Hartwick's Rule

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Abstract

Hartwick's rule for sustainability prescribes reinvesting resource rents, thus keeping the value of net investments equal to zero. The following two results hold in various classes of models, including the one-sector growth model and the model of capital accumulation and resource depletion in which John Hartwick originally formulated his rule: (1) (Hartwick's result) If along an efficient path Hartwick's rule is followed forever, then an egalitarian path is implemented. (2) (The converse of Hartwick's result) If an efficient and egalitarian path is implemented, then Hartwick's rule is followed forever.

While it is a robust result that Hartwick's rule characterizes efficient and egalitarian paths, it has proven to be an elusive goal to be able to indicate sustainability by the value of net investments as a genuine savings indicator. In particular, the value of net investments may be positive even though no positive level of consumption can be sustained. Therefore, Hartwick's result (and its converse) essentially constitutes a valuable characterization of an efficient and egalitarian path rather than establishes the basis for a useful prescriptive rule for sustainability.

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List of abbreviations:

HR: Hartwick's rule
CHR: Converse of Hartwick's rule
DHSS technology: Dasgupta-Heal-Solow-Stiglitz technology

Glossary terms:

Competitive prices: Prices to which agents in the economy can be seen to optimize.

Efficient path: A path where well-being cannot be increased for some subinterval without being decreased at some other subinterval.

Egalitarian path: A path where well-being is constant.

Dixit-Hammond-Hoel's rule: Keeping the value of net investments, measured in competitive present value prices, constant.

Golden-rule capital stock: The capital stock that maximizes sustainable well-being.

Hartwick's rule: Keeping the value of net investments, measured in competitive prices, equal to zero.

Present value prices: Deflationary nominal prices that correspond to a zero nominal interest rate.

Regular maximin path: An efficient and egalitarian path allowing for a trade-off between current well-being and the maximum sustainable level.

Resource allocation mechanism: Mechanism that assigns an attainable consumption-net investment pair to any vector of capital stocks.

Sustainable development: Development where current well-being does not exceed the sustainable level.

1 Introduction

Hartwick's rule for sustainability prescribes reinvesting resource rents, thus keeping the value of net investments equal to zero. In the article "Intergenerational equity and investing rents from exhaustible resources" that John M. Hartwick published in the *American Economic Review* in 1976 it was originally formulated as follows:

> Invest all profits or rents from exhaustible resources in reproducible capital such as machines. This injunction seems to solve the ethical problem of the current generation shortchanging future generations by "overconsuming" the current product, partly ascribable to current use of exhaustible resources.

Later research has shown that the result that Hartwick then established in a model of capital accumulation and resource depletion is not limited to this special technological environment. There is a general relationship between implementing a path which keeps well-being constant (is *egalitarian*) and where well-being cannot be increased for some subinterval of time without being decreased at some other subinterval (is *efficient*), on the one hand, and Hartwick's rule for sustainability, on the other hand. This relationship can be stated through the following two results:

- HR *Hartwick's result.* If along an efficient path Hartwick's rule is followed forever, then an egalitarian path is implemented.

- CHR *The converse of Hartwick's result.* If an efficient and egalitarian path is implemented, then Hartwick's rule is followed forever.

Sections 2–4 show how HR and CHR obtain in three different classes of technologies: (i) the one-sector growth model, (ii) the model of capital accumulation and resource depletion used by Hartwick, and (iii) a general model with multiple capital goods. Section 5 discusses the assumptions underlying these results—that the economy has constant technology and constant population and implements an efficient path—and points to ways to relax them. The concluding Section 6 summarizes conclusions from the literature on Hartwick's rule, namely that it is less a prescriptive rule for sustainability and more a characterization of investment behavior along an efficient and egalitarian path. In particular, the value of net investment is not an exact indicator of sustainability.

2 Illustration in a one-sector model

Say that a path of consumption (measuring instantaneous well-being) is *efficient* if consumption cannot be increased for some subinterval without being decreased at some other subinterval and is *egalitarian* if consumption is constant for all $t \geq 0$. Both Hartwick's result (HR) and its converse (CHR) have the feature that they are only relevant in a setting where an efficient and egalitarian path exists.

Parameterized versions of the model in which John Hartwick originally suggested the rule that bears his name (and which will be discussed in Section 3) have the property that *either* an efficient and egalitarian path does not exist from any vector of initial stocks *or* such a path exists from all vectors of (positive) initial stocks. Therefore, to illustrate the importance of this feature of HR and CHR, it is instructive to start with a simple one-sector model of economic growth.

