Fair Competition Design*

Ritxar Arlegi
Department of Economics
and
Institute for Advanced Research in Business and Economics (INARBE)
Public University of Navarre, Spain
E-mail: rarlegi@unavarra.es

Dinko Dimitrov
Chair of Economic Theory, Saarland University, Germany
E-mail: dinko.dimitrov@mx.uni-saarland.de

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Abstract

We study the impact of two basic principles of fairness on the structure of competition systems and perform our analysis by focusing on sports competitions. The first principle states that equally strong players should have the same chances of being the final winner, while the second principle requires that the competition system should not favor weaker players. We apply these requirements to a class of competitions which includes, but is not limited to, the sport tournament systems that are most commonly used in practice, such as round-robin tournaments and different kinds of knockout competitions, and we characterize the structures satisfying these requirements.

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In our results, a new competition structure that we call an *antler* is found to play a referential role. Finally, we show that the class of fair competition systems becomes rather small when both fairness principles are jointly applied.

JEL Classification: D00, D02, D60, D63, D70, Z20
Keywords: equal treatment, monotonicity, seeding, sport competition

1 Introduction

Numerous decision problems require the selection of an alternative from a set of options on the basis of the information obtained from pairwise comparisons among the available alternatives. Examples of these problems can be found in voting theory (cf. Brams and Fishburn 2002, Laslier 1997, Levin and Nalebuff 1995, Moulin 1986), multi-criteria decision making (Larichev 2001, Olson 1996), and promotion mechanisms implemented in firms (Rosen 1986). However, the most popular problem of this type is probably that of selecting a winner in a sport competition, where the alternatives are the competing "players" and the pairwise comparisons take the form of "matches".

In this work, we present an axiomatic approach to the fairness aspect of these kinds of selection problems. Although our analysis is potentially applicable in different situations, in this work we frame the study in the context of sport competitions. Our interest in this field goes beyond its use as a mere theoretical parable given the enormous economic and social relevance that the sport industry has nowadays.

Every sport competition needs a well-defined and pre-established set of basic rules that determines the "competition system": who plays against whom, at which stage of the competition, and how the final winner is decided. A competition organizer can plausibly consider different objectives when designing the competition system, such as the intensity of the matches, suspense, attracting the interest of the spectators, optimizing organizational costs, and so on. However, fairness is always a top priority within the goals of any competition designer.

Discussions about whether one or another system is more or less fair than another are often made at an intuitive and informal level. In our work, we provide a structured analysis of such discussions and we present a rigorous concept of "fair competition design". In doing so, we formally define two neat principles of fairness that respond to what is commonly pursued in real practice and translate to our context the Aristotelian Justice Principle of "treating equals equally and unequals unequally". In our framework, these principles require that the competition system should not favor weaker players, on the one hand, and that equally strong players should have the same chances of being the final winner, on the other hand. We then study to what extent different competition systems perform in relation to these principles, trying to give formal support to such informal debates. The systems that we consider can be roughly partitioned into two major classes: elimination-type competitions and league-type competitions.

In *elimination-type competitions*, which are also called "knockout tournaments", or "playoff tournaments", the competition is organized in rounds or "stages". Losers are eliminated and players progress as they win their corresponding matches in the round, being paired off in the next round, so that the final winner is the player who wins all the rounds. These competitions can be represented by binary trees, as exemplified in Figure 1.

Elimination-type competitions, such as the one displayed in Figure 1, are called *balanced* because every payer is required to win the same number of matches to become the final winner. In some cases, such as in the American National Football League, players have the right of "byes"; meaning that, on the basis of a previous qualification rating, they have the privilege to skip the initial round (or rounds) without the need of playing. In fact, "byes" becomes necessary if the number of players is not a power of 2. A special type of elimination competition with byes has a so-called "stepladder" structure (see Figure 2). This system and its variants are used in ten-pin bowling and squash, for example.

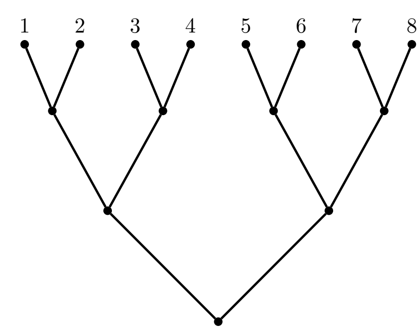


Figure 1: A balanced elimination-type competition

The design of an elimination-type competition requires a solution to the problem of "seeding"; that is, of assigning players' names to the "leaves" of the competition's tree. This involves deciding the pairing in the initial matches and, if that is the case, which player(s) deserve(s) the byes. Clearly, the seeding will have a crucial impact on the chances for a player to become the final winner.

In a *league-type competition* every player participates in a given number of matches against other players. A certain number of points is assigned to the winner of each match, and the final winner is the player with the highest score of points. The most widely used league-type competitions are “single round-robin tournaments”, where each player plays against every other player once, and “double round-robin tournaments”, where each player plays twice against every other player.

Each type of competition system has its pros and cons, related for instance with the number of matches needed to have a final winner, organizational costs, profitability for the organizer, the possibility for inconsequential matches to be played, or the manipulability by the players. While admitting the importance of all of these issues, in this work we exclusively concentrate on the analysis of competition systems from the point of view of the fairness that is intrinsically associated to their structure.

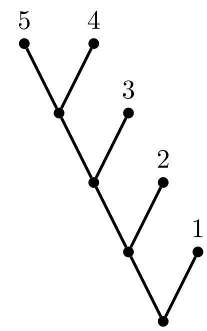


Figure 2: A stepladder competition

Literature overview

The literature about fairness in competition systems is rather disseminated. Related works usually study particular competition systems and fairness aspects related with their specificities. Most of the attention in this respect has been paid to how alternative seeding procedures in an elimination-type competition perform according to different properties, the latter being generally related with the idea of favouring the stronger players. For example, Horen and Riezman (1985) analyse balanced elimination-type competitions with four players, Hwang (1982) consider the eight-players case, Prince et al. (2013) analyse the eight-players and 16-players cases, and Schwenk (2000) looks at the general case under a special form of random seeding. Ely et al. (2015) study the performance of a stepladder competition with three players focusing on maximizing suspense and surprise.

In the case of round-robin tournaments, Briskorn and Knust (2010) study properties with a fairness flavour in relation to the schedule of the rounds. Moon and Pullman (1970) concentrate on “equalizing” handicapping methods. Rubinstein (1980) shows axiomatically that the point system used in round-robin competitions is the only one that satisfies three axioms inspired from social choice theory. Levin and Nalebuff (1995) make an analogy between round-robin punctuation systems and voting systems. Finally, fairness in sports has also been analyzed with respect to tie-breaking mechanisms (cf. Apesteguía and Palacios-Huerta 2010 and Che and Hendershott 2008).

There is a line of research that introduces exertion of effort as a strategic variable (cf. Rosen 1986, Groh et al. 2012, Krumer et al. 2017, and Pauly 2014) and which studies the possibility for manipulation by players. The analysis has always been made for a small number of players (usually four) because it is generally accepted that the extension to a larger number of players involves an excessive complexity due to the highly complicated combinatorial structure of the problem.

As to the comparison among different competition systems, most studies apply statistical simulation techniques to check the fulfilment of particular properties, or how the competition systems perform according to particular metrics (cf. Appleton 1995, McGarry and Schutz 1997, Scarf et al. 2009, Ryvkin and Ortmann 2008, and Ryvkin 2010). There is also a considerable body of literature in operations research related with sports. These works include many aspects, aside from fairness, that are out of the scope of this work. The interested reader is referred to Wright (2014) and Kendall et al. (2010) for surveys in the field.

Our contribution

We formalize these two basic ideas of fairness by means of two simple axioms that have a "rank-preserving" flavour: an "equal treatment" requirement which states that "equally strong players should have the same probability of being the final winner", and a "monotonicity in strength" condition requiring that "a weaker player should not have a higher probability of being the final winner than a stronger player".

Each of these axioms is presented in a strong form and in a weak form. The corresponding strong versions impose that a competition system should fulfill the property for every possible assignment (or seeding) of the players in the system, while the corresponding weak forms only requires the fulfilment of the property for at least one assignment of the players.

Our results include characterizations of the competition systems satisfying these fairness properties. In the case of the weak versions of the axioms, we also specify the class of seeding rules that let the structures satisfy the axioms. Generally speaking, *equal treatment* leads to balanced competitions in which every player participates in the same number of matches (Theorem 1, Theorem 4, and Theorem 5), while *monotonicity in strength* drastically restricts the number of players in the competition to two (Theorem 2 and

Theorem 6). When *weak monotonicity in strength* is under consideration, the class of competition systems fulfilling it increases. For instance, elimination-type competitions turn out to only be weakly monotonic in strength if the tree structure representing the competition does not contain a special substructure, which we call an "*antler*" (Theorem 3). This structure combines the characteristics of balanced elimination competitions with the "byes" spirit of stepladders. Moreover, we show that the seeding rule for which an antler-free competition satisfies weak monotonicity in strength is unique.

The rest of the work is organised as follows. Section 2 presents the basic elements of the formal model. Section 3 introduces the four fairness axioms. Sections 4 and 5 are devoted to the characterization results with respect to elimination-type and league-type competitions, respectively. Section 6 concludes and addresses possible extensions of the model. All of the omitted proofs are collected in Appendix A (for elimination tournaments) and in Appendix B (for leagues).

2 The model

The main ingredients of our model are the graph representation of a competition system, the description of players' strength in terms of winning probability matrices, the notion of a seeding rule and the probability for each player to be the final winner as a consequence of all the previous elements.

Graph representation of competition systems

We assume that matches always take place between two players in such a way that ties are not possible and we represent a competition system by means of a graph. In the case of elimination-type competitions, each match is represented by an *elementary binary tree*; that is, a graph with three nodes $\{a, b, w\}$ and two links $\{aw, bw\}$ with, let us say, player i being assigned to node a, player j being assigned to node b, and the winner of the match between i and j being assigned to node w. In this case, we say that i *is matched with* j.

