	9.6	5.1	22.6	6.0
0.8	0.7	1.9	1.6	3.2	2.5	4.7	3.4	6.8	4.3	10.2	5.2	29.6	6.1
0.9	0.8	2.1	1.7	3.4	2.6	4.9	3.5	7.0	4.4	10.8	5.3	46.0	6.2
1.0	0.9	2.2	1.8	3.5	2.7	5.1	3.6	7.3	4.5	11.5	5.4	1000	6.245
1.2	1.0	2.3	1.9	3.7	2.8	5.3	3.7	7.6	4.6	12.3	5.5	∞	6.251
1.3	1.1	2.5	2.0	3.9	2.9	5.6	3.8	8.0	4.7	13.3	5.6		
1.4	1.2	2.6	2.1	4.0	3.0	5.8	3.9	8.3	4.8	14.6	5.7		
1.5	1.3	2.8	2.2	4.2	3.1	6.0	4.0	8.7	4.9	16.3	5.8		
$\alpha = 5\%$													
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
0.9	0.8	2.1	1.9	3.5	3.0	5.1	4.1	7.1	5.2	10.2	6.3	20.9	7.4
1.0	0.9	2.3	2.0	3.7	3.1	5.3	4.2	7.4	5.3	10.6	6.4	24.5	7.5
1.1	1.0	2.4	2.1	3.8	3.2	5.5	4.3	7.6	5.4	11.1	6.5	30.4	7.6
1.2	1.1	2.5	2.2	3.9	3.3	5.6	4.4	7.8	5.5	11.6	6.6	41.9	7.7
1.3	1.2	2.6	2.3	4.1	3.4	5.8	4.5	8.1	5.6	12.1	6.7	73.6	7.8
1.4	1.3	2.7	2.4	4.2	3.5	6.0	4.6	8.3	5.7	12.8	6.8	1000	7.807
1.5	1.4	2.9	2.5	4.4	3.6	6.2	4.7	8.6	5.8	13.5	6.9	∞	7.815
1.6	1.5	3.0	2.6	4.5	3.7	6.3	4.8	8.9	5.9	14.4	7.0		
1.8	1.6	3.1	2.7	4.7	3.8	6.5	4.9	9.2	6.0	15.4	7.1		
1.9	1.7	3.3	2.8	4.8	3.9	6.7	5.0	9.5	6.1	16.7	7.2		
2.0	1.8	3.4	2.9	5.0	4.0	6.9	5.1	9.8	6.2	18.5	7.3		
$\alpha = 1\%$													
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
2.2	2.1	3.7	3.5	5.5	5.1	7.6	6.7	10.3	8.3	15.1	9.9	1000	11.334
2.3	2.2	3.9	3.7	5.8	5.3	7.9	6.9	10.7	8.5	16.3	10.1	∞	11.345
2.4	2.3	4.1	3.9	6.0	5.5	8.2	7.1	11.2	8.7	17.7	10.3		
2.6	2.5	4.4	4.1	6.3	5.7	8.5	7.3	11.6	8.9	19.8	10.5		
2.8	2.7	4.6	4.3	6.5	5.9	8.8	7.5	12.2	9.1	22.9	10.7		
3.0	2.9	4.8	4.5	6.8	6.1	9.2	7.7	12.8	9.3	28.3	10.9		
3.2	3.1	5.0	4.7	7.1	6.3	9.5	7.9	13.4	9.5	40.3	11.1		
3.5	3.3	5.3	4.9	7.3	6.5	9.9	8.1	14.2	9.7	85.4	11.3		

Table S.3: $1 - \alpha$ quantile of the conditional distribution, with density given in (2.12), $cv=c_{1-\alpha}(\hat{\kappa}_1, k - m_W)$ at different values of the conditioning variable $\hat{\kappa}_1$. Computed by numerical integration.

$k - m_W = 4$													
$\alpha = 10\%$													
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
0.8	0.7	2.1	1.8	3.5	2.9	5.2	4.0	7.2	5.1	10.3	6.2	19.7	7.3
0.9	0.8	2.2	1.9	3.7	3.0	5.3	4.1	7.4	5.2	10.7	6.3	22.4	7.4
1.0	0.9	2.3	2.0	3.8	3.1	5.5	4.2	7.7	5.3	11.1	6.4	26.6	7.5
1.1	1.0	2.5	2.1	3.9	3.2	5.7	4.3	7.9	5.4	11.6	6.5	33.9	7.6
1.2	1.1	2.6	2.2	4.1	3.3	5.8	4.4	8.2	5.5	12.1	6.6	49.3	7.7
1.3	1.2	2.7	2.3	4.2	3.4	6.0	4.5	8.4	5.6	12.7	6.7	1000	7.772
1.5	1.3	2.8	2.4	4.4	3.5	6.2	4.6	8.7	5.7	13.4	6.8	∞	7.779
1.6	1.4	3.0	2.5	4.5	3.6	6.4	4.7	9.0	5.8	14.2	6.9		
1.7	1.5	3.1	2.6	4.7	3.7	6.6	4.8	9.3	5.9	15.1	7.0		
1.8	1.6	3.2	2.7	4.8	3.8	6.8	4.9	9.6	6.0	16.3	7.1		
1.9	1.7	3.4	2.8	5.0	3.9	7.0	5.0	9.9	6.1	17.7	7.2		
$\alpha = 5\%$													
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
1.2	1.1	2.5	2.3	4.2	3.7	6.2	5.1	8.6	6.5	12.5	7.9	39.9	9.3
1.3	1.2	2.7	2.5	4.5	3.9	6.5	5.3	9.0	6.7	13.4	8.1	57.4	9.4
1.4	1.3	3.0	2.7	4.7	4.1	6.8	5.5	9.4	6.9	14.5	8.3	1000	9.478
1.6	1.5	3.2	2.9	5.0	4.3	7.1	5.7	9.9	7.1	15.9	8.5	∞	9.488
1.8	1.7	3.5	3.1	5.3	4.5	7.4	5.9	10.5	7.3	17.9	8.7		
2.1	1.9	3.7	3.3	5.6	4.7	7.8	6.1	11.1	7.5	20.9	8.9		
2.3	2.1	4.0	3.5	5.9	4.9	8.2	6.3	11.7	7.7	26.5	9.1		
$\alpha = 1\%$													
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
2.7	2.6	4.4	4.2	6.4	6.0	8.7	7.8	11.4	9.6	16.0	11.4	83.7	13.2
2.8	2.7	4.6	4.4	6.6	6.2	8.9	8.0	11.8	9.8	16.8	11.6	1000	13.264
2.9	2.8	4.8	4.6	6.9	6.4	9.2	8.2	12.2	10.0	17.8	11.8	∞	13.277
3.1	3.0	5.0	4.8	7.1	6.6	9.5	8.4	12.6	10.2	19.1	12.0		
3.3	3.2	5.3	5.0	7.4	6.8	9.8	8.6	13.0	10.4	20.7	12.2		
3.5	3.4	5.5	5.2	7.6	7.0	10.1	8.8	13.5	10.6	22.9	12.4		
3.7	3.6	5.7	5.4	7.9	7.2	10.4	9.0	14.0	10.8	26.3	12.6		
3.9	3.8	5.9	5.6	8.1	7.4	10.7	9.2	14.6	11.0	32.0	12.8		
4.1	4.0	6.2	5.8	8.4	7.6	11.1	9.4	15.2	11.2	44.1	13.0		

Table S.4: $1 - \alpha$ quantile of the conditional distribution, with density given in (2.12), $cv = c_{1-\alpha}(\hat{\kappa}_1, k - m_W)$ at different values of the conditioning variable $\hat{\kappa}_1$. Computed by numerical integration.

$k - m_W = 5$													
$\alpha = 10\%$													
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
0.9	0.8	2.2	2.0	4.0	3.4	6.0	4.8	8.4	6.2	12.4	7.6	34.8	9.0
1.0	0.9	2.5	2.2	4.3	3.6	6.3	5.0	8.9	6.4	13.2	7.8	74.9	9.2
1.1	1.0	2.7	2.4	4.5	3.8	6.6	5.2	9.3	6.6	14.3	8.0	1000	9.227
1.3	1.2	3.0	2.6	4.8	4.0	6.9	5.4	9.8	6.8	15.6	8.2	∞	9.236
1.5	1.4	3.2	2.8	5.1	4.2	7.3	5.6	10.4	7.0	17.4	8.4		
1.8	1.6	3.5	3.0	5.4	4.4	7.7	5.8	11.0	7.2	20.1	8.6		
2.0	1.8	3.7	3.2	5.7	4.6	8.0	6.0	11.6	7.4	24.6	8.8		
$\alpha = 5\%$													
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
1.4	1.3	2.9	2.7	4.8	4.3	6.9	5.9	9.5	7.5	13.4	9.1	31.8	10.7
1.5	1.4	3.1	2.9	5.0	4.5	7.2	6.1	9.8	7.7	14.1	9.3	49.1	10.9
1.6	1.5	3.4	3.1	5.3	4.7	7.5	6.3	10.2	7.9	15.0	9.5	73.0	11.0
1.8	1.7	3.6	3.3	5.6	4.9	7.8	6.5	10.7	8.1	16.0	9.7	1000	11.060
2.0	1.9	3.8	3.5	5.8	5.1	8.1	6.7	11.1	8.3	17.3	9.9	∞	11.070
2.2	2.1	4.1	3.7	6.1	5.3	8.4	6.9	11.6	8.5	18.9	10.1		
2.5	2.3	4.3	3.9	6.3	5.5	8.8	7.1	12.1	8.7	21.2	10.3		
2.7	2.5	4.5	4.1	6.6	5.7	9.1	7.3	12.7	8.9	24.9	10.5		
$\alpha = 1\%$													
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
3.2	3.1	5.1	4.9	7.3	6.9	9.7	8.9	12.7	10.9	17.3	12.9	62.8	14.9
3.3	3.2	5.3	5.1	7.5	7.1	10.0	9.1	13.0	11.1	18.0	13.1	90.8	15.0
3.4	3.3	5.5	5.3	7.7	7.3	10.2	9.3	13.4	11.3	18.8	13.3	1000	15.072
3.6	3.5	5.7	5.5	8.0	7.5	10.5	9.5	13.7	11.5	19.8	13.5	∞	15.086
3.8	3.7	5.9	5.7	8.2	7.7	10.8	9.7	14.1	11.7	21.1	13.7		
4.0	3.9	6.2	5.9	8.5	7.9	11.1	9.9	14.6	11.9	22.6	13.9		
4.2	4.1	6.4	6.1	8.7	8.1	11.4	10.1	15.0	12.1	24.7	14.1		
4.4	4.3	6.6	6.3	9.0	8.3	11.7	10.3	15.5	12.3	27.7	14.3		
4.6	4.5	6.8	6.5	9.2	8.5	12.0	10.5	16.0	12.5	32.5	14.5		
4.8	4.7	7.1	6.7	9.5	8.7	12.3	10.7	16.6	12.7	41.4	14.7		

Table S.5: $1 - \alpha$ quantile of the conditional distribution, with density given in (2.12), $cv = c_{1-\alpha}(\hat{\kappa}_1, k - m_W)$ at different values of the conditioning variable $\hat{\kappa}_1$. Computed by numerical integration.