Assume that the technology is given by a strictly increasing, strictly concave, and continuously differentiable gross production function $g : \mathbb{R}_+ \to \mathbb{R}_+$, satisfying $g(0) = 0$, $\lim_{k \to 0} g'(k) = \infty$ and $\lim_{k \to \infty} g'(k) = 0$. Assume also that capital k depreciates at constant rate $\delta > 0$. Denote by $f : \mathbb{R}_+ \to \mathbb{R}$ the *net production function* defined by $f(k) = g(k) - \delta k$. A continuous consumption path $\{c(t)\}$ is feasible from an initial capital stock $k_0 > 0$ at time 0 if there is an associated continuously differentiable capital path $\{k(t)\}$ such that $c(t) \geq 0$, $k(t) \geq 0$ and

$$c(t) + \dot{k}(t) = f(k(t)) \tag{1}$$

for all $t \geq 0$, with $k(0) = k_0$. Such a technology is referred to as a *Ramsey technology.*

Under the assumptions made in the previous paragraph there is a unique golden-rule capital stock, $k^* > 0$, that maximizes net production. We have that $f'(k^*) = 0$, $f'(k) > 0$ for $k < k^*$ and $f'(k) < 0$ for $k > k^*$. It is intuitively clear that if the initial capital stock satisfies $k_0 \in (0, k^*]$, then

(i) an efficient and egalitarian consumption path $\{c(t)\}$ is implemented if $\{k(t)\}$ satisfies $k(t) = k_0$ forever, and

(ii) present consumption $c(0)$ can be sustained forever if and only if $c(0) \leq f(k_0)$.

Hence, if the initial capital stock does not exceed the golden-rule size, then zero net investment corresponds to maximizing sustainable consumption. A standard method for proving these results is to support the path by competitive prices and show that it maximizes the present value of future consumption at these prices. As will be emphasized in the treatment of models with multiple capital goods, such competitive prices play a crucial role in the discussion of Hartwick's rule.

A feasible consumption path $\{c(t)\}$ from k_0 with associated capital path $\{k(t)\}$ determines a continuously differentiable path of positive supporting discount factors $\{p(t)\}$ by $p(0) = 1$ and $-\dot{p}(t)/p(t) = f'(k(t))$ for all $t \geq 0$. Here, $-\dot{p}(t)/p(t)$ is the real interest rate, which is positive if and only if $k(t) \in (0, k^*)$. Furthermore, these discount factors are *competitive* in the sense that, by the strict concavity of f, $k(t)$ maximizes real profits, $f(k) + (\dot{p}(t)/p(t))\, k$, over all $k \geq 0$. Therefore, by (1),

$$\begin{aligned} \int_0^T p(t)\left(c'(t) - c(t)\right) dt &= \int_0^T p(t)\Big(\big(f(k'(t)) - f(k(t))\big) - \big(\dot{k}'(t) - \dot{k}(t)\big)\Big)\, dt \\ &\leq \int_0^T \Big(\dot{p}(t)\big(k(t) - k'(t)\big) + p(t)\big(\dot{k}(t) - \dot{k}'(t)\big)\Big)\, dt \qquad (2) \\ &= \int_0^T \tfrac{d}{dt}\left(p(t)\left(k(t) - k'(t)\right)\right) dt = p(T)\left(k(T) - k'(T)\right) \end{aligned}$$

if $\{c'(t)\}$ is any alternative feasible path from k_0 with associated capital path $\{k'(t)\}$. If $k(t)$ is kept constant and equal to k_0, then $c(t)$ is constant and equal to $c_0 = f(k_0)$. Furthermore, with $k_0 \in (0, k^*)$, $p(t)$ becomes an exponentially decreasing function so that $p(t)k(t) \to 0$ as $t \to \infty$. Hence, for this egalitarian path it follows from (2) that

$$\limsup_{T\to\infty} \int_0^T p(t)\left(c'(t) - c_0\right) dt \leq 0$$

since $p(t) > 0$ and $k'(t) \geq 0$ for all $t \geq 0$. Thus, consumption cannot be increased above c_0 for some subinterval without falling below c_0 for some other subinterval, thereby showing that the path with $c(t) = c_0 = f(k_0)$ and $k(t) = k_0 \in (0, k^*)$ forever is efficient. This proves result (i) in the case where $k_0 \in (0, k^*)$.[1]

Result (ii) follows as a corollary, since $c_0 = f(k_0)$ is the maximum sustainable level at $t = 0$. By (1), $c(0) \leq f(k_0)$ is equivalent to $\dot{k}(0) \geq 0$ so that a non-negative net investment indicates sustainability if $k_0 \in (0, k^*]$.

Results (i) and (ii) may seem straightforward. However, it is important to note that they cannot be established in the alternative case where $k_0 > k^*$. If $k_0 > k^*$, then keeping $k(t)$ equal to k_0 forever does not lead to an efficient path. In fact, then it is feasible to maintain a consumption level equal to $f(k^*) > f(k_0)$. Moreover, even this path is wasteful in the sense that consumption can costlessly be raised above this level

[1] The case where $k_0 = k^*$ requires a separate argument which will not be presented here.

for some initial subset of time. Hence, in this case, keeping capital constant leads to an inefficient path. Furthermore, if $c(0) = f(k^*)$, then by (1) and the definition of k^*, consumption is at a sustainable level even though $c(0) > f(k_0)$ and thus $\dot{k}(0) < 0$.