Elimination-type competitions can then be represented by a graph connecting in a specific way such elementary binary trees, forming a binary tree with a finite number of nodes, such as those in Figures 1 and 2.

To perform an adequate comparative study, we represent league-type competitions consistently with the previously described representation of elimination-type competitions. In particular, league-type competitions will be represented by a collection (or *forest*) of disconnected elementary binary trees, each of them representing a match of the league (see Figure 3).[1]

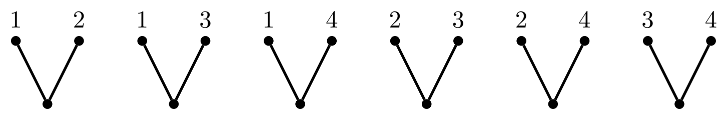


Figure 3: A league-type competition

Given a graph G of any of the two types described above, we denote by $t \in G$ a particular binary tree of the graph, and by $\#G$ the number of trees that G contains. The set of nodes (or *vertices*) of a binary tree $t \in G$ is denoted by $V(t)$. The set of leaves (or *terminal nodes*) of $t \in G$ is denoted by $\Lambda(t)$, $\Lambda(t) \subset V(t)$. The notation for the set of leaves of G is $\Lambda(G)$.

The distance between two nodes of $t \in G$ is defined by the minimal number of edges that are necessary to connect them. The *level* $\ell(v)$ of a node $v \in V(t)$ is the distance between it and the root of the binary tree t. The k-th level of a tree t is the set of all nodes of the tree of level k.[2] The *height* $h(t)$ of a binary tree t is the maximal level of its leaves, $h(t) = max_{\lambda \in \Lambda(t)}\{\ell(\lambda)\}$. By $\Lambda^k(t)$ we denote the set of leaves of t whose level is k. We say that a binary tree $t \in G$ is *balanced* (or that it *represents a balanced competition*) if the level of all of its leaves is the same. Notice that a *stepladder* competition is represented by a binary tree t with two leaves at level $h(t)$ and a unique leaf at each level ℓ for all $\ell < h(t)$.

Players' strength and winning probabilities

Let N be the finite set of competing players. We assume that the elements of N are completely ordered according to a binary relation R of *strength* so that, for all $i, j \in N$, iRj is interpreted as *"player i is at least as strong as player j"*. The corresponding asymmetric and symmetric factors of R are denoted, respectively, by P and I, so that iPj reads *"i is strictly stronger than j"* and iIj reads *"i and j are equally strong"*.

[1]The figure represents a single round-robin competition among four players. As we will see later, our definition of a league-type competition allows, and the corresponding results account for the possibility for players to participate in different numbers of matches.

[2]The different levels of a tree are usually interpreted as *rounds* of the competition. However, these are usually numbered in inverse terms; that is, the last level of the tree constitutes the first round of the competition, the second last level constitutes the second round and so on.

We attach a probabilistic meaning to the binary relation of strength in the sense that iRj presupposes that "the probability with which player i defeats in a match player j is greater than or equal to 0.5". We denote this fact by $p_{ij} \geq 0.5$. Given that R is complete, we have that iPj is accordingly interpreted as $p_{ij} > 0.5$ and iIj is interpreted as $p_{ij} = 0.5$. Throughout the next sections, we take the non-deterministic view that $0 < p_{ij} < 1$ holds for all $i, j \in N$. This assumption is taken simply to show that the presence of deterministic values is not what makes the different theorems and lemmas hold in a trivial way. It is easy check that all results also hold for the case of $p_{ij} \in [0, 1]$ for all $i, j \in N$. We adopt the convention that the players in N are ordered according to R; that is, if iPj then $i < j$ (if iIj then either $i < j$ or $j < i$).

According to this interpretation, every binary relation of strength R induces a set of *winning probability matrices*, $\mathcal{P}_R$, defined on $N \times N$ that *support* (or are compatible with) R. More precisely, $\mathcal{P}_R$ is the set of all probability matrices such that, for $\mathbf{p} \in \mathcal{P}_R$, we have that $p_{ij} \geq 0.5$ if and only if iRj. The fairness properties that we consider are required to be fulfilled for each probability matrix $\mathbf{p} \in \mathcal{P}_R$ given a binary relation R, so that the particular details of $\mathbf{p}$ are not needed for the results.

Following the related models (cf. David 1963, Hwang 1982, Horen and Riezmann 1985, and Schwenck 2000), we also assume that, given R, every probability matrix $\mathbf{p} \in \mathcal{P}_R$ satisfies the following two conditions:

$$\forall i, j \in N, \; p_{ij} + p_{ji} = 1. \tag{1}$$

$$\forall i, j \in N, \; p_{ij} \geq 0.5 \text{ implies } p_{ik} \geq p_{jk} \text{ for each } k \in N \setminus \{i, j\}. \tag{2}$$

The interpretation of (1) is straightforward. Condition (2) simply expresses the fact that any player defeats with higher probability a weaker player than a stronger player. It also implies that if two players are equally strong ($p_{ij} = 0.5$), then they should defeat with equal probability any third player.

Conditions (1) and (2) are equivalent to what is sometimes referred as "strong stochastic transitivity" of the representing probability matrix (cf. David 1963). If players are displayed in the matrix according to their strength, then strongly stochastically transitive matrices are nondecreasing in rows, nonincreasing in columns and, whenever $p_{ij} = 0.5$, the corresponding rows and columns of i and j are equal.

As the reader can easily see, if a probability matrix $\mathbf{p} \in \mathcal{P}_R$ satisfies the above two conditions (as we assume), then the binary relation R is transitive. Moreover, for $a, b, c, d \in N$ we have that

$$aRbRcRd \text{ implies } p_{ad} \geq p_{bc}. \tag{3}$$

This fact is frequently used in the proofs to follow.

Seeding and the probability of being the final winner

Given a finite set N of competing players and a graph G of the type discussed above, a *seeding rule* is a function $s : \Lambda(G) \to N$ that assigns players of N to the leaves of G. When $s(\lambda) = i$ holds for $\lambda \in \Lambda(G)$ and $i \in N$, we say that *"player i is assigned, or "seeded", to leaf λ"*. We assume that any such rule satisfies the following two properties: (1) s is a surjective function—that is, every player is seeded to at least one leaf in G; and, (2) the restriction of s to any binary tree $t \in G$ is injective—that is, no player from N is seeded to more than one leaf of t but there could be players that are not seeded to a leaf of t when $\#G > 1$.

When a seeding rule s satisfies these conditions, we will say that s is a *feasible* seeding for G. As a consequence of the two assumptions above, we have that if s is feasible for $G = \{t\}$, then each player in N is seeded to exactly one leaf of G and $|\Lambda(G)| = |N|$. Moreover, we have $2 \leq |N| \leq |\Lambda(G)|$ whenever $\#G > 1$ holds.[3]

At this point, we can define a *competition system* as a pair (G, N) consisting of the graph that represents the structure of the matches of the competition and the set N of players to be seeded to the leaves of the graph. We say that (G, N) is *admissible* if it is possible to define a feasible seeding rule for it. The set of all feasible seeding rules for a competition system (G, N) will be denoted by $\mathcal{S}^{(G,N)}$.

A seeding rule $s \in \mathcal{S}^{(G,N)}$ determines the set of potential matches that can be played at each round. Moreover, if a probability matrix $\mathbf{p}$ is given, then the set of potential matches at each round is endowed with a probability distribution. Then, given (G, N) and $s, s' \in \mathcal{S}^{(G,N)}$, we say that s and s' are *equivalent* with respect to $\mathbf{p}$ if the probability distribution associated with the set of potential matches at each round for s and for s' is the same. For instance, Figure 4 represents, for a balanced binary tree of height 2, a

[3] "Seeding" is a term typically used in elimination-type competitions but not in leagues. For consistency reasons, we set our definition of a seeding rule to apply for both cases.

situation where the two left seedings are equivalent but none of these two seedings is equivalent to the right one.

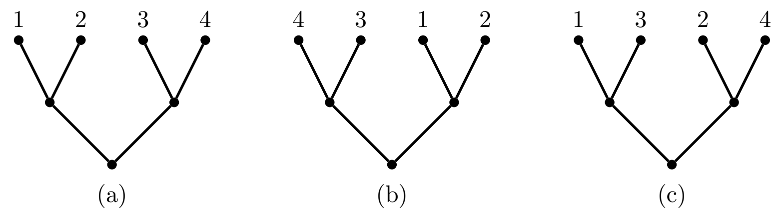


Figure 4: The seedings in (a) and (b) are equivalent, while those in (a) and (c), and in (b) and (c) are not

Given a competition system (G, N), we denote by w_t the player who reaches the root of the binary tree $t \in G$ and say that $i \in N$ is the *winner of the competition* (G, N) if $|\{t \in G : w_t = i\}| \geq |\{t \in G : w_t = j\}|$ holds for all $j \in N$. Obviously, if $G = \{t\}$, then the player who reaches the root of t is the winner of the competition.

Given a competition system (G, N), a seeding rule $s \in \mathcal{S}^{(G,N)}$and a probability matrix $\mathbf{p}$, we denote by $\varphi_i(G, s, \mathbf{p})$ the probability with which player $i \in N$ will be the final winner of the competition. The probability with which $i \in N$ reaches the root of a particular tree $t \in G$ is analogously denoted by $\varphi_i(t, s, \mathbf{p})$. Obviously, $G = \{t\}$ implies $\varphi_i(G, s, \mathbf{p}) = \varphi_i(t, s, \mathbf{p})$ for each $i \in N$. Finally, if s and s' are equivalent seeding rules, then $\varphi_i(G, s, \mathbf{p}) = \varphi_i(G, s', \mathbf{p})$ holds for each $i \in N$.