$k - m_W = 6$													
$\alpha = 10\%$													
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
1.1	1.0	2.6	2.4	4.6	4.0	6.8	5.6	9.5	7.2	13.7	8.8	39.4	10.4
1.2	1.1	2.9	2.6	4.9	4.2	7.1	5.8	9.9	7.4	14.5	9.0	81.9	10.6
1.3	1.2	3.1	2.8	5.1	4.4	7.4	6.0	10.3	7.6	15.5	9.2	1000	10.634
1.5	1.4	3.4	3.0	5.4	4.6	7.7	6.2	10.8	7.8	16.6	9.4	∞	10.645
1.7	1.6	3.6	3.2	5.7	4.8	8.0	6.4	11.3	8.0	18.1	9.6		
2.0	1.8	3.8	3.4	5.9	5.0	8.4	6.6	11.8	8.2	20.1	9.8		
2.2	2.0	4.1	3.6	6.2	5.2	8.7	6.8	12.4	8.4	23.0	10.0		
2.4	2.2	4.3	3.8	6.5	5.4	9.1	7.0	13.0	8.6	28.2	10.2		
$\alpha = 5\%$													
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
1.6	1.5	3.3	3.1	5.4	4.9	7.7	6.7	10.4	8.5	14.4	10.3	30.9	12.1
1.7	1.6	3.5	3.3	5.6	5.1	7.9	6.9	10.8	8.7	15.1	10.5	41.6	12.3
1.8	1.7	3.7	3.5	5.9	5.3	8.2	7.1	11.1	8.9	15.8	10.7	75.0	12.5
2.0	1.9	4.0	3.7	6.1	5.5	8.5	7.3	11.5	9.1	16.7	10.9	1000	12.579
2.2	2.1	4.2	3.9	6.4	5.7	8.8	7.5	11.9	9.3	17.7	11.1	∞	12.592
2.4	2.3	4.4	4.1	6.6	5.9	9.1	7.7	12.4	9.5	18.9	11.3		
2.6	2.5	4.7	4.3	6.9	6.1	9.4	7.9	12.8	9.7	20.4	11.5		
2.9	2.7	4.9	4.5	7.1	6.3	9.7	8.1	13.3	9.9	22.5	11.7		
3.1	2.9	5.1	4.7	7.4	6.5	10.1	8.3	13.9	10.1	25.7	11.9		
$\alpha = 1\%$													
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
3.7	3.6	5.5	5.4	7.8	7.5	10.3	9.6	13.1	11.7	17.1	13.8	28.7	15.9
3.8	3.7	5.9	5.7	8.2	7.8	10.7	9.9	13.6	12.0	17.9	14.1	35.5	16.2
4.0	3.9	6.2	6.0	8.5	8.1	11.1	10.2	14.1	12.3	18.8	14.4	52.4	16.5
4.3	4.2	6.5	6.3	8.9	8.4	11.5	10.5	14.6	12.6	19.8	14.7	159.8	16.8
4.6	4.5	6.8	6.6	9.2	8.7	11.9	10.8	15.2	12.9	21.1	15.0	1000	16.796
4.9	4.8	7.2	6.9	9.6	9.0	12.3	11.1	15.7	13.2	22.8	15.3	∞	16.812
5.2	5.1	7.5	7.2	9.9	9.3	12.7	11.4	16.4	13.5	25.1	15.6		

Table S.6: $1 - \alpha$ quantile of the conditional distribution, with density given in (2.12), $cv = c_{1-\alpha}(\hat{\kappa}_1, k - m_W)$ at different values of the conditioning variable $\hat{\kappa}_1$. Computed by numerical integration.

$k - m_W = 7$													
$\alpha = 10\%$													
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
1.2	1.1	2.9	2.7	5.1	4.5	7.5	6.3	10.3	8.1	14.7	9.9	38.5	11.7
1.3	1.2	3.2	2.9	5.3	4.7	7.7	6.5	10.7	8.3	15.5	10.1	65.0	11.9
1.4	1.3	3.4	3.1	5.6	4.9	8.0	6.7	11.1	8.5	16.3	10.3	112.1	12.0
1.6	1.5	3.6	3.3	5.8	5.1	8.3	6.9	11.5	8.7	17.3	10.5	1000	12.005
1.8	1.7	3.9	3.5	6.1	5.3	8.7	7.1	12.0	8.9	18.4	10.7	∞	12.017
2.0	1.9	4.1	3.7	6.4	5.5	9.0	7.3	12.4	9.1	19.9	10.9		
2.3	2.1	4.3	3.9	6.6	5.7	9.3	7.5	12.9	9.3	21.8	11.1		
2.5	2.3	4.6	4.1	6.9	5.9	9.6	7.7	13.5	9.5	24.7	11.3		
2.7	2.5	4.8	4.3	7.2	6.1	10.0	7.9	14.1	9.7	29.3	11.5		
$\alpha = 5\%$													
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
1.9	1.8	3.8	3.6	6.1	5.6	8.6	7.6	11.5	9.6	15.9	11.6	35.6	13.6
2.0	1.9	4.0	3.8	6.3	5.8	8.9	7.8	11.9	9.8	16.5	11.8	49.0	13.8
2.1	2.0	4.3	4.0	6.6	6.0	9.1	8.0	12.2	10.0	17.2	12.0	94.6	14.0
2.3	2.2	4.5	4.2	6.8	6.2	9.4	8.2	12.6	10.2	18.0	12.2	1000	14.053
2.5	2.4	4.7	4.4	7.1	6.4	9.7	8.4	13.0	10.4	19.0	12.4	∞	14.067
2.7	2.6	4.9	4.6	7.3	6.6	10.0	8.6	13.4	10.6	20.1	12.6		
2.9	2.8	5.2	4.8	7.5	6.8	10.3	8.8	13.9	10.8	21.4	12.8		
3.1	3.0	5.4	5.0	7.8	7.0	10.6	9.0	14.3	11.0	23.2	13.0		
3.4	3.2	5.6	5.2	8.1	7.2	10.9	9.2	14.8	11.2	25.6	13.2		
3.6	3.4	5.9	5.4	8.3	7.4	11.2	9.4	15.3	11.4	29.3	13.4		
$\alpha = 1\%$													
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
4.2	4.1	6.4	6.2	9.0	8.6	11.8	11.0	15.1	13.4	20.1	15.8	62.0	18.2
4.3	4.2	6.7	6.5	9.3	8.9	12.1	11.3	15.6	13.7	21.0	16.1	117.1	18.4
4.5	4.4	7.0	6.8	9.6	9.2	12.5	11.6	16.1	14.0	22.2	16.4	1000	18.459
4.8	4.7	7.3	7.1	10.0	9.5	12.9	11.9	16.6	14.3	23.6	16.7	∞	18.475
5.1	5.0	7.6	7.4	10.3	9.8	13.3	12.2	17.2	14.6	25.5	17.0		
5.4	5.3	8.0	7.7	10.7	10.1	13.7	12.5	17.8	14.9	28.1	17.3		
5.7	5.6	8.3	8.0	11.0	10.4	14.2	12.8	18.5	15.2	32.4	17.6		
6.0	5.9	8.6	8.3	11.4	10.7	14.6	13.1	19.2	15.5	40.5	17.9		

Table S.7: $1 - \alpha$ quantile of the conditional distribution, with density given in (2.12), $cv=c_{1-\alpha}(\hat{\kappa}_1, k - m_W)$ at different values of the conditioning variable $\hat{\kappa}_1$. Computed by numerical integration.

$k - m_W = 8$													
$\alpha = 10\%$													
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
1.4	1.3	3.3	3.1	5.7	5.1	8.3	7.1	11.4	9.1	16.1	11.1	47.8	13.1
1.5	1.4	3.6	3.3	5.9	5.3	8.6	7.3	11.8	9.3	16.9	11.3	93.3	13.3
1.6	1.5	3.8	3.5	6.2	5.5	8.8	7.5	12.2	9.5	17.7	11.5	1000	13.348
1.8	1.7	4.0	3.7	6.4	5.7	9.1	7.7	12.6	9.7	18.6	11.7	∞	13.362
2.0	1.9	4.3	3.9	6.7	5.9	9.4	7.9	13.0	9.9	19.7	11.9		
2.2	2.1	4.5	4.1	6.9	6.1	9.7	8.1	13.4	10.1	21.0	12.1		
2.4	2.3	4.7	4.3	7.2	6.3	10.1	8.3	13.9	10.3	22.7	12.3		
2.7	2.5	5.0	4.5	7.5	6.5	10.4	8.5	14.4	10.5	25.1	12.5		
2.9	2.7	5.2	4.7	7.7	6.7	10.7	8.7	14.9	10.7	28.7	12.7		
3.1	2.9	5.4	4.9	8.0	6.9	11.1	8.9	15.5	10.9	34.8	12.9		
$\alpha = 5\%$													
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
2.1	2.0	4.2	4.0	6.7	6.2	9.4	8.4	12.5	10.6	17.1	12.8	37.6	15.0
2.2	2.1	4.4	4.2	6.9	6.4	9.6	8.6	12.9	10.8	17.7	13.0	50.0	15.2
2.3	2.2	4.6	4.4	7.1	6.6	9.9	8.8	13.2	11.0	18.4	13.2	86.5	15.4
2.5	2.4	4.9	4.6	7.4	6.8	10.2	9.0	13.6	11.2	19.1	13.4	155.6	15.5
2.7	2.6	5.1	4.8	7.6	7.0	10.4	9.2	13.9	11.4	19.9	13.6	1000	15.492
2.9	2.8	5.3	5.0	7.9	7.2	10.7	9.4	14.3	11.6	20.9	13.8	∞	15.507
3.1	3.0	5.5	5.2	8.1	7.4	11.0	9.6	14.7	11.8	22.0	14.0		
3.3	3.2	5.8	5.4	8.4	7.6	11.3	9.8	15.1	12.0	23.4	14.2		
3.5	3.4	6.0	5.6	8.6	7.8	11.6	10.0	15.6	12.2	25.2	14.4		
3.8	3.6	6.2	5.8	8.9	8.0	11.9	10.2	16.1	12.4	27.7	14.6		
4.0	3.8	6.4	6.0	9.1	8.2	12.2	10.4	16.6	12.6	31.4	14.8		
$\alpha = 1\%$													
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
4.7	4.6	6.8	6.7	9.4	9.1	12.2	11.5	15.2	13.9	19.3	16.3	28.5	18.7
4.8	4.7	7.2	7.0	9.7	9.4	12.5	11.8	15.7	14.2	19.9	16.6	31.8	19.0
5.0	4.9	7.5	7.3	10.1	9.7	12.9	12.1	16.1	14.5	20.6	16.9	37.2	19.3
5.3	5.2	7.8	7.6	10.4	10.0	13.2	12.4	16.6	14.8	21.4	17.2	48.4	19.6
5.6	5.5	8.1	7.9	10.8	10.3	13.6	12.7	17.0	15.1	22.4	17.5	82.9	19.9
5.9	5.8	8.4	8.2	11.1	10.6	14.0	13.0	17.6	15.4	23.4	17.8	118.4	20.0
6.2	6.1	8.8	8.5	11.4	10.9	14.4	13.3	18.1	15.7	24.7	18.1	1000	20.073
6.5	6.4	9.1	8.8	11.8	11.2	14.8	13.6	18.7	16.0	26.3	18.4	∞	20.090

Table S.8: $1 - \alpha$ quantile of the conditional distribution, with density given in (2.12), $cv = c_{1-\alpha}(\hat{\kappa}_1, k - m_W)$ at different values of the conditioning variable $\hat{\kappa}_1$. Computed by numerical integration.