Why does the initial capital stock matter for results (i) and (ii) in the Ramsey technology? The key is the following two properties: There exists an efficient and egalitarian path from k_0 if and only if $k_0 \in (0, k^*]$. Furthermore, if $k_0 \in (0, k^*)$, then a uniform addition to future consumption can be implemented by sacrificing consumption now and thereby accumulating capital. These properties do not hold when $k_0 > k^*$.

To show that also the converse of result (i) holds, note first that $\dot{k}(t) = 0$ at some instant does not imply constant consumption. If a feasible consumption path $\{c(t)\}$ with associated capital path $\{k(t)\}$ is differentiable, then by (1),

$$\dot{c}(t) = f'(k(t))\dot{k}(t) - \ddot{k}(t)\,. \tag{3}$$

Hence, if $\dot{k}(t) = 0$, but $\ddot{k}(t) \neq 0$ so that net investment at time t changes from being positive to negative or vice versa, then $\dot{c}(t)$ will not equal zero. Second, eq. (3) implies that consumption can be held constant for some interval $[0, T]$ even if net investment is not zero, provided that $f'(k(t))\dot{k}(t) - \ddot{k}(t) = 0$ for $t \in [0, T]$. By the definition of $p(t)$, the latter equality is equivalent to holding the present value of net investment constant:

$$0 = \dot{c}(t) = f'(k(t))\dot{k}(t) - \ddot{k}(t) = -\frac{\dot{p}(t)}{p(t)}\dot{k}(t) - \ddot{k}(t) = -\frac{\frac{d}{dt}\big(p(t)\dot{k}(t)\big)}{p(t)}\,.$$

This is a special case of a result sometimes referred to as the Dixit-Hammond-Hoel rule, showing that constant consumption over a subinterval of time does not require zero net investment. However, in the Ramsey technology, following this investment rule with $\dot{k}(t) \neq 0$ for all $t \geq 0$ is either infeasible (if $\dot{k}(0) < 0$) or inefficient (if $\dot{k}(0) > 0$).

In the one-sector model, Hartwick's rule for sustainability prescribes keeping $k(t)$ equal to k_0, and thus having $\dot{k}(t) = 0$. The analysis above has shown both Hartwick's result (HR) and its converse (CHR). Moreover, the analysis implies that HR and CHR are relevant if and only if the initial capital stock does not exceed the golden-rule stock, because otherwise no efficient and egalitarian path exists. It also shows that a non-negative net investment indicates sustainability if the capital stock does not exceed the golden-rule stock, but not otherwise. Hence, even in the simple environment of the Ramsey technology, it is *not* straightforward to indicate sustainability. We will see how these problems are even more profound in less aggregated models.

3 Compensating resource depletion

Turn now to the *Dasgupta-Heal-Solow-Stiglitz* (DHSS) *technology* of capital accumulation and resource depletion, which was the model that John Hartwick orginally used to formulate his rule. Since this is a two capital-good model, there is no aggregate capital stock to be held constant. Instead, we are faced with the questions of how to manage the two stocks in order to implement an efficient and egalitarian path, and how much capital to accumulate along a sustainable path in order to compensate for resource depletion. To answer these questions we need to weigh capital accumulation against resource depletion.

In the DHSS technology a continuous consumption path $\{c(t)\}$ is feasible from initial stocks of capital $k_0 > 0$ and resource $s_0 > 0$ at time 0 if there exists an associated continuously differentiable capital path $\{k(t)\}$ and a continuous resource use path $\{r(t)\}$ such that $c(t) \geq 0$, $r(t) \geq 0$, $k(t) \geq 0$, $s(t) = s_0 - \int_0^t r(\tau)d\tau \geq 0$ and

$$c(t) + \dot{k}(t) = F(k(t), r(t)) \tag{4}$$

for all $t \geq 0$, with $k(t) = k_0$.