3 Fairness axioms

We introduce now in a formal way the two previously mentioned fairness principles. Each of these ideas is presented in a strong form and in a weak form. To state them, we assume that a binary relation R of strength is defined on the player set N.

Equal Treatment (ET) A competition system (G, N) satisfies ET if for all $s \in \mathcal{S}^{(G,N)}$, iIj for all $i, j \in N$ implies $\varphi_i(G, s, \mathbf{p}) = \varphi_j(G, s, \mathbf{p})$ for all $i, j \in N$.

Weak Equal Treatment (WET) A competition system (G, N) satisfies WET if there exists $s \in \mathcal{S}^{(G,N)}$ such that iIj for all $i, j \in N$ implies $\varphi_i(G, s, \mathbf{p}) = \varphi_j(G, s, \mathbf{p})$ for all $i, j \in N$.

Monotonicity in Strength (MS) A competition system (G, N) satisfies MS if for all $s \in \mathcal{S}^{(G,N)}$, for all $i, j \in N$, and for all $\mathbf{p} \in \mathcal{P}_R$ such that $p_{ij} > 0.5$, $\varphi_i(G, s, \mathbf{p}) \geq \varphi_j(G, s, \mathbf{p})$ holds.

Weak Monotonicity in Strength (WMS) A competition system (G, N) satisfies WMS if there exists $s \in \mathcal{S}^{(G,N)}$ such that, for all $i, j \in N$ and for all $\mathbf{p} \in \mathcal{P}_R$ such that $p_{ij} > 0.5$, $\varphi_i(G, s, \mathbf{p}) \geq \varphi_j(G, s, \mathbf{p})$ holds.

ET and WET express the idea that, as for the final probability of winning, the competition system should not be biased towards any particular player if all of them are equally skilled.

MS and WMS require the competition system not to benefit weaker players under any of the possible probability matrices compatible with the strength of the players. In fact, many competitions are precisely designed to avoid that worse teams win by luck: for example, round-robin tournaments and even double round-robin tournaments minimize such an effect with a high number of matches, stepladder competitions seem to be precisely aimed to benefit better players, best players are matched with the worst ones in knockout competitions, or sometimes matches (usually finals) consist of a higher number of legs at the better competitor's home, such as in basket.

Apart of the fact that WET is logically weaker than ET and WMS is logically weaker than MS, the normative power of the weaker versions versus the strong versions may depend on the concrete intended application, and in particular on the conjectures about the benevolence of the competition designer. On the one hand, ET and MS avoid the possibility of manipulation by a potentially corrupted competition designer because they ensure that there is no possibility of finding any particular seeding rule that benefits a particular player in relation with another one who is more or equally skilled. On the other hand, WET and WMS rely on the confidence in the benevolence of the competition designer, in the sense that the focus is on competition systems where he or she can always find a seeding rule that is fair, independently of the values in the probability matrices supporting the strength relation.

4 Elimination-type competitions

This section is divided in two subsections. Subsection 4.1 includes our results concerning the two equal treatment axioms (ET and WET) and the MS axiom. Subsection 4.2 contains several definitions and preliminary results that end with a characterization of the class of elimination-type competitions that satisfy the WMS axiom. Given that an elimination-type competition is represented by a unique tree, we will denote it throughout the section by the pair (t, N), where N is the fixed player set and t is the single binary tree.

4.1 Equal treatment and monotonicity in strength

The first result in this subsection connects the stronger version the equal treatment axiom with the class of balanced competition systems.

Theorem 1 *An elimination-type competition system* (t, N) *satisfies ET if and only if* t *is balanced.*

Proof. Let (t, N) be an elimination-type competition system with t being balanced and let $s \in \mathcal{S}^{(t,N)}$ be an arbitrary but fixed seeding rule. Given the balancedness of t, any of its leaves has the same level coinciding with $h(t)$. Then, by $p_{ij} = 0.5$ for all $i, j \in N$, $\varphi_i(t, s, \mathbf{p}) = (0.5)^{h(t)}$ holds for each $i \in N$. Thus, (t, N) satisfies ET.

Suppose now that (t, N) is an elimination-type competition system satisfying ET. Let $s \in \mathcal{S}^{(G,N)}$ be an arbitrary but fixed seeding rule. Suppose that iIj for all $i, j \in N$ but t is not balanced. We then have that $p_{ij} = 0.5$ holds for all $i, j \in N$. Given that t is not balanced there are leaves $\lambda, \lambda' \in \Lambda(t)$ with $\ell(\lambda) \neq \ell(\lambda')$. It follows then that $\varphi_{s(\lambda)}(t, s, \mathbf{p}) = (0.5)^{\ell(\lambda)} \neq (0.5)^{\ell(\lambda')} = \varphi_{s(\lambda')}(t, s, \mathbf{p})$ in contradiction to (t, N) satisfying ET. ■

Notice that the argument used in the proof above is independent of the characteristics of the seeding rule. Hence, we can immediately conclude that no additional competition systems emerge when ET is replaced by its weak version WET.

Corollary 1 *An elimination-type competition system* (t, N) *satisfies WET if and only if* t *is balanced.*

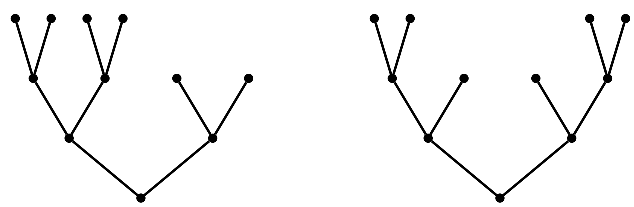
We call a (balanced) competition system *minimal* if there are only two participants in it. As it turns out, only minimal competitions satisfy MS.

Theorem 2 *An elimination-type competition system* (t, N) *satisfies MS if and only if it is minimal.*

A minimal elimination-type competition clearly consists of a unique match. Thus, Theorem 2 can be seen as having a clear flavour of an impossibility result.

4.2 Antler-free competitions and weak monotonicity in strength

When WMS is imposed instead of the strong version, we obtain a characterization of a considerably richer class of elimination-type competitions. For this characterization, a special type of binary tree (antlers) needs to be introduced. We say that a binary tree t with $h(t) = 3$ is (1) an *antler*, if $|\Lambda(t)| = 6$ with $|\Lambda^3(t)| = 4$ and $|\Lambda^2(t)| = 2$; (2) an *asymmetric antler*, if t is an antler with the leaves in $\Lambda^2(t)$ having a common immediate predecessor; and (3) a *symmetric antler*, if t is an antler with the leaves in $\Lambda^2(t)$ having distinct immediate predecessors. Figure 5(a) displays an asymmetric antler and Figure 5(b) represents a symmetric antler.



(a) (b)

Figure 5: The binary tree in (a) is an asymmetric antler, while the one in (b) is a symmetric antler

These competition structures combine the characteristics of balanced elimination trees with the "byes" spirit of stepladders. It is not easy to find such competitions in practice but extended forms can be found in some basketball tournaments and in the Basque pelota.

Antlers are of a definite theoretical interest. As later proved in Theorem 3, they constitute the minimal competition systems violating WMS in the sense that removing any match from an antler results in a system that satisfies WMS and any tree that contains (as a subgraph) an antler lets the competition system violate WMS. This naturally leads to the definition of an *antler-free* tree as a binary tree that does not contain any (symmetric or asymmetric) antler. Theorem 3 not only characterizes the set of elimination-type competition systems that satisfy WMS as those displayed by an antler-free binary tree but also uniquely specifies the seeding rule, which we call "*increasingly balanced*" as the one for which WMS is satisfied.

With respect to their graph structure, antler-free trees can be characterized as binary trees having particular features. Lemma 1 in this section provides such a characterization with the aim of facilitating the definition of the increasingly balanced rule as well as the statement and the proof of Theorem 3. Let us consider first the following definitions.

A *root-to-leaf path* connects the root of t with a leaf of t. By $\gamma(t)$ we denote a root-to-leaf path of length $h(t)$ (i.e., $\gamma(t)$ is a *maximal root-to-leaf path* in t) and by $V_{\gamma(t)}$ we denote the set of nodes of $\gamma(t)$. For $v \in V_{\gamma(t)}$, t_v denotes the subtree of t with root v and $\Lambda_{-\gamma}(t_v)$ the set of leaves of t_v for which there is a shortest path to v *not including any other node* from $V_{\gamma(t)}$. Finally, we denote by $h_{-\gamma}(t_v)$ the maximal geodesic distance between v and the leaves in $\Lambda_{-\gamma}(t_v)$. We call a binary tree t *an extended stepladder of degree* x, $x \in \{1, \ldots, h(t)\}$, if $\max_{v \in V_{\gamma(t)}} h_{-\gamma}(t_v) = x$. That is, x is the maximal distance between a node of $\gamma(t)$ and a leaf that is not in $\gamma(t)$.[4]

Any binary tree is in fact an extended stepladder of some degree. For example, balanced elimination-type competitions with four players and asymmetric antlers are extended stepladders of degree 2, balanced elimination-type competitions with eight players and symmetric antlers are extended stepladders of degree 3, while standard stepladders and elementary binary trees are extended stepladders of degree 1.

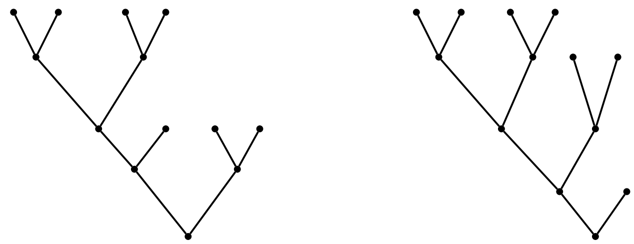
We denote by ES_x the set of extended stepladders of degree *at most* x (note that $ES_x \subseteq ES_{x'}$ for $x' \geq x$). Furthermore, we use ES_2^* to denote the subclass of ES_2 defined as follows. An extended stepladder t of degree at most 2 belongs to ES_2^* only if there exists a maximal root-to-leaf path $\gamma(t)$ such that for all $v, v' \in V_{\gamma(t)}$ with $|\ell(v) - \ell(v')| = 1$, we have that $h_{-\gamma}(t_v) = 2$ implies $h_{-\gamma}(t_{v'}) = 1$. Clearly, $ES_1 \subseteq ES_2^*$ but not every extended stepladder of degree 2 belongs to ES_2^*. Figure 6 exemplifies two extended stepladders of degree 2 with only one of them belonging to ES_2^*. Notice further that asymmetric antlers do belong to ES_2 but not to ES_2^*, while symmetric antlers do even not belong to ES_2 because they are extended stepladders of degree 3.