$k - m_W = 9$													
$\alpha = 10\%$													
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
1.5	1.4	3.4	3.2	5.7	5.2	8.2	7.2	11.0	9.2	14.6	11.2	21.6	13.2
1.6	1.5	3.6	3.4	5.9	5.4	8.4	7.4	11.3	9.4	15.1	11.4	23.0	13.4
1.7	1.6	3.9	3.6	6.2	5.6	8.7	7.6	11.6	9.6	15.6	11.6	24.8	13.6
1.9	1.8	4.1	3.8	6.4	5.8	9.0	7.8	11.9	9.8	16.1	11.8	27.3	13.8
2.1	2.0	4.3	4.0	6.7	6.0	9.2	8.0	12.3	10.0	16.6	12.0	31.0	14.0
2.3	2.2	4.5	4.2	6.9	6.2	9.5	8.2	12.6	10.2	17.3	12.2	37.3	14.2
2.5	2.4	4.8	4.4	7.2	6.4	9.8	8.4	13.0	10.4	17.9	12.4	50.3	14.4
2.8	2.6	5.0	4.6	7.4	6.6	10.1	8.6	13.4	10.6	18.7	12.6	91.8	14.6
3.0	2.8	5.2	4.8	7.7	6.8	10.4	8.8	13.8	10.8	19.5	12.8	1000	14.669
3.2	3.0	5.5	5.0	7.9	7.0	10.7	9.0	14.2	11.0	20.5	13.0	∞	14.684
$\alpha = 5\%$													
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
2.3	2.2	4.5	4.3	7.2	6.7	10.0	9.1	13.4	11.5	18.1	13.9	36.9	16.3
2.4	2.3	4.8	4.6	7.5	7.0	10.4	9.4	13.8	11.8	18.9	14.2	53.5	16.6
2.6	2.5	5.1	4.9	7.8	7.3	10.8	9.7	14.3	12.1	19.9	14.5	154.4	16.9
2.9	2.8	5.5	5.2	8.2	7.6	11.2	10.0	14.9	12.4	21.0	14.8	1000	16.903
3.2	3.1	5.8	5.5	8.6	7.9	11.6	10.3	15.4	12.7	22.4	15.1	∞	16.919
3.5	3.4	6.1	5.8	8.9	8.2	12.0	10.6	16.0	13.0	24.1	15.4		
3.8	3.7	6.5	6.1	9.3	8.5	12.5	10.9	16.6	13.3	26.5	15.7		
4.2	4.0	6.8	6.4	9.7	8.8	12.9	11.2	17.3	13.6	30.2	16.0		
$\alpha = 1\%$													
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
5.2	5.1	7.6	7.5	10.5	10.2	13.6	12.9	17.1	15.6	22.1	18.3	44.1	21.0
5.3	5.2	8.0	7.8	10.9	10.5	14.0	13.2	17.6	15.9	22.9	18.6	61.9	21.3
5.5	5.4	8.3	8.1	11.2	10.8	14.4	13.5	18.1	16.2	23.8	18.9	143.4	21.6
5.8	5.7	8.6	8.4	11.5	11.1	14.7	13.8	18.5	16.5	24.8	19.2	1000	21.647
6.1	6.0	8.9	8.7	11.9	11.4	15.1	14.1	19.0	16.8	26.0	19.5	∞	21.666
6.4	6.3	9.2	9.0	12.2	11.7	15.5	14.4	19.6	17.1	27.5	19.8		
6.7	6.6	9.6	9.3	12.6	12.0	15.9	14.7	20.1	17.4	29.4	20.1		
7.0	6.9	9.9	9.6	12.9	12.3	16.3	15.0	20.7	17.7	32.2	20.4		
7.3	7.2	10.2	9.9	13.3	12.6	16.7	15.3	21.4	18.0	36.4	20.7		

Table S.9: $1 - \alpha$ quantile of the conditional distribution, with density given in (2.12), $cv=c_{1-\alpha}(\hat{\kappa}_1, k - m_W)$ at different values of the conditioning variable $\hat{\kappa}_1$. Computed by numerical integration.

$k - m_W = 10$													
$\alpha = 10\%$													
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
1.7	1.6	3.9	3.7	6.7	6.1	9.7	8.5	13.2	10.9	18.4	13.3	54.4	15.7
1.8	1.7	4.3	4.0	7.0	6.4	10.1	8.8	13.7	11.2	19.4	13.6	98.3	15.9
2.0	1.9	4.6	4.3	7.4	6.7	10.5	9.1	14.2	11.5	20.5	13.9	1000	15.971
2.3	2.2	4.9	4.6	7.8	7.0	10.9	9.4	14.8	11.8	21.9	14.2	∞	15.987
2.6	2.5	5.3	4.9	8.1	7.3	11.3	9.7	15.4	12.1	23.6	14.5		
2.9	2.8	5.6	5.2	8.5	7.6	11.8	10.0	16.1	12.4	26.0	14.8		
3.3	3.1	6.0	5.5	8.9	7.9	12.2	10.3	16.8	12.7	29.7	15.1		
3.6	3.4	6.3	5.8	9.3	8.2	12.7	10.6	17.5	13.0	36.6	15.4		
$\alpha = 5\%$													
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
2.5	2.4	5.0	4.8	8.0	7.5	11.2	10.2	15.0	12.9	20.8	15.6	183.9	18.3
2.6	2.5	5.3	5.1	8.3	7.8	11.6	10.5	15.5	13.2	21.8	15.9	1000	18.289
2.8	2.7	5.6	5.4	8.7	8.1	12.0	10.8	16.0	13.5	23.0	16.2	∞	18.307
3.1	3.0	6.0	5.7	9.0	8.4	12.4	11.1	16.6	13.8	24.4	16.5		
3.4	3.3	6.3	6.0	9.4	8.7	12.8	11.4	17.1	14.1	26.3	16.8		
3.7	3.6	6.6	6.3	9.7	9.0	13.2	11.7	17.7	14.4	28.9	17.1		
4.0	3.9	7.0	6.6	10.1	9.3	13.6	12.0	18.4	14.7	32.9	17.4		
4.4	4.2	7.3	6.9	10.5	9.6	14.1	12.3	19.1	15.0	40.4	17.7		
4.7	4.5	7.6	7.2	10.8	9.9	14.5	12.6	19.9	15.3	59.3	18.0		
$\alpha = 1\%$													
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
5.7	5.6	8.1	8.0	11.0	10.7	14.0	13.4	17.4	16.1	21.6	18.8	30.6	21.5
5.8	5.7	8.5	8.3	11.3	11.0	14.4	13.7	17.8	16.4	22.2	19.1	33.0	21.8
6.0	5.9	8.8	8.6	11.7	11.3	14.7	14.0	18.2	16.7	22.9	19.4	36.7	22.1
6.3	6.2	9.1	8.9	12.0	11.6	15.1	14.3	18.6	17.0	23.6	19.7	42.7	22.4
6.6	6.5	9.4	9.2	12.3	11.9	15.5	14.6	19.1	17.3	24.4	20.0	54.9	22.7
6.9	6.8	9.7	9.5	12.7	12.2	15.8	14.9	19.6	17.6	25.2	20.3	90.9	23.0
7.2	7.1	10.0	9.8	13.0	12.5	16.2	15.2	20.0	17.9	26.2	20.6	221.2	23.2
7.5	7.4	10.4	10.1	13.3	12.8	16.6	15.5	20.5	18.2	27.4	20.9	1000	23.190
7.8	7.7	10.7	10.4	13.7	13.1	17.0	15.8	21.1	18.5	28.8	21.2	∞	23.209

Table S.10: $1 - \alpha$ quantile of the conditional distribution, with density given in (2.12), $cv=c_{1-\alpha}(\hat{\kappa}_1, k - m_W)$ at different values of the conditioning variable $\hat{\kappa}_1$. Computed by numerical integration.

$k - m_W = 11$													
$\alpha = 10\%$													
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
1.8	1.7	4.0	3.8	6.7	6.2	9.6	8.6	12.8	11.0	17.1	13.4	25.7	15.8
1.9	1.8	4.3	4.1	7.1	6.5	10.0	8.9	13.3	11.3	17.7	13.7	28.3	16.1
2.1	2.0	4.7	4.4	7.4	6.8	10.3	9.2	13.8	11.6	18.5	14.0	32.4	16.4
2.4	2.3	5.0	4.7	7.8	7.1	10.7	9.5	14.2	11.9	19.3	14.3	40.1	16.7
2.7	2.6	5.3	5.0	8.1	7.4	11.1	9.8	14.8	12.2	20.2	14.6	60.3	17.0
3.0	2.9	5.7	5.3	8.5	7.7	11.6	10.1	15.3	12.5	21.2	14.9	112.6	17.2
3.4	3.2	6.0	5.6	8.8	8.0	12.0	10.4	15.8	12.8	22.4	15.2	1000	17.258
3.7	3.5	6.4	5.9	9.2	8.3	12.4	10.7	16.4	13.1	23.9	15.5	∞	17.275
$\alpha = 5\%$													
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
2.8	2.7	5.3	5.1	8.2	7.8	11.4	10.5	14.9	13.2	19.7	15.9	32.8	18.6
2.9	2.8	5.6	5.4	8.6	8.1	11.7	10.8	15.4	13.5	20.4	16.2	38.3	18.9
3.1	3.0	5.9	5.7	8.9	8.4	12.1	11.1	15.8	13.8	21.2	16.5	49.8	19.2
3.4	3.3	6.3	6.0	9.3	8.7	12.5	11.4	16.3	14.1	22.0	16.8	86.7	19.5
3.7	3.6	6.6	6.3	9.6	9.0	12.9	11.7	16.8	14.4	23.0	17.1	127.1	19.6
4.0	3.9	6.9	6.6	10.0	9.3	13.3	12.0	17.3	14.7	24.1	17.4	1000	19.656
4.3	4.2	7.2	6.9	10.3	9.6	13.7	12.3	17.9	15.0	25.5	17.7	∞	19.675
4.6	4.5	7.6	7.2	10.7	9.9	14.1	12.6	18.4	15.3	27.2	18.0		
5.0	4.8	7.9	7.5	11.0	10.2	14.5	12.9	19.1	15.6	29.4	18.3		
$\alpha = 1\%$													
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
6.1	6.0	8.9	8.8	12.3	12.0	16.0	15.2	20.1	18.4	26.5	21.6	208.0	24.7
6.2	6.1	9.4	9.2	12.8	12.4	16.4	15.6	20.8	18.8	27.7	22.0	1000	24.705
6.5	6.4	9.8	9.6	13.2	12.8	16.9	16.0	21.4	19.2	29.3	22.4	∞	24.725
6.9	6.8	10.2	10.0	13.7	13.2	17.4	16.4	22.1	19.6	31.3	22.8		
7.3	7.2	10.6	10.4	14.1	13.6	17.9	16.8	22.8	20.0	34.2	23.2		
7.7	7.6	11.1	10.8	14.6	14.0	18.5	17.2	23.6	20.4	38.9	23.6		
8.1	8.0	11.5	11.2	15.0	14.4	19.0	17.6	24.4	20.8	48.3	24.0		
8.5	8.4	11.9	11.6	15.5	14.8	19.6	18.0	25.4	21.2	75.7	24.4		

Table S.11: $1 - \alpha$ quantile of the conditional distribution, with density given in (2.12), $cv=c_{1-\alpha}(\hat{\kappa}_1, k - m_W)$ at different values of the conditioning variable $\hat{\kappa}_1$. Computed by numerical integration.

$k - m_W = 12$													
$\alpha = 10\%$													
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
2.0	1.9	4.5	4.3	7.5	7.0	10.8	9.7	14.5	12.4	19.7	15.1	37.6	17.8
2.1	2.0	4.9	4.6	7.9	7.3	11.2	10.0	15.0	12.7	20.4	15.4	49.3	18.1
2.3	2.2	5.2	4.9	8.2	7.6	11.6	10.3	15.5	13.0	21.3	15.7	89.6	18.4
2.6	2.5	5.5	5.2	8.6	7.9	12.0	10.6	16.0	13.3	22.3	16.0	139.0	18.5
2.9	2.8	5.9	5.5	9.0	8.2	12.4	10.9	16.5	13.6	23.4	16.3	1000	18.531
3.2	3.1	6.2	5.8	9.3	8.5	12.8	11.2	17.1	13.9	24.8	16.6	∞	18.549
3.6	3.4	6.5	6.1	9.7	8.8	13.2	11.5	17.7	14.2	26.5	16.9		
3.9	3.7	6.9	6.4	10.0	9.1	13.6	11.8	18.3	14.5	28.8	17.2		
4.2	4.0	7.2	6.7	10.4	9.4	14.1	12.1	18.9	14.8	32.0	17.5		
$\alpha = 5\%$													
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
3.0	2.9	5.8	5.6	9.1	8.6	12.5	11.6	16.6	14.6	22.2	17.6	56.7	20.6
3.1	3.0	6.1	5.9	9.4	8.9	12.9	11.9	17.0	14.9	23.0	17.9	109.1	20.9
3.3	3.2	6.4	6.2	9.7	9.2	13.3	12.2	17.5	15.2	23.9	18.2	181.6	21.0
3.6	3.5	6.8	6.5	10.1	9.5	13.7	12.5	18.0	15.5	25.0	18.5	1000	21.006
3.9	3.8	7.1	6.8	10.4	9.8	14.1	12.8	18.5	15.8	26.2	18.8	∞	21.026
4.2	4.1	7.4	7.1	10.8	10.1	14.5	13.1	19.0	16.1	27.6	19.1		
4.5	4.4	7.7	7.4	11.1	10.4	14.9	13.4	19.6	16.4	29.5	19.4		
4.8	4.7	8.1	7.7	11.5	10.7	15.3	13.7	20.2	16.7	32.0	19.7		
5.2	5.0	8.4	8.0	11.8	11.0	15.7	14.0	20.8	17.0	35.8	20.0		
5.5	5.3	8.7	8.3	12.2	11.3	16.1	14.3	21.5	17.3	42.4	20.3		
$\alpha = 1\%$													
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
6.6	6.5	9.4	9.3	12.8	12.5	16.4	15.7	20.4	18.9	25.7	22.1	45.5	25.3
6.7	6.6	9.9	9.7	13.2	12.9	16.8	16.1	20.9	19.3	26.7	22.5	61.7	25.7
7.0	6.9	10.3	10.1	13.7	13.3	17.3	16.5	21.5	19.7	27.7	22.9	136.8	26.1
7.4	7.3	10.7	10.5	14.1	13.7	17.8	16.9	22.1	20.1	28.9	23.3	232.0	26.2
7.8	7.7	11.1	10.9	14.6	14.1	18.3	17.3	22.7	20.5	30.4	23.7	1000	26.197
8.2	8.1	11.5	11.3	15.0	14.5	18.8	17.7	23.4	20.9	32.2	24.1	∞	26.217
8.6	8.5	12.0	11.7	15.5	14.9	19.3	18.1	24.1	21.3	34.8	24.5		
9.0	8.9	12.4	12.1	15.9	15.3	19.8	18.5	24.9	21.7	38.6	24.9		