Consider a path that is *interior* in the sense that $c(t)$, $k(t)$ and $r(t)$ are all positive for $t \geq 0$. If the consumption path is not efficient, then there are no well-defined scarcity values for capital and resource use, as the same consumption path is feasible even with smaller initial stocks of capital and resource. This is an argument for restricting attention to efficient consumption paths. Hotelling's rule for no-arbitrage is a necessary condition for efficiency along an interior path: $\dot{F}_2(\cdot)/F_2(\cdot) = F_1(\cdot)$, where F_1 and F_2 denote partial derivatives with respect to k and r respectively. Furthermore, along such a path Hartwick's rule for sustainability prescribes keeping the value of net investment, $\dot{k}(t) - F_2(\cdot)r(t)$, equal to zero; that is, letting capital accumulation compensate for resource depletion in terms of scarcity values at each point in time.[2]

The following Buchholz-Dasgupta-Mitra equation establishes the fundamental relationship that holds along an interior path in the DHSS model between constant consumption and Hotelling's and Hartwick's rules:

$$0 = \dot{c}(t) + F_2(\cdot)\frac{d}{dt}\left[\frac{\dot{k}(t)}{F_2(\cdot)} - r(t)\right] + \left[\frac{\dot{F}_2(\cdot)}{F_2(\cdot)} - F_1(\cdot)\right]\dot{k}(t)\,. \tag{5}$$

[2] Note that Hartwick's rule does not prescribe keeping the value of capital constant. By Hartwick's and Hotelling's rules, the rate of change of the real value of capital equals $F_1(\cdot)F_2(\cdot)s(t) > 0$.

Here, Hotelling's rule implies that the right-hand side bracket in eq. (5) is zero while Hartwick's rule imples that the center bracket in eq. (5) is zero. Hence, along an interior and efficient path observing both Hotelling's and Hartwick's rules, consumption is constant.

Provided that Hotelling's rule is satisfied for all $t \geq 0$ along an interior and efficient path, we can determine a continuously differentiable path of positive and decreasing supporting discount factors $\{p(t)\}$ such that $p(t) = 1/F_2(\cdot)$ and $-\dot{p}(t)/p(t) = F_1(\cdot)$ for all $t \geq 0$. By substituting $p(t)$ for $1/F_2(\cdot)$ and $-\dot{p}(t)/p(t)$ for $F_1(\cdot)$ eq. (5) becomes:

$$p(t)\dot{c}(t) = -\tfrac{d}{dt}\big(p(t)\dot{k}(t) + \dot{s}(t)\big)\,. \tag{6}$$

Eq. (6) shows that the Dixit-Hammond-Hoel rule holds also in the DHSS technology: Consumption is constant over a subinterval of time if and only if the present value of net investments is constant. In particular, as in the Ramsey technology, the value of net investments need not be zero.

Still, it is the case that both Hartwick's result (HR) and its converse (CHR) hold also in the DHSS technology. HR follows directly from (5), as efficiency implies that Hotelling's rule is satisfied so that consumption is constant if also $\dot{k}(t) = F_2(\cdot)r(t)$ holds at each point in time. Hence, following Hartwick's rule for sustainability is clearly sufficient for an efficient path to be egalitarian.

To understand that following Hartwick's rule for sustainability is also necessary, it is useful to consider a parameterized version of the DHSS technology. In this technology, net production is a Cobb-Douglas function $F(k, r) = k^{\alpha} r^{\beta}$, where $\alpha > 0$, $\beta > 0$ and $\alpha + \beta \leq 1$. Then there exists an efficient and egalitarian consumption path if and only if $\alpha > \beta$, in which case the maximum sustainable consumption level is given by

$$c_0 = m(k_0, s_0) = (1-\beta)\left[(\alpha-\beta)^{\beta} k_0^{\alpha-\beta} s_0^{\beta}\right]^{\frac{1}{1-\beta}}\,.$$

If on the other hand $\alpha \leq \beta$, then no efficient and egalitarian consumption path exists and the maximum sustainable consumption level equals 0.

If $\alpha > \beta$ and an efficient and egalitarian path is implemented, then Hotelling's rule is satisfied at any point in time, and moreover, the resource is exhausted as time goes to infinity: $\lim_{t\to\infty} \int_0^t r(\tau)d\tau = s_0$. From eq. (6) we know that $p(t)\dot{k}(t) + \dot{s}(t)$ is constant. However, this result does not show that $p(t)\dot{k}(t) + \dot{s}(t)$ is equal to zero, so that Hartwick's rule is followed. So why is Hartwick's rule necessary for an efficient and egalitarian path in the DHSS technology?

[FIGURE 1 ABOUT HERE]

As shown by Figure 1, the efficient and egalitarian path—depicted by the solid line—has the property that accumulated capital compensates for the depleted resource and allows consumption to be maintained at its maximin level c_0. The rate at which capital is accumulated equals $\dot{k}(t) = F(k(t), r(t)) - c_0$ and the rate at which the resource is depleted equals $r(t)$. To follow the solid line and not divert along the dashed line leading to resource exhaustion in finite time, the ratio of capital accumulation and resource depletion must be maximized wrt. r for $t \geq 0$, so that $r(t)$ satisfies:

$$0 = \frac{d}{dr}\left(\frac{F(k(t), r) - c_0}{r}\right) = \frac{F_2(\cdot)}{r} - \frac{F(\cdot) - c_0}{r^2} .$$

As first pointed out by Wolfgang Buchholz this is equivalent to $\dot{k}(t) = F(\cdot) - c_0 = F_2(\cdot)r(t)$, and shows that observing Hartwick's rule for sustainability is necessary for husbanding the finite resource stock! Since by assumption the path is efficient—implying that c_0 is the highest consumption level for which the solid line in Figure 1 asymptotically approaches but never crosses the vertical axis—any diversion from Hartwick's rule would result in resource exhaustion in finite time.