The next lemma characterizes antler-free binary trees.

Lemma 1 *A binary tree belongs to* ES_2^* *if and only if it is antler-free.*

Let us now introduce the increasingly balanced seeding rule. This rule takes into account the following two characteristics of antler-free binary trees:

[4]Clearly, by t being a binary tree, there are at least two maximal root-to-leaf paths in t. Despite this fact, it can be easily shown that the degree of an extended stepladder is *robust* with respect to the selection of any of the maximal root-to-leaf paths.



(a) (b)

Figure 6: Extended stepladders of degree 2. Only the one displayed in (a) belongs to ES_2^*

(1) they allow for four players to be involved in a balanced elimination-type competition, and (2) they also incorporate byes at different levels.

We formally define the increasingly balanced rule for binary trees in ES_2 and, therefore, by $ES_2^* \subset ES_2$ and Lemma 1, for antler-free trees as well. Since the rule makes use of the notion of a balanced seeding for balanced elimination-type competitions with four players, we first introduce this type of seeding. Given an admissible elimination-type competition system (t, N) with $|N| = 4$ and a binary relation of strength R, we say that a seeding rule $s : \Lambda(t) \to N$ is *balanced* (and we denote it by s_{b4}) if there are players $i, j \in N$ who are initially playing against each other under s such that iRk and kRj holds for each $k \in N \setminus \{i, j\}$. Thus, a seeding that matches 1 with 4 and 2 with 3 is always balanced. But a seeding that matches 1 with 3 and 2 with 4 would also be balanced if (and only if) $1I2$ or $3I4$. Similarly, a seeding that matches 1 with 2 and 3 with 4 would also be balanced if (and only if) $2I3I4$.

We are now prepared to define the increasingly balanced seeding rule. Let (t, N) be an elimination-type competition system with $t \in ES_2$. Given a binary relation of strength R, we say that a seeding rule $s : \Lambda(t) \to N$ is *increasingly balanced* (and we denote it by s_{ib}) if the following three conditions hold:

(1) For all $\lambda, \lambda' \in \Lambda(t)$, $\lambda \in \Lambda^{\ell}(t)$ and $\lambda' \in \Lambda^{\ell'}(t)$ with $\ell > \ell'$ implies $s(\lambda')Rs(\lambda)$;

(2) For all $\ell \in \{1, \ldots, h(t) - 1\}$, $\Lambda^{\ell}(t) = \{\lambda, \lambda', \lambda''\}$ with λ' and λ'' having a common intermediate predecessor implies $s(\lambda)Rs(\lambda''')$ for each $\lambda''' \in \{\lambda', \lambda''\}$;

(3) $\left|\Lambda^{h(t)}(t)\right| = 4$ implies that: (a) iRj holds for each $i \in N$ with $\ell(s^{-1}(i)) < h(t)$ and $j \in N$ with $\ell(s^{-1}(j)) = h(t)$, and (b) $s(\lambda) = s_{b4}(\lambda)$ for each $\lambda \in \Lambda^{h(t)}(t)$.

In other words, s_{ib} assigns players to leaves in such a way that weaker players are seeded to higher levels in the tree. When more than one player is seeded at the same level, then the rule distinguishes between two possibilities: (a) if the level is not the maximal one, then the best player among those seeded at that level is seeded to the leaf that is closest to the root-to-leaf path $\gamma(t)$; and (b) if the level is the maximal one and there are four leaves at it, then among the weakest four players, the weakest one is matched with the fourth weakest and the other two are matched together (see Figure 7).

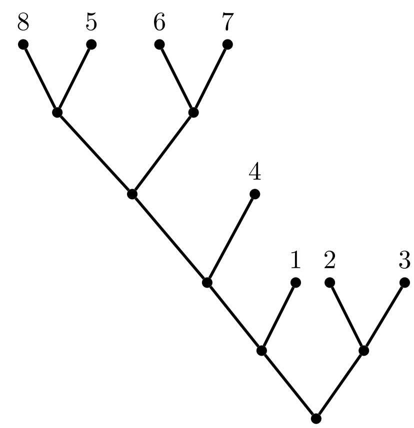


Figure 7: An increasingly balanced seeding in an extended stepladder of degree 2

Note also that there are two cases in which s_{ib} is silent. The first case is when there are three leaves of t at the same level and, therefore, the two weakest players among the three seeded at that level play their initial match. Clearly, in such a case, the two possible seedings of these players are equivalent. The second case is when there are only two leaves of t at level $h(t)$. In this case, the two seedings of the two weakest players are equivalent.

It should also be noted that, due to the structure of the extended stepladder competition of degree 2, if there are three or four leaves at a certain level, then four players are playing a balanced elimination-type sub-competition. The key feature of s_{ib} is that it ensures that the strongest of the newly seeded players at that level will play against the survivor of the previous elimination process who, by the construction of s_{ib}, is necessarily weaker than any of the newly seeded players. In other words, s_{ib} ensures that in any balanced elimination-type sub-competition played by four players, the strongest player is matched with the weakest player.

Theorem 3 *An elimination-type competition system* (t, N) *satisfies WMS with respect to* $s \in \mathcal{S}^{(t,N)}$ *if and only if* t *is antler-free and* $s = s_{ib}$.

In many real situations there are 2^q players participating in balanced elimination-type competitions. Then, extended forms of the balanced seeding are profusely taken because they are broadly considered to be a fair solution.[5] A remarkable corollary of Theorem 3 is that no balanced competition satisfies

[5]In the case of eight players the "balanced" seeding would consist of matching 1 with

WMS if $q \geq 3$, that is, if eight or more players compete, then even if the balanced seeding is used it is possible that weaker players have a strictly higher probability of being the final winners. The reason for this is that the binary trees that represent those competition systems do contain an antler. Broadly speaking, it is impossible to find a fair balanced playoff competition with more than four participants.

5 League-type competitions

As already advanced in the introduction, a league-type competition consists of a pair (G, N), where N is the player set and G is a forest of elementary binary trees. Recall that, as a consequence of the two assumptions on seeding rules, $s \in \mathcal{S}^{(G,N)}$ implies $2 \leq |N| \leq |\Lambda(G)|$.

5.1 Equal participation and equal treatment

Similar to the case of elimination-type competitions, the fulfilment of the two equal treatment axioms (ET and WET) by league-type competitions is closely related with the fact that players should play the same number of matches. The first result in this section shows that a league satisfies ET if and only if either each player plays a unique match or there are only two players participating in all matches.

Theorem 4 *A league-type competition system* (G, N) *satisfies ET if and only if either* $|\Lambda(G)| = |N|$ *or* $|\Lambda(G)| > |N| = 2$.

It is sometimes suggested that leagues are fair competition systems. However, this is not true in our framework because our definition of leagues is much broader than what is popularly understood as “leagues”, which is usually identified with round-robin tournaments. In fact, the theorem given above excludes leagues as fair structures, unless every player plays exactly one match or there are only two players. We will next show that weakening ET to WET extends the class of league-type competitions that are fair to those where each player has the possibility to participate the same number of times in at least $|N|-1$ matches. In particular, for $m \geq 1$ being an integer, we propose a seeding rule that lets each player participate in exactly m matches

8 and 4 with 5 in one branch, and 2 with 7 and 3 with 6 in the other branch of the binary tree.

against every other player. We call this class of competitions *m-round-robin tournaments*, which includes as particular cases of single round-robin tournaments ($m = 1$) and double round-robin tournaments ($m = 2$).

Theorem 5 *Any league-type competition* (G, N) *with* $|\Lambda(G)| = |N|$ *or* $|\Lambda(G)| = m \cdot (|N| - 1) \cdot |N|$ *for some integer* $m \geq 1$ *satisfies WET.*

It should be noted that Theorem 5 is not vacuous in the sense that not every league-type competition satisfies WET. For example, it is easy to check that there is no way to seed three players in a two-match competition so that WET is fulfilled.

5.2 Leagues and monotonicity in strength

The next result shows that the monotonicity in strength requirement restricts the number of participants to two for league-type competitions, again illustrating an analogy between the results for elimination tournaments and for leagues.

Theorem 6 *A league-type competition system* (G, N) *satisfies MS if and only if it is minimal.*

When MS is weakened to WMS, the class of competition systems that are fair is considerably enlarged because any league-type competition turns to satisfy this axiom provided that either each player participates in exactly one match or the total number of matches is at least $|N| - 1$. In particular, this implies that any league competition covered by Theorem 5 satisfies the weak versions of both the equal treatment and monotonicity in strength properties.

Theorem 7 *Any league-type competition* (G, N) *with either* $|\Lambda(G)| = |N|$ *or* $|\Lambda(G)| \geq 2\,(|N| - 1)$ *satisfies WMS.*

It is not difficult to prove that the seeding used in the proof of Theorem 7 (see Appendix B) when $|\Lambda(G)| = |N|$ is the only one for which (G, N) satisfies WMS. However, there are other seedings that differ from those used when $|\Lambda(G)| \geq 2(|N| - 1)$ holds, for which (G, N) satisfies WMS.