Table S.12: $1 - \alpha$ quantile of the conditional distribution, with density given in (2.12), $cv = c_{1-\alpha}(\hat{\kappa}_1, k - m_W)$ at different values of the conditioning variable $\hat{\kappa}_1$. Computed by numerical integration.

$k - m_W = 13$													
$\alpha = 10\%$													
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
2.1	2.0	4.9	4.7	8.3	7.7	11.9	10.7	16.1	13.7	22.2	16.7	109.7	19.7
2.2	2.1	5.3	5.0	8.6	8.0	12.2	11.0	16.5	14.0	23.1	17.0	192.2	19.8
2.4	2.3	5.6	5.3	9.0	8.3	12.6	11.3	17.0	14.3	24.1	17.3	1000	19.793
2.7	2.6	5.9	5.6	9.3	8.6	13.0	11.6	17.6	14.6	25.4	17.6	∞	19.812
3.0	2.9	6.3	5.9	9.7	8.9	13.4	11.9	18.1	14.9	26.8	17.9		
3.3	3.2	6.6	6.2	10.0	9.2	13.8	12.2	18.7	15.2	28.7	18.2		
3.6	3.5	6.9	6.5	10.4	9.5	14.3	12.5	19.3	15.5	31.1	18.5		
4.0	3.8	7.3	6.8	10.7	9.8	14.7	12.8	19.9	15.8	34.8	18.8		
4.3	4.1	7.6	7.1	11.1	10.1	15.1	13.1	20.6	16.1	41.2	19.1		
4.6	4.4	7.9	7.4	11.5	10.4	15.6	13.4	21.4	16.4	55.3	19.4		
$\alpha = 5\%$													
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
3.2	3.1	6.0	5.8	9.2	8.8	12.6	11.8	16.4	14.8	21.3	17.8	32.0	20.8
3.3	3.2	6.3	6.1	9.5	9.1	13.0	12.1	16.8	15.1	21.9	18.1	34.9	21.1
3.5	3.4	6.6	6.4	9.9	9.4	13.3	12.4	17.3	15.4	22.6	18.4	39.3	21.4
3.8	3.7	6.9	6.7	10.2	9.7	13.7	12.7	17.7	15.7	23.3	18.7	47.4	21.7
4.1	4.0	7.3	7.0	10.6	10.0	14.1	13.0	18.2	16.0	24.0	19.0	66.2	22.0
4.4	4.3	7.6	7.3	10.9	10.3	14.5	13.3	18.6	16.3	24.9	19.3	154.5	22.3
4.7	4.6	7.9	7.6	11.2	10.6	14.8	13.6	19.1	16.6	25.9	19.6	1000	22.341
5.0	4.9	8.2	7.9	11.6	10.9	15.2	13.9	19.6	16.9	27.0	19.9	∞	22.362
5.3	5.2	8.6	8.2	11.9	11.2	15.6	14.2	20.2	17.2	28.3	20.2		
5.7	5.5	8.9	8.5	12.3	11.5	16.0	14.5	20.7	17.5	29.9	20.5		
$\alpha = 1\%$													
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
7.1	7.0	9.9	9.8	13.3	13.0	16.8	16.2	20.7	19.4	25.5	22.6	35.6	25.8
7.2	7.1	10.3	10.2	13.7	13.4	17.3	16.6	21.2	19.8	26.2	23.0	38.9	26.2
7.5	7.4	10.8	10.6	14.2	13.8	17.7	17.0	21.7	20.2	27.1	23.4	44.4	26.6
7.9	7.8	11.2	11.0	14.6	14.2	18.2	17.4	22.3	20.6	28.0	23.8	55.8	27.0
8.3	8.2	11.6	11.4	15.0	14.6	18.7	17.8	22.9	21.0	29.0	24.2	90.7	27.4
8.7	8.6	12.0	11.8	15.5	15.0	19.2	18.2	23.5	21.4	30.2	24.6	162.1	27.6
9.1	9.0	12.4	12.2	15.9	15.4	19.6	18.6	24.1	21.8	31.6	25.0	1000	27.668
9.5	9.4	12.9	12.6	16.4	15.8	20.1	19.0	24.8	22.2	33.3	25.4	∞	27.688

Table S.13: $1 - \alpha$ quantile of the conditional distribution, with density given in (2.12), $cv = c_{1-\alpha}(\hat{\kappa}_1, k - m_W)$ at different values of the conditioning variable $\hat{\kappa}_1$. Computed by numerical integration.

$k - m_W = 14$													
$\alpha = 10\%$													
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
2.3	2.2	5.1	4.9	8.4	7.9	11.9	10.9	15.9	13.9	21.1	16.9	34.9	19.9
2.4	2.3	5.4	5.2	8.8	8.2	12.3	11.2	16.4	14.2	21.8	17.2	39.9	20.2
2.6	2.5	5.8	5.5	9.1	8.5	12.7	11.5	16.8	14.5	22.6	17.5	49.5	20.5
2.9	2.8	6.1	5.8	9.5	8.8	13.1	11.8	17.3	14.8	23.4	17.8	75.4	20.8
3.2	3.1	6.4	6.1	9.8	9.1	13.5	12.1	17.8	15.1	24.3	18.1	145.2	21.0
3.5	3.4	6.8	6.4	10.1	9.4	13.8	12.4	18.3	15.4	25.3	18.4	1000	21.044
3.8	3.7	7.1	6.7	10.5	9.7	14.2	12.7	18.8	15.7	26.4	18.7	∞	21.064
4.2	4.0	7.4	7.0	10.9	10.0	14.6	13.0	19.3	16.0	27.8	19.0		
4.5	4.3	7.8	7.3	11.2	10.3	15.1	13.3	19.9	16.3	29.5	19.3		
4.8	4.6	8.1	7.6	11.6	10.6	15.5	13.6	20.5	16.6	31.7	19.6		
$\alpha = 5\%$													
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
3.4	3.3	6.5	6.3	10.0	9.6	13.8	12.9	18.0	16.2	23.7	19.5	43.4	22.8
3.5	3.4	6.8	6.6	10.4	9.9	14.1	13.2	18.4	16.5	24.3	19.8	53.7	23.1
3.7	3.6	7.1	6.9	10.7	10.2	14.5	13.5	18.9	16.8	25.1	20.1	80.3	23.4
4.0	3.9	7.4	7.2	11.0	10.5	14.9	13.8	19.3	17.1	25.9	20.4	145.9	23.6
4.3	4.2	7.8	7.5	11.4	10.8	15.2	14.1	19.8	17.4	26.8	20.7	1000	23.663
4.6	4.5	8.1	7.8	11.7	11.1	15.6	14.4	20.3	17.7	27.9	21.0	∞	23.685
4.9	4.8	8.4	8.1	12.0	11.4	16.0	14.7	20.8	18.0	29.1	21.3		
5.2	5.1	8.7	8.4	12.4	11.7	16.4	15.0	21.3	18.3	30.5	21.6		
5.5	5.4	9.1	8.7	12.7	12.0	16.8	15.3	21.8	18.6	32.3	21.9		
5.9	5.7	9.4	9.0	13.1	12.3	17.2	15.6	22.4	18.9	34.6	22.2		
6.2	6.0	9.7	9.3	13.4	12.6	17.6	15.9	23.0	19.2	38.0	22.5		
$\alpha = 1\%$													
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
7.6	7.5	10.8	10.7	14.6	14.3	18.6	17.9	23.1	21.5	29.3	25.1	75.0	28.7
7.7	7.6	11.3	11.1	15.1	14.7	19.1	18.3	23.6	21.9	30.3	25.5	216.0	29.1
8.0	7.9	11.7	11.5	15.5	15.1	19.5	18.7	24.2	22.3	31.5	25.9	1000	29.120
8.4	8.3	12.1	11.9	15.9	15.5	20.0	19.1	24.8	22.7	32.8	26.3	∞	29.141
8.8	8.7	12.5	12.3	16.4	15.9	20.5	19.5	25.5	23.1	34.4	26.7		
9.2	9.1	12.9	12.7	16.8	16.3	21.0	19.9	26.1	23.5	36.6	27.1		
9.6	9.5	13.4	13.1	17.2	16.7	21.5	20.3	26.8	23.9	39.5	27.5		
10.0	9.9	13.8	13.5	17.7	17.1	22.0	20.7	27.6	24.3	44.2	27.9		
10.4	10.3	14.2	13.9	18.2	17.5	22.5	21.1	28.4	24.7	53.0	28.3		

Table S.14: $1 - \alpha$ quantile of the conditional distribution, with density given in (2.12), $cv = c_{1-\alpha}(\hat{\kappa}_1, k - m_W)$ at different values of the conditioning variable $\hat{\kappa}_1$. Computed by numerical integration.