These arguments suggest why Hartwick's rule for sustainability is followed along an efficient and egalitarian path whenever such a path exists, thereby establishing CHR. Hartwick's result and its converse are relevant in the Cobb-Douglas version of the DHSS technology if and only if $\alpha > \beta$. If this condition fails, then no efficient and egalitarian path exists.

Hartwick's result cannot be used to establish the claim that the value of net investments (often referred to as the *genuine savings indicator*) indicates whether present consumption is sustainable. This is easy to see in the Cobb-Douglas version of the DHSS technology with $\alpha \leq \beta$ so that the maximum sustainable consumption level equals 0. There may still exist efficient paths with both $c(t) > 0$ and $\dot{k}(t) > F_2(\cdot)r(t)$ to begin with. Hence, even though the value of net investments is positive, consumption exceeds the maximum sustainable level. As shown in my article on "Net national product as an indicator of sustainability", this negative result carries over to the case where $\alpha > \beta$: The value of net investments can be positive while consumption exceeds the maximum sustainable level even in a model where an efficient and egalitarian path with positive consumption exists. It is true though—as shown by John Pezzey—that, under the assumption that the economy implements a discounted utilitarian optimum, the value of net investments is non-negative if present consumption is sustainable. However, this one-sided test does not apply to any efficient path in the DHSS model.

In relation to these negative results it is worth emphasizing that the present relative scarcity values of different capital stocks depend on the property of the whole future path. The counter-examples in the references of the previous paragraph show how the present scarcity of the resource depends positively on the amount of future capital with which the resource flow will be combined. Thus, the future development—in particular, the distribution of consumption between the intermediate and the distant future—affects the present value of net investments and, thereby, the usefulness of this measure as an indicator of sustainability.

One can argue that the "correct" relative price for indicating sustainability is the ratio of the partial derivatives of $m(k, s)$ with respect to the stocks:

$$m_1(k, s) = \frac{\alpha - \beta}{1 - \beta} \frac{m(k, s)}{k} \quad \text{and} \quad m_2(k, s) = \frac{\beta}{1 - \beta} \frac{m(k, s)}{s} .$$

If $\dot{m}(\cdot) = m_1(\cdot)\dot{k}(t) + m_2(\cdot)\dot{s}(t) = 0$ for all $t \geq 0$, then the path moves along the solid path in Figure 1. However, when Hartwick's rule (prescribing $\dot{k}(t) - F_2(\cdot)r(t) = 0$) is followed along an egalitarian path, then also the speed of the movement is determined. As showed by eq. (5), this ensures that the Hotelling's rule of no-arbitrage is satisfied. Thus, Hartwick's rule is more than keeping $\dot{m}(\cdot)$ equal to zero, since moving along the solid line at a different speed would reduce consumption below the maximum sustainable level. This is an argument for the approach chosen here, where Hartwick's rule is defined as zero net investment in terms of competitive prices.[3] Rather, it is a result that following Hartwick's rule along the efficient and egalitarian path implies that the competitive scarcity value of the resource, $F_2(\cdot)$, in terms of capital coincides with the ratio, $m_2(\cdot)/m_1(\cdot)$, of the partial derivatives of $m(k, s)$.[4]

Note that $\dot{m}(\cdot)$ does not exactly indicate whether present consumption is sustainable: $\dot{m}(\cdot) \geq 0$ is a sufficient but not a necessary condition for present consumption to be sustainable. The reason is the splicing together of the path actually being followed, with the efficient and egalitarian path from then on, need not be efficient. In particular, present consumption is not necessarily maximized subject to rate of growth (or decline), $\dot{m}(\cdot)$, of the sustainable consumption level. Therefore, the sign of $\dot{m}(\cdot)$ is an

[3] This was how Hartwick's rule was formulated in the seminal contributions by Hartwick and Dixit, Hammond and Hoel. There is near consensus for this approach in the subsequent literature.

[4] It can be shown that this property need not hold if Hartwick's rule is followed for $t \in [0, T]$ along an efficient and *non*-egalitarian path.

interesting, but alternative, indicator of sustainability. From a practical point of view, its main drawback is that $m_1(\cdot)$ and $m_2(\cdot)$ are not currently available prices unless an efficient and egalitarian path is actually followed.

Summing up, the analysis of the DHSS technology in this section highlights that Hartwick's rule has three different interpretations:

(i) Hartwick's rule is related to the implementation of an efficient and egalitarian path through HR and CHR.