Although round-robin tournaments are certainly competitions where $|\Lambda(G)| \geq 2(|N| - 1)$ holds, the seeding rule that is used to prove the fulfilment of WMS in this case does not correspond to a round-robin tournament. It is most likely that league-type competitions also satisfy WMS with respect to the round-robin tournament seeding. However, providing a formal proof is much more complex than what it may appear at first sight. Just to have an impression with respect to the difficulties, we note that Saarinen et al. (2015) shows that calculating the final winning probability in a round-robin tournament is $\#P$-complete even if all values in the probability matrix belong to $\{0, 1/2, 1\}$.

Finally, it should be noted that Theorem 7 is not vacuous in the sense that there exist league-type competitions that do not satisfy WMS. For example, it can be proven that a league-type competition consisting of three matches and five players does not satisfy WMS.

6 Concluding remarks and further research

The results that are presented in the current work enable the evaluation and comparison of different competition systems on the basis of two reasonable principles of fairness. Our model connects with the specific line of research in Management Mathematics, which is devoted to the study of seeding procedures in elimination-type competitions with few players (cf. Horen and Riezmann 1985, Hwang 1982, Prince et al. 2013, and Schwenk 2000). We see as especially remarkable the way in which weak monotonicity leads to a singular structure, which we have called an *antler* and which was found to play a referential role in our analysis.

In general, our results show that there are limited numbers of competition systems that are *fair* in the sense of *simultaneously* satisfying both types of fairness in their corresponding strong or weak forms.

In the case of elimination-type competitions, MS already restricts the set of fair competitions to the minimal ones so that no additional restriction is at place when either ET or WET is added. Weakening MS to WMS (Theorem 3) does not heavily enlarge that set because the only balanced trees that are antler-free are either elementary or balanced of height 2. Thus, replacing MS by WMS in the above combinations adds only four-players balanced competitions as "fair".

In the case of leagues, MS combined with either ET or WET produces again a degenerate competition consisting of a two-player league (Theorem 4). Weakening MS to WMS and imposing it together with ET slightly enlarges the class of admissible leagues to those where each participant plays a unique match. In contrast, the combination of WMS and WET results in an expansion of the mentioned class of admissible leagues to include leagues allowing for players to participate the same number of times in at least $|N|-1$ matches (Theorems 5 and 7).

In Theorem 6, we prove that m-round-robin tournaments (which include single and double round-robin tournaments as special cases) satisfy WET. Most likely they also satisfy WMS, although the formal proof remains an open question.

For possible extensions of our model, we note that the stochastic transitivity condition assumed with respect to the probability matrices is sufficient but not necessary for the associated binary relation of strength to be transitive. Notice that any weakening of this condition would result in a larger number of probability matrices satisfying it and, thus, in even smaller class of competition systems fulfilling the corresponding fairness axioms.

We have excluded from the analysis the possibility of random seedings and reseedings, in addition to the study of double elimination competitions and the typical two-stage competitions consisting of qualification parallel round-robin tournaments followed by a knockout competition. However, we believe that our model sets the fundamentals for approaching such problems.

Another intriguing extension of our setup concerns the analysis of competition systems representable by forests of non-elementary binary trees. For example, one could imagine a variant of a round-robin competition where, at each round, the players do not play a single match but are grouped to play four-player (or larger) knockout or stepladder competitions with the final winner being the player who wins the most sub-competitions. David (1959) considers "repeated knockout tournaments" as interesting systems to be studied but, to the best of our knowledge, these kinds of competitions have neither been applied in sports nor have they been theoretically analyzed.

A Appendix: Omitted proofs from Section 4

We start by noting that in elimination-type competitions, provided that two players i and j are equally strong ($p_{ij} = 0.5$), it is always possible to exchange the leaves they have been assigned by some seeding rule without affecting the probabilities of winning of any player. The reason for this fact is simple and it is based on condition (2), which implies that $p_{ik} = p_{jk}$ holds for each $k \in N \setminus \{i, j\}$. Thus, we are generally allowed to fix a particular player from those who are equally strong in a given situation without loss of generality.

Proof of Theorem 2. We first show that if $N = \{1, 2\}$, then (t, N) satisfies

MS. Let $\mathbf{p}$ be an arbitrary but fixed probability matrix and note that $s \in \mathcal{S}^{(t,N)}$ implies $\varphi_1(t,s,\mathbf{p}) = p_{12}$ and $\varphi_2(t,s,\mathbf{p}) = p_{21}$. Hence, $p_{12} > 0.5$ implies $\varphi_1(t,s,\mathbf{p}) > \varphi_2(t,s,\mathbf{p})$ as required for MS to be satisfied.

Suppose now that (t,N) satisfies MS. We have to prove that $|N| = 2$ holds in such a case. We split the proof into three steps referring to the possible cases when t is not an elementary binary tree. In each of these steps, we reach a contradiction by showing that (t,N) violates MS.

Step 1 If t is balanced with $h(t) = 2$, then (t,N) violates MS.

Proof. Note that by the definition of a feasible seeding rule, $h(t) = 2$ implies that N consists of four players, $N = \{1,2,3,4\}$. Consider the seeding rule $s \in \mathcal{S}^{(t,N)}$ assigning the players to the leaves of t in such a way that the initial matches are between players 1 and 2, and 3 and 4, respectively. Take the probability matrix $\mathbf{p}$ as specified below.

$$\mathbf{p} = \begin{pmatrix} 0.5 & 0.85 & 0.86 & 0.90 \\ & 0.5 & 0.60 & 0.70 \\ & & 0.5 & 0.60 \\ & & & 0.5 \end{pmatrix}$$

We have $\varphi_2(t,s,\mathbf{p}) = (p_{21} \cdot p_{34} \cdot p_{23}) + (p_{21} \cdot p_{43} \cdot p_{24}) = 0.096$ and $\varphi_3(t,s,\mathbf{p}) = (p_{34} \cdot p_{12} \cdot p_{31}) + (p_{34} \cdot p_{21} \cdot p_{32}) = 0.1074$. Thus, MS is violated since $p_{23} = 0.60 > 0.5$ and $\varphi_3(t,s,\mathbf{p}) > \varphi_2(t,s,\mathbf{p})$.

Step 2 If t is a stepladder with $h(t) = 2$, then (t,N) violates MS.

Proof. Again by the definition of a feasible seeding rule, $h(t) = 2$ implies that N consists of three players, $N = \{1,2,3\}$. Consider the seeding rule $s \in \mathcal{S}^{(t,N)}$ assigning the players to the leaves of t in such a way that the initial match is between player 1 and player 2. Let $\mathbf{p}$ be a probability matrix such that $p_{12} = 0.51$, $p_{13} = 0.53$ and $p_{23} = 0.52$. Then $\varphi_2(t,s,\mathbf{p}) = p_{12} \cdot p_{23} = 0.2548$ and $\varphi_3(t,s,\mathbf{p}) = (p_{12} \cdot p_{31}) + (p_{21} \cdot p_{32}) = 0.4749$. Thus, MS is violated since $p_{23} = 0.52 > 0.5$ and $\varphi_3(t,s,\mathbf{p}) > \varphi_2(t,s,\mathbf{p})$.

Step 3 If t is such that $h(t) > 1$, then (t,N) violates MS.

Proof. Let $\Lambda^{h(t)}$ be the set of all leaves of t at the maximal level $h(t)$. Note that one of the following two situations necessary happens: (i) there exists a subtree t' of t which is a stepladder with $h(t') = 2$ and $\Lambda^{h(t')} \subseteq \Lambda^{h(t)}$, or (ii) there exists a balanced subtree t'' of t with $h(t'') = 2$ and $\Lambda^{h(t'')} \subseteq \Lambda^{h(t)}$.

The reason for these two possibilities is as follows. Because t is a binary tree, $\left|\Lambda^{h(t)}\right| \geq 2$. Take $\lambda_1, \lambda_2 \in \Lambda^{h(t)}$ to be such that they have a common immediate predecessor and denote it by a. Again by t being a binary tree, a has a unique immediate predecessor, which we call b. Given that b is not a leaf, it should have an intermediate successor $a' \neq a$. There are then two possibilities. First, if a' has no successors, then the set of nodes $\{\lambda_1, \lambda_2, a, b, a'\}$ and the corresponding edges form a stepladder t' with $h(t') = 2$ and $\Lambda^{h(t')} = \{\lambda_1, \lambda_2\} \subseteq \Lambda^{h(t)}$. Second, if a' does have successors, then there are exactly two of them, which we call λ_3 and λ_4. Note that λ_3 and λ_4 are indeed leaves of t as by assumption, $\lambda_1, \lambda_2 \in \Lambda^{h(t)}$. Hence, in this case, the set of nodes $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4, a, a', b\}$ and the corresponding edges form a balanced tree t'' with $h(t'') = 2$ and $\Lambda^{h(t'')} = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} \subseteq \Lambda^{h(t)}$.

Let us now separately consider the two addressed possibilities:

Case (i) There exists a subtree t' of t which is a stepladder with $h(t') = 2$ and $\Lambda^{h(t')} \subseteq \Lambda^{h(t)}$.

The situation in which $t' = t$ has already been considered in Step 2. Therefore, we will assume that $h(t) \geq 3$ holds. Let $N = \{1, \ldots, n\}$ and take the seeding rule $s \in \mathcal{S}^{(t,N)}$ assigning players 1, 2, and 3 to the leaves of t' as in the proof of Step 2. For any probability matrix $\mathbf{p}$, denote by $\varphi_w(t', s, \mathbf{p})$, $w \in \{1, 2, 3\}$, the probability with which w wins the subcompetition t'. We will prove that there exists a probability matrix $\mathbf{p}^*$ such that $p^*_{23} > 0.5$ and $\varphi_3(t, s, \mathbf{p}^*) > \varphi_2(t, s, \mathbf{p}^*)$.