$k - m_W = 15$													
$\alpha = 10\%$													
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
2.4	2.3	5.5	5.3	9.2	8.6	13.0	11.9	17.4	15.2	23.5	18.5	55.6	21.8
2.5	2.4	5.9	5.6	9.5	8.9	13.4	12.2	17.9	15.5	24.3	18.8	91.2	22.1
2.7	2.6	6.2	5.9	9.8	9.2	13.8	12.5	18.3	15.8	25.1	19.1	224.8	22.3
3.0	2.9	6.5	6.2	10.2	9.5	14.1	12.8	18.8	16.1	26.1	19.4	1000	22.286
3.3	3.2	6.8	6.5	10.5	9.8	14.5	13.1	19.3	16.4	27.1	19.7	∞	22.307
3.6	3.5	7.2	6.8	10.9	10.1	14.9	13.4	19.8	16.7	28.4	20.0		
3.9	3.8	7.5	7.1	11.2	10.4	15.3	13.7	20.4	17.0	29.9	20.3		
4.2	4.1	7.8	7.4	11.6	10.7	15.7	14.0	20.9	17.3	31.7	20.6		
4.6	4.4	8.1	7.7	11.9	11.0	16.1	14.3	21.5	17.6	34.2	20.9		
4.9	4.7	8.5	8.0	12.3	11.3	16.6	14.6	22.1	17.9	37.8	21.2		
5.2	5.0	8.8	8.3	12.6	11.6	17.0	14.9	22.8	18.2	43.7	21.5		
$\alpha = 5\%$													
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
3.7	3.6	7.0	6.8	10.9	10.4	14.9	14.0	19.6	17.6	26.2	21.2	104.1	24.8
3.8	3.7	7.4	7.2	11.3	10.8	15.4	14.4	20.2	18.0	27.2	21.6	146.4	24.9
4.1	4.0	7.8	7.6	11.7	11.2	15.9	14.8	20.8	18.4	28.5	22.0	1000	24.973
4.5	4.4	8.3	8.0	12.2	11.6	16.4	15.2	21.4	18.8	29.9	22.4	∞	24.996
4.9	4.8	8.7	8.4	12.6	12.0	16.9	15.6	22.1	19.2	31.7	22.8		
5.3	5.2	9.1	8.8	13.1	12.4	17.4	16.0	22.8	19.6	34.1	23.2		
5.7	5.6	9.5	9.2	13.5	12.8	18.0	16.4	23.5	20.0	37.5	23.6		
6.1	6.0	10.0	9.6	14.0	13.2	18.5	16.8	24.3	20.4	43.4	24.0		
6.6	6.4	10.4	10.0	14.5	13.6	19.0	17.2	25.2	20.8	56.2	24.4		
$\alpha = 1\%$													
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
8.1	8.0	11.3	11.2	15.1	14.8	19.0	18.4	23.4	22.0	29.0	25.6	44.5	29.2
8.2	8.1	11.7	11.6	15.5	15.2	19.5	18.8	23.9	22.4	29.8	26.0	51.8	29.6
8.5	8.4	12.2	12.0	16.0	15.6	20.0	19.2	24.5	22.8	30.7	26.4	67.8	30.0
8.9	8.8	12.6	12.4	16.4	16.0	20.4	19.6	25.0	23.2	31.6	26.8	129.2	30.4
9.3	9.2	13.0	12.8	16.8	16.4	20.9	20.0	25.6	23.6	32.8	27.2	185.6	30.5
9.7	9.6	13.4	13.2	17.3	16.8	21.4	20.4	26.2	24.0	34.1	27.6	1000	30.557
10.1	10.0	13.8	13.6	17.7	17.2	21.9	20.8	26.8	24.4	35.7	28.0	∞	30.578
10.5	10.4	14.3	14.0	18.1	17.6	22.4	21.2	27.5	24.8	37.7	28.4		
10.9	10.8	14.7	14.4	18.6	18.0	22.9	21.6	28.2	25.2	40.4	28.8		

Table S.15: $1 - \alpha$ quantile of the conditional distribution, with density given in (2.12), $cv=c_{1-\alpha}(\hat{\kappa}_1, k - m_W)$ at different values of the conditioning variable $\hat{\kappa}_1$. Computed by numerical integration.

$k - m_W = 16$													
$\alpha = 10\%$													
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
2.6	2.5	5.7	5.5	9.3	8.8	13.1	12.1	17.3	15.4	22.7	18.7	34.9	22.0
2.7	2.6	6.0	5.8	9.6	9.1	13.5	12.4	17.8	15.7	23.4	19.0	38.0	22.3
2.9	2.8	6.4	6.1	10.0	9.4	13.8	12.7	18.2	16.0	24.0	19.3	42.9	22.6
3.2	3.1	6.7	6.4	10.3	9.7	14.2	13.0	18.6	16.3	24.7	19.6	51.8	22.9
3.5	3.4	7.0	6.7	10.7	10.0	14.6	13.3	19.1	16.6	25.5	19.9	73.1	23.2
3.8	3.7	7.3	7.0	11.0	10.3	15.0	13.6	19.6	16.9	26.3	20.2	184.3	23.5
4.1	4.0	7.7	7.3	11.4	10.6	15.3	13.9	20.0	17.2	27.2	20.5	1000	23.519
4.4	4.3	8.0	7.6	11.7	10.9	15.7	14.2	20.5	17.5	28.3	20.8	∞	23.542
4.8	4.6	8.3	7.9	12.0	11.2	16.1	14.5	21.0	17.8	29.5	21.1		
5.1	4.9	8.7	8.2	12.4	11.5	16.5	14.8	21.6	18.1	30.9	21.4		
5.4	5.2	9.0	8.5	12.8	11.8	16.9	15.1	22.1	18.4	32.6	21.7		
$\alpha = 5\%$													
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
3.9	3.8	7.2	7.0	11.0	10.6	15.0	14.2	19.5	17.8	25.3	21.4	40.8	25.0
4.0	3.9	7.6	7.4	11.5	11.0	15.5	14.6	20.1	18.2	26.1	21.8	48.1	25.4
4.3	4.2	8.0	7.8	11.9	11.4	16.0	15.0	20.6	18.6	27.0	22.2	65.4	25.8
4.7	4.6	8.4	8.2	12.3	11.8	16.5	15.4	21.2	19.0	28.0	22.6	153.6	26.2
5.1	5.0	8.9	8.6	12.8	12.2	17.0	15.8	21.8	19.4	29.1	23.0	1000	26.272
5.5	5.4	9.3	9.0	13.2	12.6	17.4	16.2	22.4	19.8	30.5	23.4	∞	26.296
5.9	5.8	9.7	9.4	13.7	13.0	17.9	16.6	23.1	20.2	32.0	23.8		
6.3	6.2	10.2	9.8	14.1	13.4	18.5	17.0	23.8	20.6	34.0	24.2		
6.8	6.6	10.6	10.2	14.6	13.8	19.0	17.4	24.5	21.0	36.7	24.6		
$\alpha = 1\%$													
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
8.5	8.4	12.0	11.9	16.2	15.9	20.6	19.9	25.5	23.9	32.5	27.9	175.4	31.9
8.6	8.5	12.6	12.4	16.8	16.4	21.2	20.4	26.2	24.4	33.8	28.4	1000	31.979
9.0	8.9	13.1	12.9	17.3	16.9	21.8	20.9	27.0	24.9	35.4	28.9	∞	32.000
9.5	9.4	13.6	13.4	17.8	17.4	22.4	21.4	27.7	25.4	37.4	29.4		
10.0	9.9	14.1	13.9	18.4	17.9	23.0	21.9	28.5	25.9	40.0	29.9		
10.5	10.4	14.6	14.4	18.9	18.4	23.6	22.4	29.4	26.4	44.0	30.4		
11.0	10.9	15.2	14.9	19.5	18.9	24.2	22.9	30.3	26.9	51.4	30.9		
11.5	11.4	15.7	15.4	20.0	19.4	24.9	23.4	31.4	27.4	69.6	31.4		

Table S.16: $1 - \alpha$ quantile of the conditional distribution, with density given in (2.12), $cv = c_{1-\alpha}(\hat{\kappa}_1, k - m_W)$ at different values of the conditioning variable $\hat{\kappa}_1$. Computed by numerical integration.

$k - m_W = 17$													
$\alpha = 10\%$													
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
2.7	2.6	6.0	5.8	9.9	9.4	14.1	13.0	18.7	16.6	24.8	20.2	44.6	23.8
2.8	2.7	6.4	6.2	10.4	9.8	14.5	13.4	19.3	17.0	25.7	20.6	58.2	24.2
3.1	3.0	6.9	6.6	10.8	10.2	15.0	13.8	19.8	17.4	26.7	21.0	112.6	24.6
3.5	3.4	7.3	7.0	11.3	10.6	15.5	14.2	20.5	17.8	27.8	21.4	166.2	24.7
3.9	3.8	7.7	7.4	11.7	11.0	16.0	14.6	21.1	18.2	29.1	21.8	1000	24.745
4.3	4.2	8.2	7.8	12.2	11.4	16.5	15.0	21.7	18.6	30.6	22.2	∞	24.769
4.7	4.6	8.6	8.2	12.6	11.8	17.1	15.4	22.4	19.0	32.5	22.6		
5.2	5.0	9.0	8.6	13.1	12.2	17.6	15.8	23.2	19.4	34.9	23.0		
5.6	5.4	9.5	9.0	13.6	12.6	18.1	16.2	24.0	19.8	38.5	23.4		
$\alpha = 5\%$													
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
4.1	4.0	7.4	7.2	11.2	10.8	15.2	14.4	19.5	18.0	24.7	21.6	34.3	25.2
4.2	4.1	7.8	7.6	11.6	11.2	15.6	14.8	20.0	18.4	25.5	22.0	36.5	25.6
4.5	4.4	8.2	8.0	12.1	11.6	16.1	15.2	20.5	18.8	26.2	22.4	39.6	26.0
4.9	4.8	8.6	8.4	12.5	12.0	16.6	15.6	21.1	19.2	27.0	22.8	44.5	26.4
5.3	5.2	9.1	8.8	12.9	12.4	17.0	16.0	21.6	19.6	27.9	23.2	53.8	26.8
5.7	5.6	9.5	9.2	13.4	12.8	17.5	16.4	22.2	20.0	28.8	23.6	78.9	27.2
6.1	6.0	9.9	9.6	13.8	13.2	18.0	16.8	22.8	20.4	29.9	24.0	167.6	27.5
6.5	6.4	10.3	10.0	14.3	13.6	18.5	17.2	23.4	20.8	31.1	24.4	1000	27.562
7.0	6.8	10.8	10.4	14.7	14.0	19.0	17.6	24.1	21.2	32.5	24.8	∞	27.587
$\alpha = 1\%$													
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
9.0	8.9	12.5	12.4	16.7	16.4	21.1	20.4	25.8	24.4	32.1	28.4	55.9	32.4
9.1	9.0	13.0	12.9	17.2	16.9	21.6	20.9	26.5	24.9	33.2	28.9	79.5	32.9
9.5	9.4	13.6	13.4	17.8	17.4	22.2	21.4	27.2	25.4	34.4	29.4	298.9	33.4
10.0	9.9	14.1	13.9	18.3	17.9	22.8	21.9	27.9	25.9	35.8	29.9	1000	33.388
10.5	10.4	14.6	14.4	18.8	18.4	23.4	22.4	28.6	26.4	37.4	30.4	∞	33.409
11.0	10.9	15.1	14.9	19.4	18.9	24.0	22.9	29.4	26.9	39.6	30.9		
11.5	11.4	15.7	15.4	19.9	19.4	24.6	23.4	30.3	27.4	42.5	31.4		
12.0	11.9	16.2	15.9	20.5	19.9	25.2	23.9	31.1	27.9	47.1	31.9		

Table S.17: $1 - \alpha$ quantile of the conditional distribution, with density given in (2.12), $cv=c_{1-\alpha}(\hat{\kappa}_1, k - m_W)$ at different values of the conditioning variable $\hat{\kappa}_1$. Computed by numerical integration.