(ii) Given the consumption level of the efficient and egalitarian path, Hartwick's rule maximizes the ratio of capital accumulation to resource depletion.

(iii) Along the efficient and egalitarian path Hartwick's rule in terms of competitive prices coincides with an alternative formulation of Hartwick's rule in terms of the stocks' marginal contributions to maximal sustainable consumption.

4 Results in a general multiple-capital-good technology

While Hartwick used the DHSS technology to formulate his rule, Dixit, Hammond and Hoel applied a general framework to establish its broad applicability. This section presents their general approach which includes the technologies of Sections 2 and 3—as well as many other models—as special cases.

Let the vector of consumption flows at time t (≥ 0) be denoted $\mathbf{c}(t)$, the vector of capital stocks at time t be denoted $\mathbf{k}(t)$, and the vector of investment flows at time t be denoted $\dot{\mathbf{k}}(t)$. Here, consumption includes both ordinary material consumption goods, as well as environmental amenities, while the vector of capital stocks comprises not only different kinds of man-made capital, but also stocks of natural capital and stocks of accumulated knowledge. Let $\mathbf{k}_0$ denote the initial stocks at time 0.

The *technology* is described by a time-independent set $\mathcal{F}$. The triple $(\mathbf{c}(t), \mathbf{k}(t), \dot{\mathbf{k}}(t))$ is attainable at time t if $(\mathbf{c}(t), \mathbf{k}(t), \dot{\mathbf{k}}(t)) \in \mathcal{F}$. A continuous consumption path $\{\mathbf{c}(t)\}$ is feasible from $\mathbf{k}_0$ if there is an associated continuously differentiable capital path $\{\mathbf{k}(t)\}$ such that $\mathbf{c}(t) \geq 0$, $\mathbf{k}(t) \geq 0$ and $(\mathbf{c}(t), \mathbf{k}(t), \dot{\mathbf{k}}(t))$ is attainable for $t \geq 0$, with $\mathbf{k}(0) = \mathbf{k}_0$.

Assume that there is a constant population, where each generation lives for one instant. Hence, generations are not overlapping nor infinitely lived, implying that any intertemporal issue is of an intergenerational nature. The vector of consumption goods generates well-being, $u(\mathbf{c})$, where u is a time-invariant, strictly increasing, concave, and differentiable utility function. Write $u(t) = u(\mathbf{c}(t))$ for well-being at time t.

Assume that there are market prices for all consumption goods and capital goods. The discussion of HR and CHR is facilitated by using *present value prices*; i.e. deflationary nominal prices that correspond to a zero nominal interest rate. Hence, prices of future deliveries are measured in a numeraire at the present time. Let the vector of present value prices of consumption flows at time t be denoted $\mathbf{p}(t)$, and the vector of present value prices of investment flows at time t be denoted $\mathbf{q}(t)$. It follows that $-\dot{\mathbf{q}}(t)$ is the vector of rental prices for capital stocks at time t, entailing that $\mathbf{p}(t)\mathbf{c}(t) + \mathbf{q}(t)\dot{\mathbf{k}}(t) + \dot{\mathbf{q}}(t)\mathbf{k}(t)$ can be interpreted at the instantaneous profit at time t.

A feasible path $\{\mathbf{c}(t), \mathbf{k}(t)\}$ is *competitive* at discount factors $\{\mu(t)\}$ and prices $\{\mathbf{p}(t), \mathbf{q}(t)\}$ if $\mu(t) > 0$, $(\mathbf{p}(t), \mathbf{q}(t)) \geq 0$ and the following conditions are satisfied for $t \geq 0$.

Instantaneous *well-being* is maximized subject to a budget constraint:
$\mathbf{c}(t)$ maximizes $\mu(t)u(\mathbf{c}) - \mathbf{p}(t)\mathbf{c}$. (C1)

Instantaneous *profit* is maximized subject to the technological constraint:
$(\mathbf{c}(t), \mathbf{k}(t), \dot{\mathbf{k}}(t))$ maximizes $\mathbf{p}(t)\mathbf{c} + \mathbf{q}(t)\dot{\mathbf{k}} + \dot{\mathbf{q}}(t)\mathbf{k}$ subject to $(\mathbf{c}, \mathbf{k}, \dot{\mathbf{k}}) \in \mathcal{F}$. (C2)

Hartwick's rule for sustainability in this general setting becomes: $\mathbf{q}(t)\dot{\mathbf{k}}(t) = 0$.

It can be shown that the result of eq. (6) generalizes to competitive paths in this general technology:

$$\mu(t)\dot{u}(t) = \mathbf{p}(t)\dot{\mathbf{c}}(t) = -\tfrac{d}{dt}\big(\mathbf{q}(t)\dot{\mathbf{k}}(t)\big) \,. \tag{7}$$

This results means that the Dixit-Hammond-Hoel rule carries over: Well-being is constant if and only if the present value of net investments is constant. Since, essentially, competitiveness is a necessary condition for efficiency, eq. (7) establishes Hartwick's result in this multiple-capital-good technology: If along an efficient path Hartwick's rule is followed forever, then an egalitarian path is implemented.