For this, take $\mathbf{p}^*$ to be such that $p^*_{12} = 0.51$, $p^*_{13} = 0.53$, $p^*_{23} = 0.52$. Note that these are the same probability values that were previously used in the proof of Step 2. Thus, we already know that $\varphi_3((t', \{1,2,3\}), s, \mathbf{p}^*_{|\{1,2,3\}}) > \varphi_2((t', \{1,2,3\}), s, \mathbf{p}^*_{|\{1,2,3\}})$ though $p^*_{23} > 0.5$. Moreover, we further assume that $p^*_{2i} > p^*_{3i}$ and $\frac{p^*_{2i}}{p^*_{3i}} \approx 1$ holds for each $i \in N \setminus \{1, 2, 3\}$.

Let $V_{t',t}$ be the set of all nodes of t which are on the shortest path from the root of t' to the root of t. Note that for each $v \in V_{t',t}$, $\ell(v) \in \{0, 1, \ldots, h(t) - 2\}$ with $\ell(v) = 0$ indicating that v is the root of t and $\ell(v) = h(t) - 2$ indicating that node v is the root of t'. Because t is a binary tree, each $v \in V_{t',t}$ has exactly one successor node v' with $\ell(v') = \ell(v) + 1$ and $v' \in V(t) \setminus V_{t',t}$. Clearly, v could either be a terminal node or not.

We denote by $\Lambda_{v'}$ the set of all leaves of t whose shortest path to the root of t contains v' and by $N_{v'}$ the set of players seeded by s to some leaf from $\Lambda_{v'}$. For $k \in N_{v'}$, $\varphi^{v'}_k(t, s, \mathbf{p}^*)$ stands for the probability with which a player k seeded to a leaf from $\Lambda_{v'}$ reaches the node v'.

Suppose now that some player $w \in N$ has reached the node $\overline{v} \in V_{t',t}$ and let $v \in V_{t',t}$ be such that $\ell(\overline{v}) = \ell(v) + 1$. Given the above additional notation, it is then clear that $\sum\limits_{k \in N_{v'}} p^*_{wk} \varphi^{v'}_k(t, s, \mathbf{p}^*)$ expresses the probability with which w reaches v. Hence,

$$\varphi_2(t, s, \mathbf{p}^*) = \varphi_2((t', \{1, 2, 3\}), s, \mathbf{p}^*_{|\{1,2,3\}}) \cdot \prod_{v':\ell(v')=1}^{h(t)-2} \sum_{k \in N_{v'}} p^*_{2k} \varphi^{v'}_k(t, s, \mathbf{p}^*)$$

and

$$\varphi_3(t, s, \mathbf{p}^*) = \varphi_3((t', \{1, 2, 3\}), s, \mathbf{p}^*_{|\{1,2,3\}}) \cdot \prod_{v':\ell(v')=1}^{h(t)-2} \sum_{k \in N_{v'}} p^*_{3k} \varphi^{v'}_k(t, s, \mathbf{p}^*).$$

We already know that $\varphi_3((t', \{1, 2, 3\}), s, \mathbf{p}^*_{|\{1,2,3\}}) > \varphi_2((t', \{1, 2, 3\}), s, \mathbf{p}^*_{|\{1,2,3\}})$ and that $p^*_{2k} > p^*_{3k}$ holds for each $k \in N$ by construction. Moreover, since $\frac{p^*_{2i}}{p^*_{3i}} \approx 1$ holds for each $i \in N \setminus \{1, 2, 3\}$, we have

$$\prod_{v':\ell(v')=1}^{h(t)-2} \sum_{k \in N_{v'}} p^*_{2k} \varphi^{v'}_k(t, s, \mathbf{p}^*) \approx \prod_{v':\ell(v')=1}^{h(t)-2} \sum_{k \in N_{v'}} p^*_{3k} \varphi^{v'}_k(t, s, \mathbf{p}^*)$$

and thus, $\varphi_3(t, s, \mathbf{p}^*) > \varphi_2(t, s, \mathbf{p}^*)$ holds. Because $p^*_{23} > 0.5$, we conclude that (t, N) violates MS.

Case (ii) There exists a balanced subtree t'' of t with $h(t'') = 2$ and $\Lambda^{h(t'')} \subseteq \Lambda^{h(t)}$.

Consider the seeding rule $s \in \mathcal{S}^{(t,N)}$ assigning players 1, 2, 3, and 4 to the leaves of t'' in such a way that the initial matches are between players 1 and 2, and 3 and 4, respectively. Let $\mathbf{p}^*$ be such that $\mathbf{p}^*_{|\{1,2,3,4\}} = \mathbf{p}$ as defined in the proof of Step 1. As already shown in Step 1, $\varphi_3((t'', \{1, 2, 3, 4\}), s, \mathbf{p}^*_{|\{1,2,3,4\}}) > \varphi_2((t'', \{1, 2, 3, 4\}), s, \mathbf{p}^*_{|\{1,2,3,4\}})$. We can then further proceed as in the proof of Case (i) showing that $\varphi_3(t, s, \mathbf{p}^*) > \varphi_2(t, s, \mathbf{p}^*)$ holds though $p^*_{23} > 0.5$. Thus, (t, N) violates MS. ■

Proof of Lemma 1. The proof consists of the following tree steps.

Step 1 If t is symmetric-antler-free, then $t \in ES_2$.

Proof. Let t be a symmetric-antler-free binary tree and suppose that $t \notin ES_2$. The latter implies that t is an extended stepladder of degree $x \geq 3$ with respect to a maximal root-to-leaf path $\gamma(t)$, as defined previously. Therefore, there exists a node $v \in V_{\gamma(t)}$ and a leaf $\lambda \in \Lambda_{-\gamma}(t_v)$ such that the distance between v and λ is x. Denote by π the path connecting v and λ. Let y be the immediate successor of v in π, y' the immediate successor of y in π and y'' the immediate successor of y' in π (notice that such nodes exist because $x \geq 3$). Because t is a binary tree, y has another immediate successor $z' \neq y'$ and y' has another immediate successor $z' \neq y''$. Meanwhile, given that $v \in V_{\gamma(t)}$ and that $\gamma(t)$ is a maximal root-to-leaf path, there are at least three consecutive successor nodes x, x' and x'' that belong to $V_{\gamma(t)}$ (otherwise, there would be a longer root-to-leaf path connecting the root with λ). Again, since t is a binary tree, x has another immediate successor $w' \neq x'$ and x' has another immediate successor $w'' \neq y''$. Now, notice that the set of nodes $\{v, x, y, x', y', x'', y'', z', w', z'', w''\}$ and the corresponding edges form a symmetric antler, which is a contradiction.

Step 2 If t is antler-free, then $t \in ES_2^*$.

Proof. Notice first that if t is antler-free then t is both symmetric-antler free and asymmetric-antler free. Given that $ES_2^* \subset ES_2$ and in view of Step 1 it suffices to show that if $t \in ES_2$ does not contain an asymmetric antler, then $t \in ES_2^*$. Suppose not and let $\gamma(t)$ be a maximal root-to-leaf path in t. If $t \in ES_2 \setminus ES_2^*$, then there are two nodes $v, v' \in V_{\gamma(t)}$ with $\ell(v') = \ell(v) + 1$ and $h_{-\gamma}(t_v) = h_{-\gamma}(t_{v'}) = 2$.

Moreover, given that $\gamma(t)$ is a maximal root-to-leaf path, v' has at least two consecutive successors x and x' belonging to $V_{\gamma(t)}$, and given that t is a binary tree, x has another immediate successor $y \neq x'$. Consider then the set of nodes consisting of v, v', x, x', y, the immediate successors of v and v', as well as the leaves in $\Lambda_{-\gamma}(t_v) \cup \Lambda_{-\gamma}(t_{v'})$. Note that this set of nodes together with the corresponding edges form an asymmetric antler, which is a contradiction.

Step 3 If $t \in ES_2^*$, then t is antler-free.

Proof. Note first that if $t \in ES_1 \subseteq ES_2^*$, then it is antler-free. Suppose then that t is an extended stepladder of degree 2 belonging to ES_2^*. Clearly, t does not contain a symmetric antler t' because each symmetric antler is an extended stepladder of degree 3 and, thus, t containing t' implies that t should be an extended stepladder of degree at least 3, which is a contradiction. Let us show now that $t \in ES_2^*$ implies that t does not contain an asymmetric antler.

Suppose that, to the contrary, t contains an asymmetric antler t^A. Let

$\gamma(t)$ and $\gamma'(t^A)$ be maximal root-to-leaf paths in t and t^A, respectively. There are two possibilities:

(i) $V_{\gamma'(t^A)} \cap V_{\gamma(t)} = \emptyset$. Consider the root v_0^A of t^A and the closest predecessor v of v_0^A such that $v \in V_{\gamma(t)}$. Let d be the distance between v_0^A and v. Then we have that $h_{-\gamma}(t_v) > d + 3$ in contradiction to t being an extended stepladder of degree 2.

(ii) $V_{\gamma'(t^A)} \cap V_{\gamma(t)} \neq \emptyset$. Let $V_{\gamma'(t^A)} = \{v_0^A, v_1^A, v_2^A, v_3^A\}$ be such that, for all $i \in \{1, 2, 3\}$, v_i^A is the immediate successor of v_{i-1}^A and v_0^A is the root of t^A. Given that $V_{\gamma'(t^A)} \cap V_{\gamma(t)} \neq \emptyset$ there exists $v_i^A \in V_{\gamma'(t^A)} \cap V_{\gamma(t)}$. Note that $v_i^A \in V_{\gamma(t)}$ implies $v_j^A \in V_{\gamma(t)}$ for all $j < i$. Therefore $v_0^A \in V_{\gamma(t)}$. We distinguish then two cases: either $v_0^A \in V_{\gamma(t)}$ and $v_1^A \notin V_{\gamma(t)}$ or $v_0^A, v_1^A \in V_{\gamma(t)}$. If $v_0^A \in V_{\gamma(t)}$ and $v_1^A \notin V_{\gamma(t)}$, then $h_{-\gamma}(t_{v_0^A}) \geq 3$ in contradiction to t being an extended stepladder of degree 2. If $v_0^A, v_1^A \in V_{\gamma(t)}$, then by the structure of an asymmetric antler, and given that t is an extended stepladder of degree 2, we know that $h_{-\gamma}(t_{v_0^A}) = h_{-\gamma}(t_{v_1^A}) = 2$, which is a contradiction to $t \in ES_2^*$.
■

Proof of Theorem 3. We start with two additional lemmas. Lemma 2 states that a four-player balanced elimination-type competition satisfies WMS only for the balanced seeding as defined in Section 4, while Lemma 3 shows that, for a competition system to satisfy WMS, better players should not be seeded to leaves that are further away from the root of the tree. The proof of Theorem 3 is then structured as follows. We start with the previously mentioned lemmas and their corresponding proofs, we then continue with the proof of the sufficiency part of the theorem, and we conclude with the proof of the necessity part.