$k - m_W = 18$													
$\alpha = 10\%$													
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
2.9	2.8	6.2	6.0	10.1	9.6	14.2	13.2	18.6	16.8	24.2	20.4	35.2	24.0
3.0	2.9	6.6	6.4	10.5	10.0	14.6	13.6	19.2	17.2	25.0	20.8	38.2	24.4
3.3	3.2	7.1	6.8	11.0	10.4	15.1	14.0	19.7	17.6	25.8	21.2	42.8	24.8
3.7	3.6	7.5	7.2	11.4	10.8	15.6	14.4	20.3	18.0	26.7	21.6	51.6	25.2
4.1	4.0	7.9	7.6	11.9	11.2	16.1	14.8	20.9	18.4	27.6	22.0	75.2	25.6
4.5	4.4	8.4	8.0	12.3	11.6	16.6	15.2	21.5	18.8	28.7	22.4	158.1	25.9
4.9	4.8	8.8	8.4	12.8	12.0	17.1	15.6	22.1	19.2	29.9	22.8	1000	25.965
5.4	5.2	9.2	8.8	13.2	12.4	17.6	16.0	22.8	19.6	31.3	23.2	∞	25.989
5.8	5.6	9.7	9.2	13.7	12.8	18.1	16.4	23.5	20.0	33.0	23.6		
$\alpha = 5\%$													
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
4.3	4.2	8.0	7.8	12.2	11.8	16.7	15.8	21.6	19.8	27.9	23.8	48.8	27.8
4.4	4.3	8.4	8.2	12.7	12.2	17.1	16.2	22.1	20.2	28.8	24.2	61.3	28.2
4.7	4.6	8.8	8.6	13.1	12.6	17.6	16.6	22.7	20.6	29.7	24.6	101.2	28.6
5.1	5.0	9.2	9.0	13.5	13.0	18.1	17.0	23.2	21.0	30.7	25.0	190.8	28.8
5.5	5.4	9.7	9.4	14.0	13.4	18.5	17.4	23.8	21.4	31.8	25.4	1000	28.843
5.9	5.8	10.1	9.8	14.4	13.8	19.0	17.8	24.4	21.8	33.1	25.8	∞	28.869
6.3	6.2	10.5	10.2	14.8	14.2	19.5	18.2	25.1	22.2	34.7	26.2		
6.7	6.6	10.9	10.6	15.3	14.6	20.0	18.6	25.7	22.6	36.6	26.6		
7.1	7.0	11.4	11.0	15.7	15.0	20.5	19.0	26.4	23.0	39.1	27.0		
7.6	7.4	11.8	11.4	16.2	15.4	21.0	19.4	27.2	23.4	42.7	27.4		
$\alpha = 1\%$													
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
9.5	9.4	13.0	12.9	17.2	16.9	21.5	20.9	26.2	24.9	32.0	28.9	45.2	32.9
9.6	9.5	13.5	13.4	17.7	17.4	22.1	21.4	26.8	25.4	32.9	29.4	50.5	33.4
10.0	9.9	14.1	13.9	18.2	17.9	22.6	21.9	27.5	25.9	33.9	29.9	61.4	33.9
10.5	10.4	14.6	14.4	18.8	18.4	23.2	22.4	28.1	26.4	35.0	30.4	93.8	34.4
11.0	10.9	15.1	14.9	19.3	18.9	23.8	22.9	28.8	26.9	36.3	30.9	1000	34.785
11.5	11.4	15.6	15.4	19.9	19.4	24.4	23.4	29.6	27.4	37.7	31.4	∞	34.805
12.0	11.9	16.1	15.9	20.4	19.9	25.0	23.9	30.3	27.9	39.5	31.9		
12.5	12.4	16.7	16.4	20.9	20.4	25.6	24.4	31.1	28.4	41.9	32.4		

Table S.18: $1 - \alpha$ quantile of the conditional distribution, with density given in (2.12), $cv=c_{1-\alpha}(\hat{\kappa}_1, k - m_W)$ at different values of the conditioning variable $\hat{\kappa}_1$. Computed by numerical integration.

$k - m_W = 19$													
$\alpha = 10\%$													
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
3.0	2.9	6.7	6.5	11.0	10.5	15.6	14.5	20.7	18.5	27.4	22.5	57.3	26.5
3.1	3.0	7.1	6.9	11.5	10.9	16.1	14.9	21.2	18.9	28.4	22.9	90.4	26.9
3.4	3.3	7.6	7.3	11.9	11.3	16.5	15.3	21.8	19.3	29.4	23.3	280.9	27.2
3.8	3.7	8.0	7.7	12.4	11.7	17.0	15.7	22.4	19.7	30.5	23.7	1000	27.178
4.2	4.1	8.4	8.1	12.8	12.1	17.5	16.1	23.0	20.1	31.8	24.1	∞	27.204
4.6	4.5	8.9	8.5	13.3	12.5	18.0	16.5	23.7	20.5	33.3	24.5		
5.0	4.9	9.3	8.9	13.7	12.9	18.5	16.9	24.3	20.9	35.1	24.9		
5.5	5.3	9.7	9.3	14.2	13.3	19.0	17.3	25.1	21.3	37.5	25.3		
5.9	5.7	10.2	9.7	14.6	13.7	19.6	17.7	25.8	21.7	40.9	25.7		
6.3	6.1	10.6	10.1	15.1	14.1	20.1	18.1	26.6	22.1	46.3	26.1		
$\alpha = 5\%$													
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
4.5	4.4	8.2	8.0	12.4	12.0	16.8	16.0	21.6	20.0	27.4	24.0	39.0	28.0
4.6	4.5	8.6	8.4	12.8	12.4	17.2	16.4	22.1	20.4	28.1	24.4	41.9	28.4
4.9	4.8	9.0	8.8	13.3	12.8	17.7	16.8	22.6	20.8	28.9	24.8	46.3	28.8
5.3	5.2	9.4	9.2	13.7	13.2	18.2	17.2	23.1	21.2	29.7	25.2	54.0	29.2
5.7	5.6	9.8	9.6	14.1	13.6	18.6	17.6	23.7	21.6	30.6	25.6	71.5	29.6
6.1	6.0	10.3	10.0	14.6	14.0	19.1	18.0	24.3	22.0	31.5	26.0	146.7	30.0
6.5	6.4	10.7	10.4	15.0	14.4	19.6	18.4	24.8	22.4	32.6	26.4	229.3	30.1
6.9	6.8	11.1	10.8	15.4	14.8	20.1	18.8	25.4	22.8	33.8	26.8	1000	30.116
7.3	7.2	11.5	11.2	15.9	15.2	20.6	19.2	26.1	23.2	35.2	27.2	∞	30.144
7.8	7.6	12.0	11.6	16.3	15.6	21.0	19.6	26.7	23.6	36.9	27.6		
$\alpha = 1\%$													
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
10.0	9.9	14.0	13.9	18.7	18.4	23.6	22.9	29.1	27.4	36.9	31.9	201.8	36.1
10.1	10.0	14.6	14.4	19.3	18.9	24.2	23.4	29.8	27.9	38.2	32.4	1000	36.171
10.5	10.4	15.1	14.9	19.8	19.4	24.8	23.9	30.5	28.4	39.8	32.9	∞	36.191
11.0	10.9	15.6	15.4	20.3	19.9	25.4	24.4	31.2	28.9	41.7	33.4		
11.5	11.4	16.1	15.9	20.9	20.4	25.9	24.9	32.0	29.4	44.3	33.9		
12.0	11.9	16.6	16.4	21.4	20.9	26.5	25.4	32.8	29.9	48.1	34.4		
12.5	12.4	17.1	16.9	22.0	21.4	27.2	25.9	33.7	30.4	54.5	34.9		
13.0	12.9	17.7	17.4	22.5	21.9	27.8	26.4	34.7	30.9	68.2	35.4		
13.5	13.4	18.2	17.9	23.1	22.4	28.4	26.9	35.7	31.4	116.7	35.9		

Table S.19: $1 - \alpha$ quantile of the conditional distribution, with density given in (2.12), $cv = c_{1-\alpha}(\hat{\kappa}_1, k - m_W)$ at different values of the conditioning variable $\hat{\kappa}_1$. Computed by numerical integration.

$k - m_W = 20$													
$\alpha = 10\%$													
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
3.1	3.0	6.8	6.6	11.1	10.6	15.6	14.6	20.5	18.6	26.6	22.6	39.9	26.6
3.2	3.1	7.2	7.0	11.5	11.0	16.0	15.0	21.0	19.0	27.4	23.0	43.8	27.0
3.5	3.4	7.7	7.4	12.0	11.4	16.5	15.4	21.6	19.4	28.2	23.4	50.3	27.4
3.9	3.8	8.1	7.8	12.4	11.8	17.0	15.8	22.1	19.8	29.1	23.8	64.4	27.8
4.3	4.2	8.5	8.2	12.9	12.2	17.5	16.2	22.7	20.2	30.1	24.2	114.8	28.2
4.7	4.6	8.9	8.6	13.3	12.6	18.0	16.6	23.3	20.6	31.1	24.6	273.1	28.4
5.1	5.0	9.4	9.0	13.8	13.0	18.5	17.0	23.9	21.0	32.3	25.0	1000	28.385
5.5	5.4	9.8	9.4	14.2	13.4	19.0	17.4	24.6	21.4	33.7	25.4	∞	28.412
6.0	5.8	10.2	9.8	14.7	13.8	19.5	17.8	25.2	21.8	35.3	25.8		
6.4	6.2	10.7	10.2	15.1	14.2	20.0	18.2	25.9	22.2	37.3	26.2		
$\alpha = 5\%$													
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
4.8	4.7	8.9	8.7	13.5	13.1	18.4	17.5	23.8	21.9	30.8	26.3	64.7	30.7
4.9	4.8	9.3	9.1	14.0	13.5	18.8	17.9	24.3	22.3	31.7	26.7	101.7	31.1
5.2	5.1	9.7	9.5	14.4	13.9	19.3	18.3	24.8	22.7	32.6	27.1	299.7	31.4
5.6	5.5	10.1	9.9	14.8	14.3	19.8	18.7	25.4	23.1	33.7	27.5	1000	31.382
6.0	5.9	10.6	10.3	15.3	14.7	20.3	19.1	26.0	23.5	34.8	27.9	∞	31.410
6.4	6.3	11.0	10.7	15.7	15.1	20.7	19.5	26.6	23.9	36.2	28.3		
6.8	6.7	11.4	11.1	16.1	15.5	21.2	19.9	27.2	24.3	37.8	28.7		
7.2	7.1	11.8	11.5	16.6	15.9	21.7	20.3	27.9	24.7	39.8	29.1		
7.6	7.5	12.2	11.9	17.0	16.3	22.2	20.7	28.5	25.1	42.4	29.5		
8.1	7.9	12.7	12.3	17.5	16.7	22.7	21.1	29.3	25.5	46.1	29.9		
8.5	8.3	13.1	12.7	17.9	17.1	23.2	21.5	30.0	25.9	52.3	30.3		
$\alpha = 1\%$													
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
10.4	10.3	14.4	14.3	19.1	18.8	24.0	23.3	29.3	27.8	36.3	32.3	72.0	36.8
10.5	10.4	14.9	14.8	19.6	19.3	24.5	23.8	29.9	28.3	37.3	32.8	126.7	37.3
10.9	10.8	15.5	15.3	20.2	19.8	25.1	24.3	30.6	28.8	38.5	33.3	231.5	37.5
11.4	11.3	16.0	15.8	20.7	20.3	25.7	24.8	31.3	29.3	39.9	33.8	1000	37.547
11.9	11.8	16.5	16.3	21.2	20.8	26.2	25.3	32.0	29.8	41.5	34.3	∞	37.566
12.4	12.3	17.0	16.8	21.8	21.3	26.8	25.8	32.8	30.3	43.6	34.8		
12.9	12.8	17.5	17.3	22.3	21.8	27.4	26.3	33.6	30.8	46.3	35.3		
13.4	13.3	18.1	17.8	22.9	22.3	28.0	26.8	34.4	31.3	50.2	35.8		
13.9	13.8	18.6	18.3	23.4	22.8	28.6	27.3	35.3	31.8	57.1	36.3		

Table S.20: $1 - \alpha$ quantile of the conditional distribution, with density given in (2.12), $cv = c_{1-\alpha}(\hat{\kappa}_1, k - m_W)$ at different values of the conditioning variable $\hat{\kappa}_1$. Computed by numerical integration.

k	α			k	α		
	0.1	0.05	0.01		0.1	0.05	0.01
2	0.1000	0.0500	0.0100	12	0.1003	0.0504	0.0101
3	0.1000	0.0504	0.0100	13	0.1004	0.0504	0.0102
4	0.1000	0.0500	0.0100	14	0.1007	0.0506	0.0102
5	0.1000	0.0500	0.0100	15	0.1007	0.0503	0.0102
6	0.1000	0.0500	0.0100	16	0.1013	0.0507	0.0101
7	0.1000	0.0500	0.0100	17	0.1006	0.0509	0.0101
8	0.1000	0.0502	0.0100	18	0.1014	0.0508	0.0101
9	0.1000	0.0500	0.0101	19	0.1017	0.0508	0.0101
10	0.1001	0.0505	0.0101	20	0.1014	0.0511	0.0102
11	0.1005	0.0504	0.0100	21	0.1019	0.0510	0.0102

Table S.21: Size of the conditional subvector AR test with nominal size α for different k with $m_W = 1$, using critical values given in Tables S.1 to S.20 and linear interpolation. Computed using 1 million Monte Carlo replications.