To establish the converse of Hartwick's result is a more delicate task. For this purpose, say that a competitive path $\{\mathbf{c}(t), \mathbf{k}(t)\}$ is a *regular maximin path* at discount factors $\{\mu(t)\}$ and prices $\{\mathbf{p}(t), \mathbf{q}(t)\}$ if

$$u(t) = u_0 \text{ (constant) for } t \geq 0 \,, \tag{R1}$$

$$\int_0^\infty \mu(t)dt < 0 \,, \tag{R2}$$

$$\lim_{t\to\infty} \mathbf{q}(t)\mathbf{k}(t) \to 0 \,. \tag{R3}$$

Here (R1) requires that the path is egalitarian, (R2) signifies that a uniform addition to future well-being can be implemented by sacrificing well-being now, while (R3) ensures that capital stocks are not over-accumulated, in particular, non-renewable resources without stock value are asymptotically exhausted. It can be shown that a regular maximin path is efficient.

In certain technologies like the Cobb-Douglas version of the DHSS technology of Section 4, all efficient and egalitarian paths are regular maximin paths. For such technologies, the CHR was shown by Cees Withagen and myself.

For other technologies, there may exist efficient and egalitarian paths which are not regular maximin paths. This is indeed the case with the golden-rule path from $k_0 = k^*$ in the Ramsey technology of Section 2. A version of CHR that also includes such cases was established by Tapan Mitra. His proof is based on the result that, for any time T along an efficient path $\{\mathbf{c}(t), \mathbf{k}(t)\}$ with competitive capital prices $\{\mathbf{q}(t)\}$, maintaining $u'(t) \geq u(t)$ from T onwards requires a vector of capital stocks $\mathbf{k}'(T)$ at time T which, when measured in the competitive prices $\mathbf{q}(T)$, costs as much as $\mathbf{k}(T)$: $\mathbf{q}(T)\mathbf{k}'(T) \geq \mathbf{q}(T)\mathbf{k}(T)$. Hence, along an egalitarian path where $u(t)$ is constant, $\mathbf{q}(T)\mathbf{k}(t)$ is minimized at T among all t in a neighborhood of T. Then Hartwick's rule, $\mathbf{q}(T)\dot{\mathbf{k}}(T) = 0$, is simply the necessary first-order condition of such a minimum.

The reservations of earlier sections—concerning the relevance of Hartwick's rule and its validity and usefulness as a sustainability indicator—are of course as pertinent in this more general environment.

5 Generalizations

The previous section has served to make explicit three assumption underlying Hartwick's result and its converse: (i) Constant technology, (ii) constant population, and (iii) implementation of a competitive path. Without a constant technology and a constant population, the setting is not stationary so that zero value of net investments is not appropriate for conserving per capita productive capacity. If the path is not competitive, then the path is inefficient and accounting prices for the stocks cannot be derived from preferences and technological considerations alone. Can HR and CHR be generalized to a situation where these assumptions are not fulfilled?

The assumption of constant technology corresponds to the time-independency of the set $\mathcal{F}$. It means that all technological progress is endogenous, being captured by accumulated stocks of knowledge. If there is exogenous technological progress in the sense of a time-dependent technology, we may capture this within the formalism of the previous section by including time as an additional stock: The triple $(\mathbf{c}(t), \mathbf{k}(t), \dot{\mathbf{k}}(t))$ is attainable at time t if $(\mathbf{c}(t), (\mathbf{k}(t), t), (\dot{\mathbf{k}}(t), 1)) \in \mathcal{F}$. This formulation, which has been widely applied, leads however to the challenge of calculating the present value price associated with the passage of time through a forward-looking term.

When applying Hartwick's rule in open economies, changing terms-of-trade leads to time-dependence. Hotelling's rule suggests that terms-of-trade will improve for resource-exporting countries and deteriorate for resource-importing countries. If Hartwick's rule is generalized to open economies in the context of the Cobb-Douglas version of the DHSS technology, then it follows that it is the resource-consuming—not the resource-producing—countries' responsibility to reinvest resource rents.

The case where population is exponentially increasing instead of constant can be handled in a straightforward manner in certain models, like the Ramsey technology of Section 2. By assuming that there is an underlying constant-returns-to-scale production function of capital and labor and appropriately re-defining the rate of depreciation δ, the model can be interpreted in per-capita terms. Thus, maintaining a constant per-capita consumption along an efficient path can be associated with keeping per-capita capital constant. However, in other models, like the Cobb-Douglas version of the DHSS technology, exponential population growth is incompatible with the existence of an efficient and egalitarian path.