Lemma 2 *A balanced elimination-type competition system* (t, N) *with* $h(t) = 2$ *satisfies WMS with respect to a seeding rule* $s \in \mathcal{S}^{(t,N)}$ *if and only if* $s = s_{b4}$.

Proof. Let (t, N) be as above with $N = \{1, 2, 3, 4\}$ and recall that $1R2R3R4$ holds. Assume, w.l.o.g., that s_{b4} is such that player 1 is matched with player 4 and player 2 is matched with player 3. Consider then the seeding rule $s = s_{b4}$ and fix *any* probability matrix $\mathbf{p} \in \mathcal{P}_R$. We have to show that $\varphi_1(t, s_{b4}, \mathbf{p}) \geq \varphi_2(t, s_{b4}, \mathbf{p}) \geq \varphi_3(t, s_{b4}, \mathbf{p}) \geq \varphi_4(t, s_{b4}, \mathbf{p})$ holds.

Note first that for the final winner's probabilities we have $\varphi_1(t, s_{b4}, \mathbf{p}) = (p_{14} \cdot p_{23} \cdot p_{12}) + (p_{14} \cdot p_{32} \cdot p_{13})$, $\varphi_2(t, s_{b4}, \mathbf{p}) = (p_{23} \cdot p_{14} \cdot p_{21}) + (p_{23} \cdot p_{41} \cdot p_{24})$, $\varphi_3(t, s_{b4}, \mathbf{p}) = (p_{32} \cdot p_{14} \cdot p_{31}) + (p_{32} \cdot p_{41} \cdot p_{34})$, $\varphi_4(t, s_{b4}, \mathbf{p}) = (p_{41} \cdot p_{23} \cdot p_{42}) + (p_{41} \cdot p_{32} \cdot p_{43})$.

First, when comparing $\varphi_1(t, s_{b4}, \mathbf{p})$ with $\varphi_2(t, s_{b4}, \mathbf{p})$ we have from $p_{12} \geq 0.5$ and condition (1) that $p_{21} \leq p_{12}$ holds. Thus, $p_{14} \cdot p_{23} \cdot p_{12} \geq p_{23} \cdot p_{14} \cdot p_{21}$. In contrast, $p_{14} \cdot p_{32} \cdot p_{13} \geq p_{23} \cdot p_{41} \cdot p_{24}$ holds by $p_{32} \geq p_{41}$ (due to (3)) and by $p_{14} \geq p_{24}$ and $p_{13} \geq p_{23}$ (following from $p_{12} \geq 0.5$ and condition (2)). Thus, $\varphi_1(t, s_{b4}, \mathbf{p}) \geq \varphi_2(t, s_{b4}, \mathbf{p})$ holds.

Following analogous reasoning, it can be proven that $\varphi_2(t, s_{b4}, \mathbf{p}) > \varphi_3(t, s_{b4}, \mathbf{p})$ and $\varphi_3(t, s_{b4}, \mathbf{p}) > \varphi_4(t, s_{b4}, \mathbf{p})$, concluding that (t, N) satisfies WMS with respect to s_{b4}.

Let us now consider a seeding rule $s \in \mathcal{S}^{(t,N)}$ which differs from $s \neq s_{b4}$. Let $\varepsilon > 0$ be arbitrarily small and $\mathbf{p} \in \mathcal{P}_R$ be defined as follows:

$$\mathbf{p} = \begin{pmatrix} 0.5 & 0.5+\varepsilon & 0.5+2\varepsilon & 1-\varepsilon \\ & 0.5 & 0.5+\varepsilon & 1-2\varepsilon \\ & & 0.5 & 1-3\varepsilon \\ & & & 0.5 \end{pmatrix}$$

There are two possible cases with respect to the seeding produced by s.

Case 1 (the initial matches are between 1 and 2, and 3 and 4, respectively). We have in this case:
$\varphi_3(t, s, \mathbf{p}) = (p_{34} \cdot p_{12} \cdot p_{31}) + (p_{34} \cdot p_{21} \cdot p_{32}) \approx (1 \cdot 0.5 \cdot 0.5) + (1 \cdot 0.5 \cdot 0.5) \approx 0.5$
and
$\varphi_2(t, s, \mathbf{p}) = (p_{21} \cdot p_{34} \cdot p_{23}) + (p_{21} \cdot p_{43} \cdot p_{24}) \approx (0.5 \cdot 1 \cdot 0.5) + (0.5 \cdot 0 \cdot 1) \approx 0.25$, in contradiction to $p_{23} > 0.5$ and (t, N) satisfying WMS.

Case 2 (the initial matches are between 1 and 3, and 2 and 4, respectively). Considering again the probability matrix shown previously, we have $\varphi_1(t, s, \mathbf{p}) = (p_{13} \cdot p_{24} \cdot p_{12}) + (p_{13} \cdot p_{42} \cdot p_{14}) \approx (0.5 \cdot 1 \cdot 0.5) + (0.5 \cdot 0 \cdot 1) \approx 0.25$ and $\varphi_2(t, s, \mathbf{p}) = (p_{24} \cdot p_{13} \cdot p_{21}) + (p_{24} \cdot p_{31} \cdot p_{23}) \approx (1 \cdot 0.5 \cdot 0.5) + (1 \cdot 0.5 \cdot 0.5) \approx 0.5$, which is in contradiction to $p_{12} > 0.5$ and (t, N) satisfying WMS. ■

Lemma 3 *Let R be a strength relation defined on N, (t, N) an elimination-type competition system, and $s \in \mathcal{S}^{(t,N)}$. If (t, N) satisfies WMS with respect to s, then $\ell(\lambda) > \ell(\lambda')$ for $\lambda, \lambda' \in \Lambda(t)$ implies $s(\lambda')Rs(\lambda)$.*

Proof. Suppose that the implication is false. That is, given R, let (t, N) satisfy WMS with respect to s such that $s(\lambda)Ps(\lambda')$ holds for some $\lambda, \lambda' \in \Lambda(t)$ with $\ell(\lambda) > \ell(\lambda')$. For (t, N) to satisfy WMS, it is necessary that $\varphi_{s(\lambda)}(t, s, \mathbf{p}') \geq \varphi_{s(\lambda')}(t, s, \mathbf{p}')$ for all probability matrices $\mathbf{p}' \in \mathcal{P}_R$ such that $p'_{s(\lambda),s(\lambda')} > 0.5$. Let us consider a probability matrix $\mathbf{p} \in \mathcal{P}_R$ such that, for all $i, j \in N$, $p_{ij} \approx 0.5$ with $p_{s(\lambda),s(\lambda')} > 0.5$. Then $\varphi_{s(\lambda)}(t, s, \mathbf{p}) \approx 0.5^{\ell(\lambda)}$ and

$\varphi_{s(\lambda')}(t,s,\mathbf{p}) \approx 0.5^{\ell(\lambda')}$. Because $\ell(\lambda) > \ell(\lambda')$, $\varphi_{s(\lambda)}(t,s,\mathbf{p}) < \varphi_{s(\lambda')}(t,s,\mathbf{p})$. Taking into account that $p_{s(\lambda),s(\lambda')} > 0.5$, the latter inequality implies that (t,N) violates WMS with respect to s, which is a contradiction. ■

Before moving to the proof of the sufficiency and necessity parts of Theorem 3, let us introduce the following additional concept. We say that a binary tree t with $h(t) = 3$ is a *one-bye antler*, if $|\Lambda(t)| = 7$ with $|\Lambda^3(t)| = 6$ and $|\Lambda^2(t)| = 1$. Clearly, any one-bye antler is an extended stepladder of degree 3. Further, for t and t' being binary trees, we say that (1) t' is an *extension from the leaves* of t if t' and t have the same root and $\Lambda(t) \subseteq \Lambda(t')$; (2) t' is an *extension from the root* of t if t is a subtree of t'; (3) t' is a *limited extension from the root* of t, if t is a subtree of t' and $\Lambda^{h(t)}(t) \subseteq \Lambda^{h(t')}(t')$. Thus, a limited extension from the root of a tree t never has leaves at a height that is greater than the height of any of the leaves of t.

Proof of Theorem 3 (Sufficiency). Given a strength relation R defined on the player set N, then we have to prove that an elimination-type competition system (t,N) with t being antler-free satisfies WMS with respect to s_{ib}.

Let (t,N) be such that t is antler-free and $s = s_{ib}$. By Lemma 1, $t \in ES_2^*$. Take a maximal root-to-leaf path $\gamma(t)$ and note that $t \in ES_2^*$ irrespective of the choice of $\gamma(t)$. For $s \in \mathcal{S}^{(t,N)}$, $v \in V_{\gamma(t)}$, and any probability matrix $\mathbf{p}$, we denote by $p_i^v(s)$ the probability with which player $i \in N$ reaches v under a given seeding s and by v_h the unique leaf in $V_{\gamma(t)}$. Moreover, we collect in the set $S_v^1(s)$ all players whose first match in the competition is against a player who has already reached some $v' \in V_{\gamma(t)}$ with $\ell(v') > \ell(v)$; correspondingly, $S_v^2(s)$ stands for the set of all players who had to play an initial match before having the possibility to meet a player who has already reached some node from $V_{\gamma(t)}$ at a higher level than v. Note that for each $i \in N$ we have that, due to $t \in ES_2^*$, either $i = s(v_h)$ or $i \in S_v^1(s) \cup S_v^2(s)$ holds for some $v \in V_{\gamma(t)}$.