S.4 Additional numerical results

S.4.1 Size

We computed the size of φ_c at significance levels 1%, 5% and 10% for $k = 2, \dots, 21$ and $m_W = 1$ using a grid of 42 points in κ_1 equally spaced in log-scale between 0 and 100. The reported size is the maximum of α or the estimated NRPs. The results are reported in Table S.21. In all cases, the size of the test is controlled to two decimals, in accordance with Theorem 2.

S.4.2 Power

Here, we report supplementary power comparisons for Section 2.4. The power of the conditional subvector AR test φ_c and the unconditional test φ_{GKMC} are compared to the ALFD estimate of the point-optimal power envelope for $k = 2, 5, 10$, and 20.

Figure S.5 gives the difference between the power of 5% level φ_c test and the point-optimal ALFD power bound $\tilde{\pi}$ defined in the step 9 of the algorithm in Section S.2.3.2, across all alternatives. The power of φ_c is well within 1% of the power bound except for alternatives very close to H_0 . The largest deviations from the power bound occur when $\kappa_1 = \kappa_2$.

Figure S.6 repeats the comparison of the power of φ_c but with $\tilde{\pi}$ replaced by $\min(\tilde{\pi}^{AMS}, \tilde{\pi}^{EMW})$, where $\tilde{\pi}^{AMS}, \tilde{\pi}^{EMW}$ are computed using the algorithms in sections S.2.3.1 and S.2.3.2, respectively. Since $\min(\tilde{\pi}^{AMS}, \tilde{\pi}^{EMW}) \geq \tilde{\pi}$, the differences are larger than in Figure S.5, but not by much.

Figures S.7 through S.10 report power comparisons in 2D, where $\kappa_1 - \kappa_2$ is kept fixed in each figure, and the alternative only varies across κ_2 . The figures plot the power curves of both test $\varphi_c, \varphi_{GKMC}$ at 5% level, and both power bounds, $\min(\tilde{\pi}^{AMS}, \tilde{\pi}^{EMW})$ and $\tilde{\pi}$. We notice that the power of φ_c is very close to both power bounds, which are in turn very close to each other, while the power of the unconditional subvector AR test φ_{GKMC} is noticeably below the power bounds. As $\kappa_1 - \kappa_2$ increases, both power curves get closer to the power bounds, and they essentially collapse on top of each other when $\kappa_1 - \kappa_2 = 64$. This is why we do not consider values higher than that in the simulations. The distance of φ_{GKMC} from φ_c and the power bounds is also somewhat increasing in k .

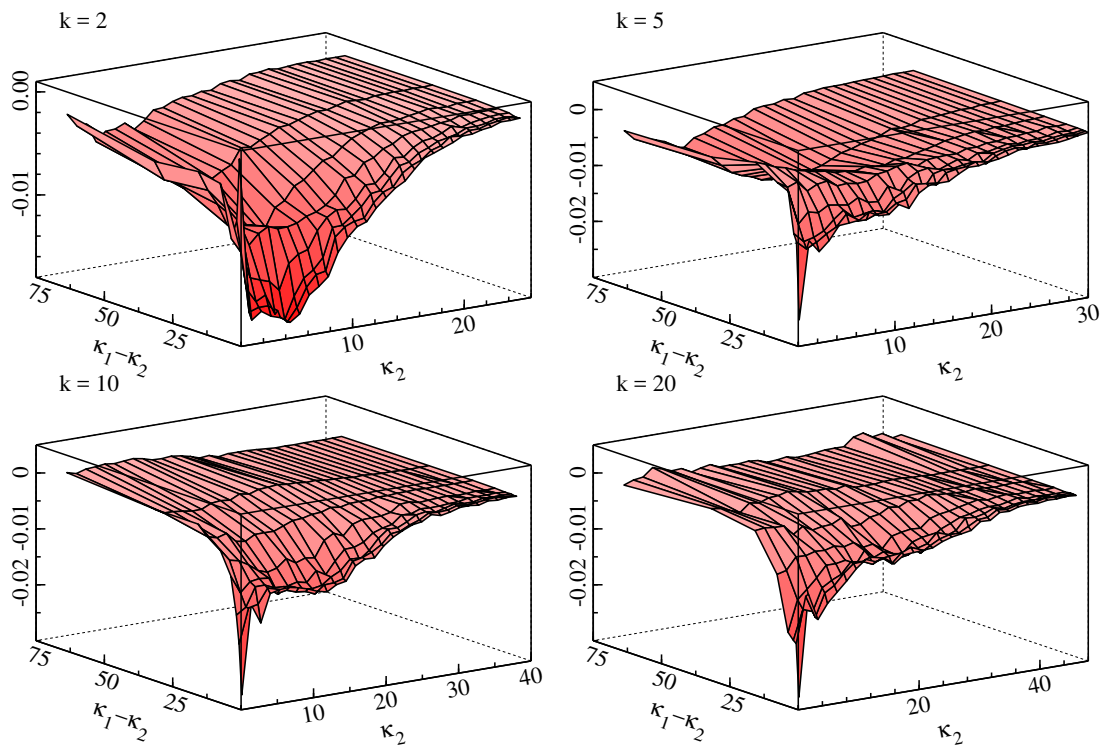


Figure S.5: Power of 5% level conditional subvector AR test φ_c minus the ALFD power bound $\tilde{\pi}$ computed by the algorithm in Section S.2.

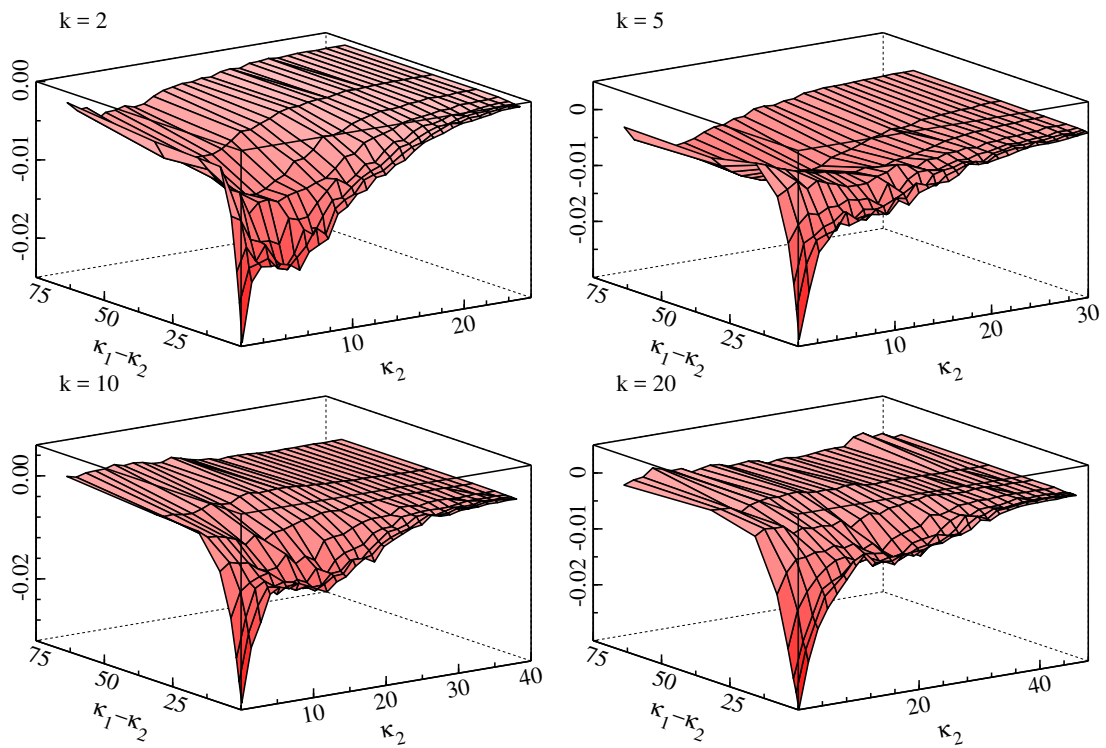


Figure S.6: Power of 5% level conditional subvector AR test φ_c minus $\min(\bar{\pi}^{AMS}, \bar{\pi}^{EMW})$ power bound, where $\bar{\pi}^{AMS}, \bar{\pi}^{EMW}$ were computed by the algorithms in Section S.2.

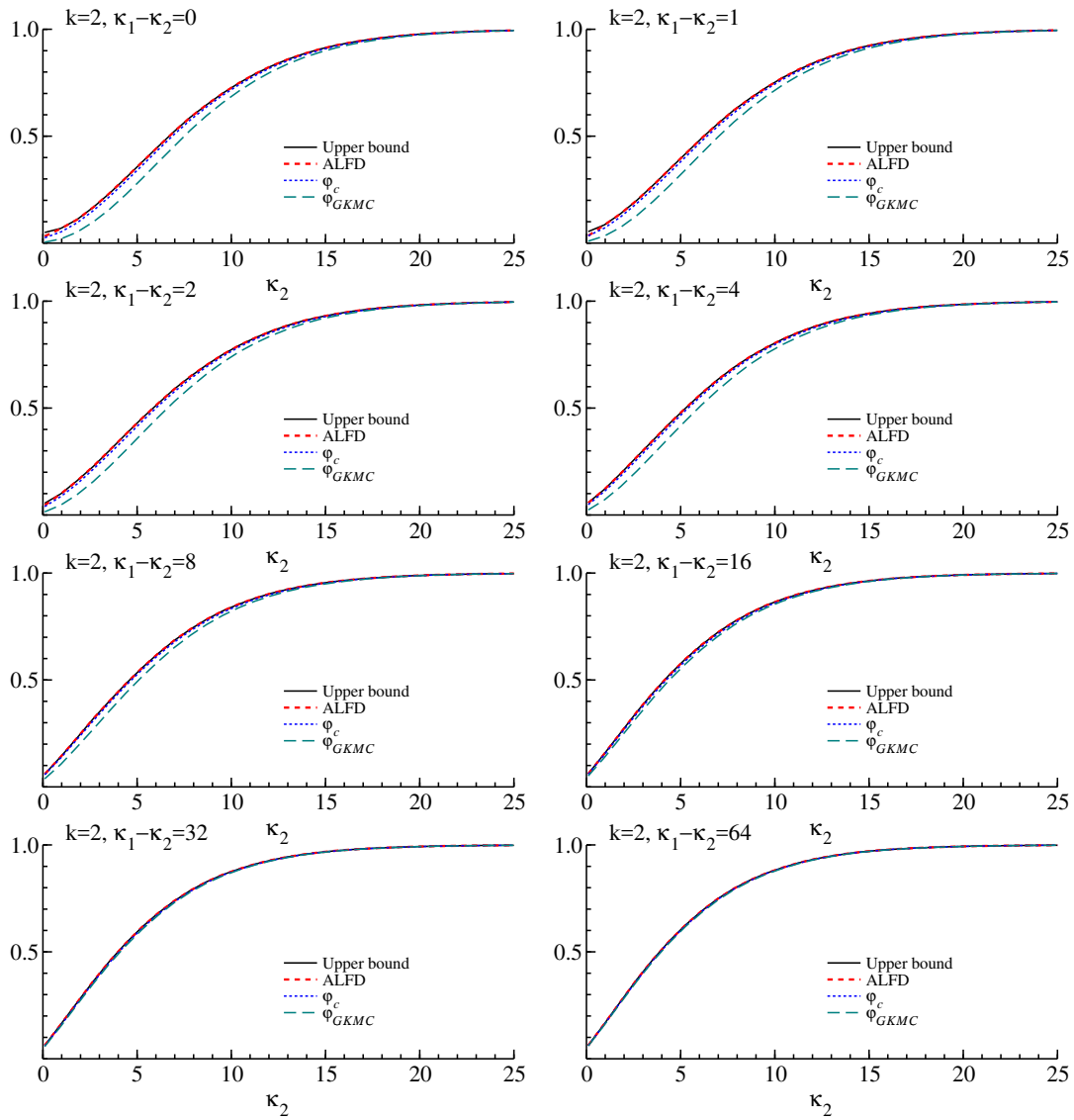


Figure S.7: Power curves and power bounds of 5% level φ_c and φ_{GKMC} tests as a function of κ_2 , at different values of $\kappa_1 - \kappa_2$ when the number of instruments $k = 2$.