Non-exponential population growth is a source of non-stationarity even when models are interpreted in per-capita terms. This makes the formulation of a version of Hartwick's rule for sustainability in such a context an interesting challenge, but no general analysis seems to have been published yet. Results from green national accounting under non-exponential exogenous population growth may provide a useful basis for these investigations.

In the real world, environmental externalities are not always internalized. This is one of many causes that prevent real economies from being competitive. Furthermore, for many capital stocks (e.g. stocks of natural and environmental resources or stocks of accumulated knowledge) it is hard to find market prices (or to calculate accounting prices) that can be used to estimate the value of such stocks. It seems challenging to formulate Hartwick's rule in a setting without competitive prices.

One approach, suggested by Arrow, Dasgupta and Mäler, is to assume that the economy's actual decisions are taken according to a possibly inefficient *resource allocation mechanism* that assigns some attainable consumption-net investment pair to any vector of capital stocks. Combined with some welfare objective (e.g. maximizing a discounted utilitarian welfare function) one can in principle estimate accounting prices for the investment flows and thereby calculate the social value of net investments. However, one cannot assume that this indicator will be zero along the implemented path, unless a maximin objective is implemented. Moreover, the social value of net investment calculated according to a discounted utilitarian objective may well be positive even if resource depletion seriously undermines the long-run livelihood of future generations, so that current well-being far exceeds the level that can be sustained forever.

Lastly, note that HR and CHR are results obtained in continuous-time models; as shown by Swapan Dasgupta and Tapan Mitra, the generalization to discrete-time models is not straightforward.

6 Concluding remarks

What is the status of Hartwick's rule for sustainbility 35 years after John M. Hartwick's original contribution appeared?

While it is a robust result that Hartwick's rule characterizes efficient and egalitarian paths, it has proven to be an elusive goal to be able to indicate sustainability by the value of net investments as a genuine savings indicator. The value of net investments may be positive even though no positive level of consumption can be sustained. Moreover, even when a positive consumption level is sustainable, neither the scarcity values along the path that the economy actually implements nor the scarcity values along an hypothetical efficient and egalitarian path will produce exact indicators of sustainability. Finally, current markets may not correctly forecast the real scarcity of different capital and resource stocks, in which case the implemented path is not even efficient.

From this it follows that Hartwick's result (and its converse) essentially constitutes a valuable characterization of an efficient and egalitarian path rather than establishes the basis for a useful prescriptive rule for sustainability.

Further reading

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Author biography and photograph

Author biography

After studying mathematics and economics at the University of Bergen, Norway, Geir B. Asheim received his PhD in economics at University of California, Santa Barbara. Since 1994 he has been professor of economics at the University of Oslo. He has had longer visits to several US universities, including Cornell, Harvard, Northwestern and Stanford. His main research fields are: (1) Game theory, in which he has published the book *The Consistent Preferences Approach to Deductive Reasoning in Games* (2006), in addition to a number of journal articles (one awarded the Royal Economic Society Prize). (2) Intergenerational equity, in which he has published numerous articles during the last 25 years. Geir B. Asheim is currently working on axiomatic analysis of intergenerational equity, motivated by the need to resolve the intergenerational conflict that climate change leads to.

Photograph

Available at:
http://www.sv.uio.no/econ/personer/vit/gasheim/gasheim.jpg?vrtx=view-as-webpage

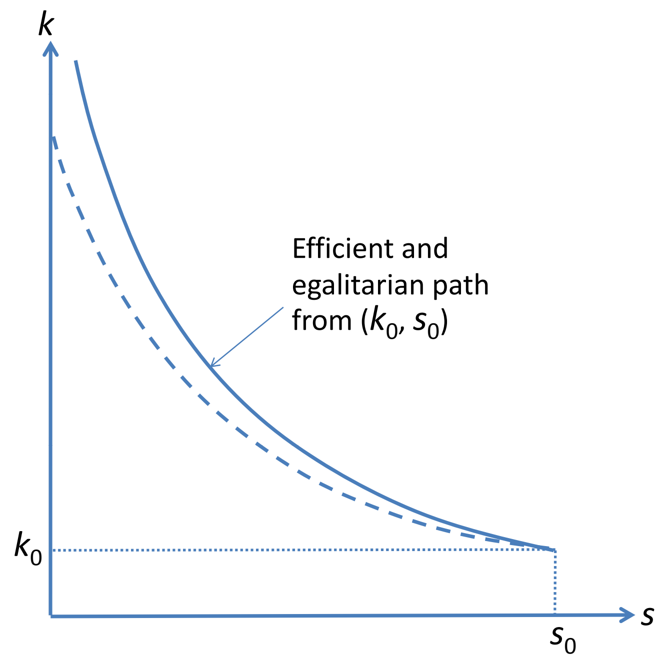

Figure 1: Maximizing the capital accumulation/resource depletion ratio