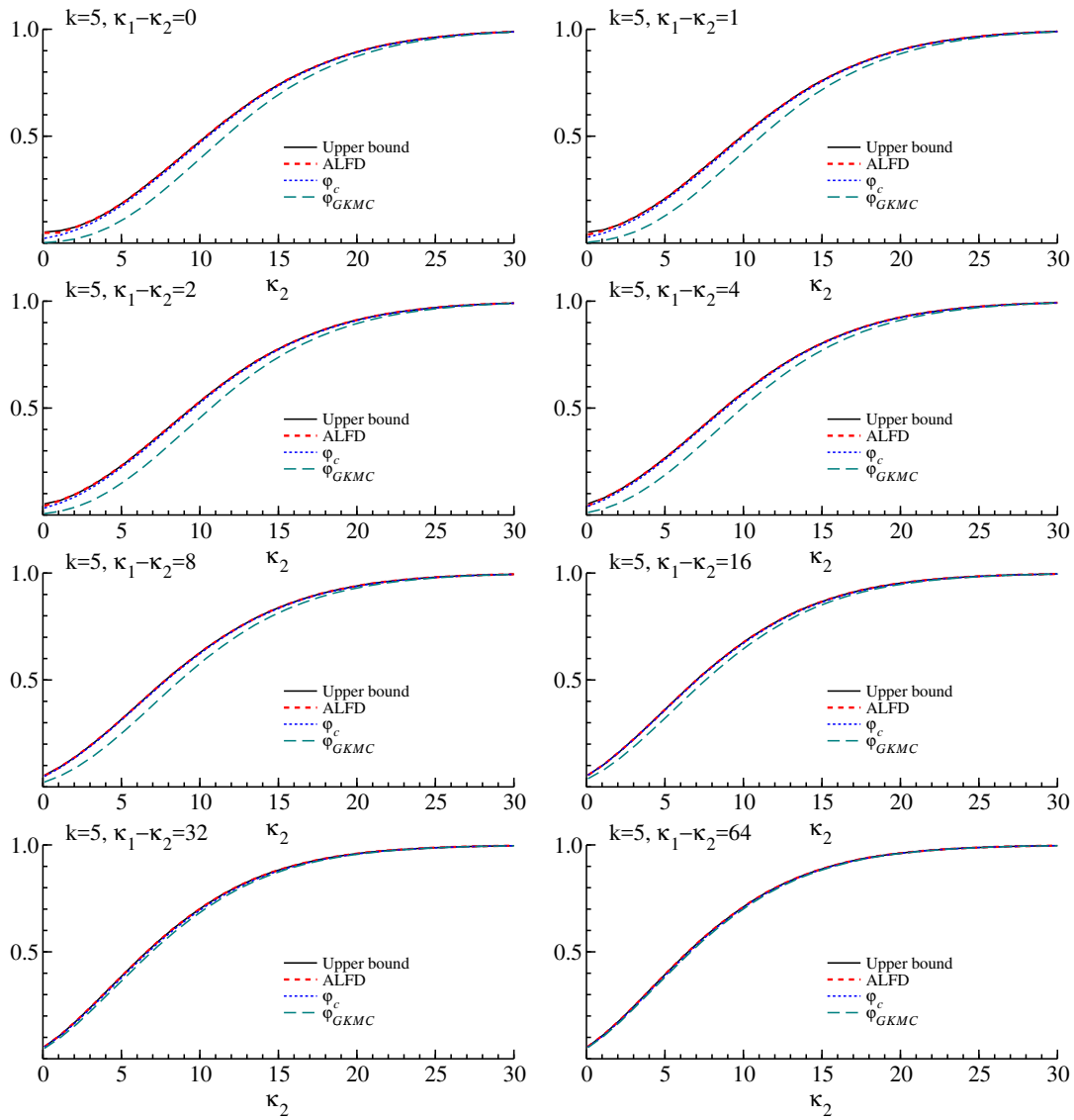


Figure S.8: Power curves and power bounds of 5% level φ_c and φ_{GKMC} tests as a function of κ_2 , at different values of $\kappa_1 - \kappa_2$ when the number of instruments $k = 5$.

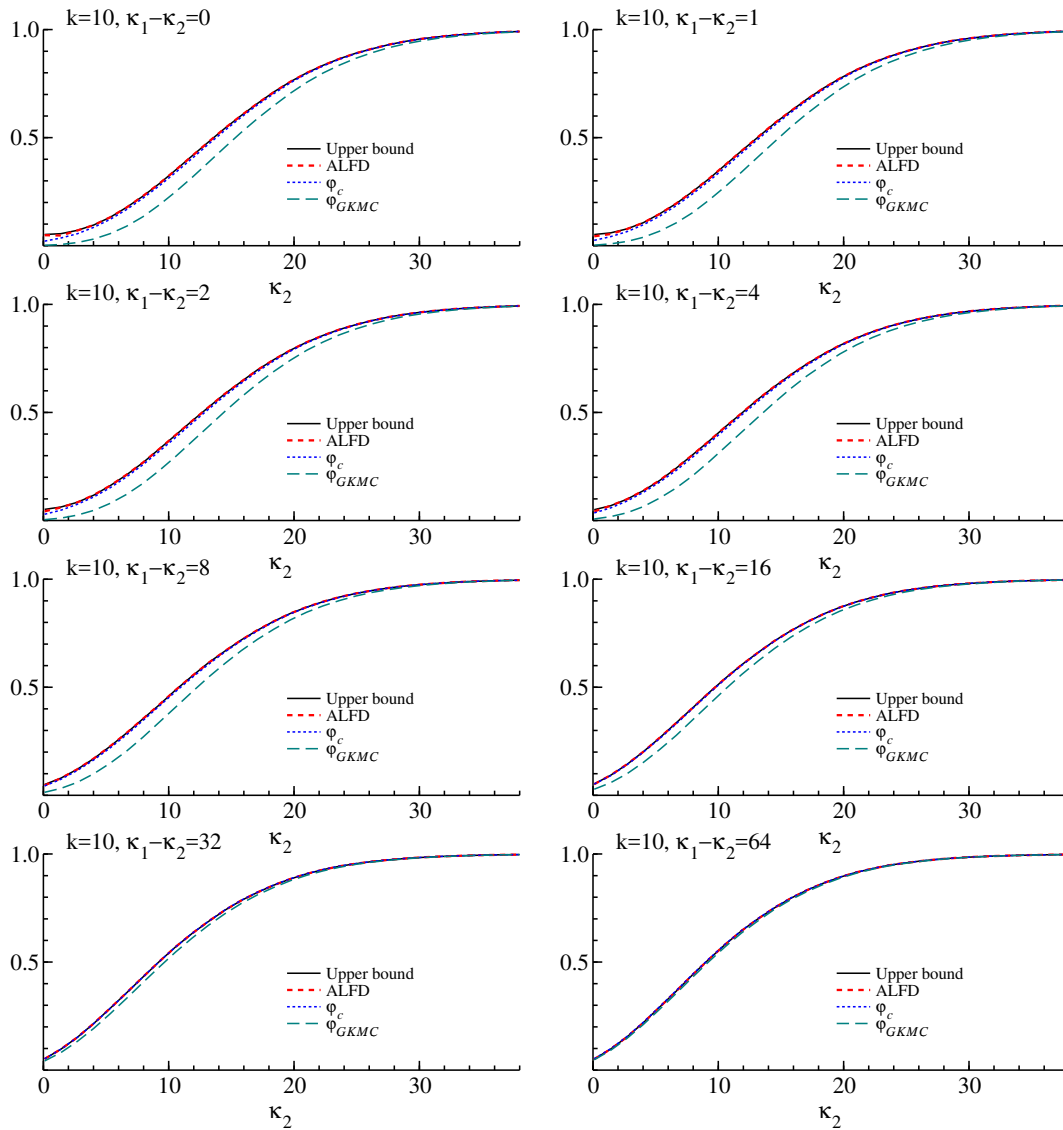


Figure S.9: Power curves and power bounds of 5% level φ_c and φ_{GKMC} tests as a function of κ_2 , at different values of $\kappa_1 - \kappa_2$ when the number of instruments $k = 10$.

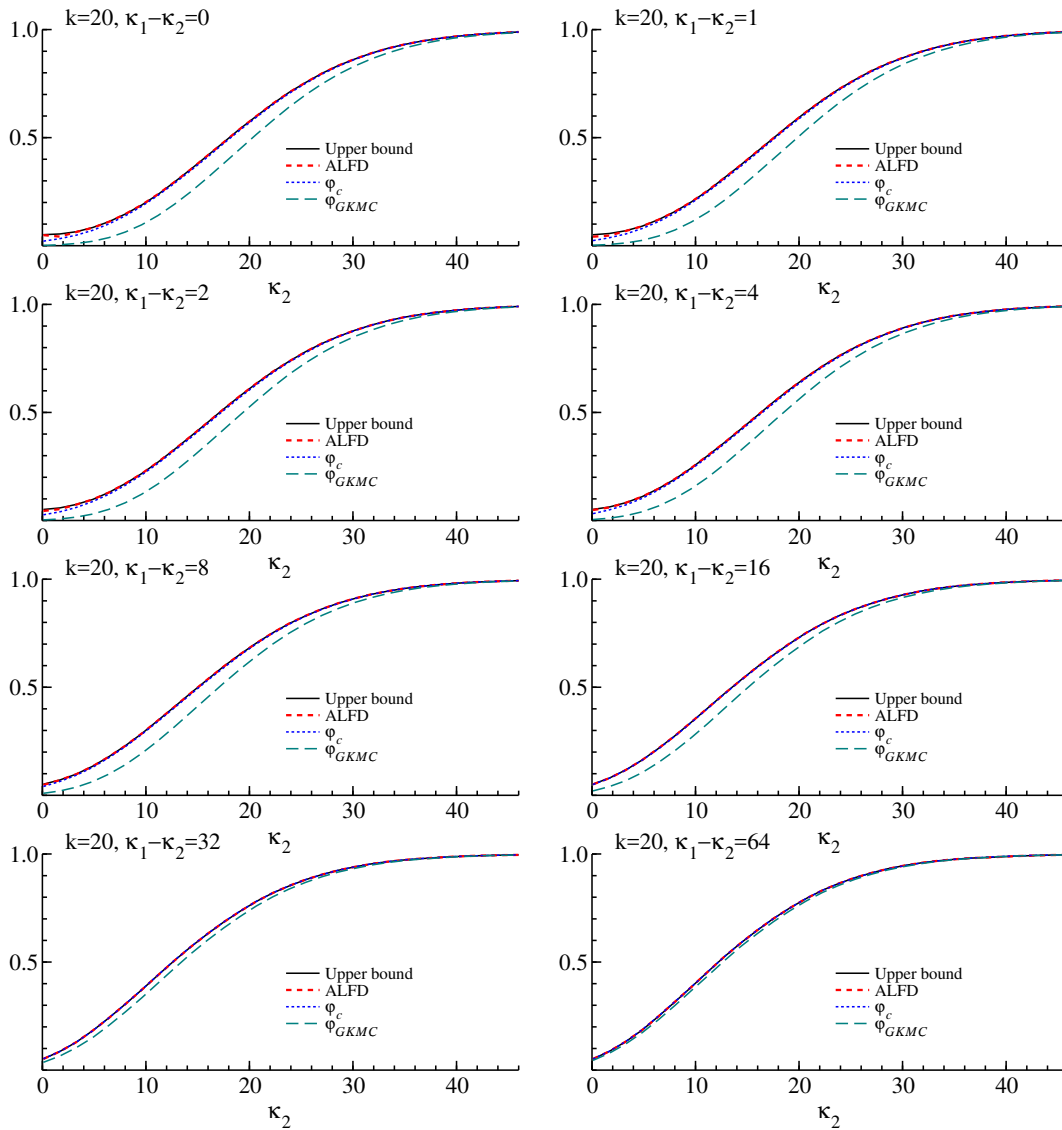


Figure S.10: Power curves and power bounds of 5% level φ_c and φ_{GKMC} tests as a function of κ_2 , at different values of $\kappa_1 - \kappa_2$ when the number of instruments $k = 20$.