We denote by v^x the closest predecessor belonging to $V_{\gamma(t)}$ of $x = s(\lambda)$ for some $\lambda \in \Lambda(t)$. Note that, for each $v \in V_{\gamma(t)}$, any probability matrix $\mathbf{p}$, and any two players $k, j \in N$ with $p_{kj} > 0.5$ and $s^{-1}(k), s^{-1}(j) \in \Lambda(t_v)$, we have that $p_k^v(s) > p_j^v(s)$ implies $\varphi_k(t,s,\mathbf{p}) > \varphi_j(t,s,\mathbf{p})$. The reason is that for each $i \in N$ with $s^{-1}(i) \in \Lambda(t_v)$ we have

$$\varphi_i(t,s,\mathbf{p}) = p_i^v(s) \cdot \prod_{x \in S^1_{v'}(s):\ell(v')<\ell(v)} p_{ix} \cdot \prod_{y,z \in S^2_{v'}(s):\ell(v')<\ell(v)} (p_{iy}p_{yz} + p_{iz}p_{zy}).$$

Hence, $\varphi_k(t,s,\mathbf{p}) > \varphi_j(t,s,\mathbf{p})$ is implied by $p_k^v(s) > p_j^v(s)$, $p_{kx} \geq p_{jx}$ for each

$x \in N$ following from condition (2), and p_{yz} (p_{zy}) being independent of any other parameter in the respective formulae for k and j.

Thus, to prove the sufficiency part of Theorem 3, let us now consider the increasingly balanced rule s_{ib}. We have to show that (t, N) satisfies WMS with respect to s_{ib}. In view of the argument that was just explained, assuming that $p_{kj} > 0.5$, then it is enough to find a node $v \in V_{\gamma(t)}$ with $s_{ib}^{-1}(k), s_{ib}^{-1}(j) \in \Lambda(t)$ and $p_k^v(s_{ib}) > p_j^v(s_{ib})$. We distinguish the following three possible cases:

(i) $\ell(s_{ib}^{-1}(k)) < \ell(s_{ib}^{-1}(j))$ and there is no $m \in N$ with $\ell(s_{ib}^{-1}(m)) = \ell(s_{ib}^{-1}(k))$. Clearly, player k does not need to win any match to reach $v^k \in V_{\gamma(t)}$. Therefore, given that s_{ib} seeds worse players to higher levels, $p_k^{v^k}(s_{ib}) > 0.5$ and because j has to defeat k to reach v^k, $p_j^{v^k}(s_{ib}) < 0.5$.

(ii) $\ell(s_{ib}^{-1}(k)) < \ell(s_{ib}^{-1}(j))$ and there exists $m \in N$ with $\ell(s_{ib}^{-1}(m)) = \ell(s_{ib}^{-1}(k))$. In this case, player k is involved in a balanced sub-competition of four players. Let v^* be the root of the sub-competition (note that $v^* \in V_{\gamma(t)}$ with $p_k^{v^*}(s_{ib})$ being the probability for player k to win the sub-competition). For player j, $p_j^{v^*}(s_{ib})$ is the product of two probabilities: the probability to reach the sub-competition, that is, to reach the node $v \in V_{\gamma(t)}$ such that $\ell(v) = \ell(s_{ib}^{-1}(k))$; and, the probability to win the sub-competition. Given that the sub-competition is played under a balanced seeding, we know by Lemma 2 that for any probability matrix with $p_{kj} > 0.5$, then the probability for k to win the sub-competition is weakly greater than the one for j. We conclude that $p_k^{v^*}(s_{ib}) > p_j^{v^*}(s_{ib})$ should hold.

(iii) $\ell(s_{ib}^{-1}(k)) = \ell(s_{ib}^{-1}(j))$. Also in this case, players k and j are involved in a balanced sub-competition of four players. Following the same reasoning as in (ii), we obtain $p_k^{v^*}(s_{ib}) > p_j^{v^*}(s_{ib})$.

Proof of Theorem 3 (Necessity). We have to prove that if (t, N) satisfies WMS with respect to some seeding rule s, then t is antler-free and $s = s_{ib}$. To prove that t is antler-free in such a case, we will show that if t contains an antler, then the competition system (t, N) violates WMS. More precisely, in Steps 1 to 9 of the proof, we show progressively and in an exhaustive way that all of the different types of structures that can contain an antler violate WMS. In Step 10 we finally prove that the seeding rule s with respect to which (t, N) satisfies WMS is necessarily $s = s_{ib}$.

Step 1 Let (t, N) be an elimination-type competition system with t being a symmetric antler. Then (t, N) violates WMS.

Proof. Note that $N = \{1, \dots, 6\}$ holds in this case. Let λ_ℓ^2 and λ_r^2 be the two leaves of t that are at level 2 of its left and right branch, respectively. Similarly, let λ_ℓ^{3a} and λ_ℓ^{3b} be the two leaves at level 3 of t's left branch, while λ_r^{3a} and λ_r^{3b} be the two leaves at level 3 of t's right branch. We proceed by reduction to the absurd; that is, we assume that (t, N) satisfies WMS and then prove that we reach a contradiction. By Lemma 3, any $s \in \mathcal{S}^{(t,N)}$ with respect to which (t, N) satisfies WMS should be such that the two strongest players are seeded to λ_ℓ^2 and λ_r^2. Assume w.l.o.g. that these players are 1 and 2, and that $s(\lambda_\ell^2) = 1$ and $s(\lambda_r^2) = 2$. There are then six possible non-equivalent seedings for the remaining players:
(i) $s(\lambda_\ell^{3a}) = 3$, $s(\lambda_\ell^{3b}) = 4$, $s(\lambda_r^{3a}) = 5$, $s(\lambda_r^{3b}) = 6$.
(ii) $s(\lambda_\ell^{3a}) = 3$, $s(\lambda_\ell^{3b}) = 5$, $s(\lambda_r^{3a}) = 4$, $s(\lambda_r^{3b}) = 6$.
(iii) $s(\lambda_\ell^{3a}) = 3$, $s(\lambda_\ell^{3b}) = 6$, $s(\lambda_r^{3a}) = 4$, $s(\lambda_r^{3b}) = 5$.
(iv) $s(\lambda_\ell^{3a}) = 4$, $s(\lambda_\ell^{3b}) = 5$, $s(\lambda_r^{3a}) = 3$, $s(\lambda_r^{3b}) = 6$.
(v) $s(\lambda_\ell^{3a}) = 4$, $s(\lambda_\ell^{3b}) = 6$, $s(\lambda_r^{3a}) = 3$, $s(\lambda_r^{3b}) = 5$.
(vi) $s(\lambda_\ell^{3a}) = 5$, $s(\lambda_\ell^{3b}) = 6$, $s(\lambda_r^{3a}) = 3$, $s(\lambda_r^{3b}) = 4$.

To prove that (t, N) violates WMS, we next show that for each of the six possible seedings we can find a probability matrix $\mathbf{p} \in \mathcal{P}_R$ defined on N such that there exists $i \in N$ with $p_{i-1,i} > 0.5$ (and, therefore, $(i-1)Pi$) and $\varphi_i(t, s, \mathbf{p}) > \varphi_{i-1}(t, s, \mathbf{p})$.

(i) Take $\mathbf{p}$ as follows: $p_{jk} > 0.5$ if $j < k$; $p_{j6} \approx 1$ for all $j < 6$, and $p_{jk} \approx 0.5$ for all $j, k < 6$. We have then $\varphi_5(t, s, \mathbf{p}) \approx 0.25 > 0.125 \approx \varphi_4(t, s, \mathbf{p})$ while $p_{45} > 0.5$.

(ii) Consider the same probability matrix $\mathbf{p}$ as in case (i), then $\varphi_4(t, s, \mathbf{p}) \approx 0.25 > 0.125 \approx \varphi_3(t, s, \mathbf{p})$ while $p_{34} > 0.5$.

(iii) Let $\mathbf{p}$ be such that $p_{jk} > 0.5$ if $j < k$; $p_{jk} \approx 1$ if $j \in \{1, 2\}$ and $k \in \{4, 5, 6\}$, and $p_{jk} \approx 0.5$, otherwise. Then $\varphi_2(t, s, \mathbf{p}) \approx 0.5 > 0.375 \approx \varphi_1(t, s, \mathbf{p})$ while $p_{12} > 0.5$.

(iv) Take $\mathbf{p}$ as follows: $p_{jk} > 0.5$ if $j < k$; $p_{jk} \approx 1$ if $j \in \{1, 2\}$ and $k = 6$, and $p_{jk} \approx 0.5$, otherwise. Then $\varphi_2(t, s, \mathbf{p}) \approx 0.375 > 0.25 \approx \varphi_1(t, s, \mathbf{p})$ while $p_{12} > 0.5$.

(v) Let $\mathbf{p}$ be as follows: $p_{jk} > 0.5$ if $j < k$; $1 \approx p_{15} \approx p_{16} \approx p_{25} \approx p_{26} \approx p_{36} \approx p_{46}$, and $p_{jk} \approx 0.5$, otherwise. Then $\varphi_2(t, s, \mathbf{p}) \approx 0.375 > 0.25 \approx \varphi_1(t, s, \mathbf{p})$ while $p_{12} > 0.5$.

(vi) Consider the same probability matrix $\mathbf{p}$ as in cases (i) and (ii), then $\varphi_5(t, s, \mathbf{p}) \approx 0.25 > 0.125 \approx \varphi_4(t, s, \mathbf{p})$ while $p_{45} > 0.5$.

For later steps in the proof, it is important to remark that, according to the probability matrices shown above and the one shown in the proof of

Lemma 3, whatever seeding $s \in \mathcal{S}^{(t,N)}$ we consider in a symmetric antler t, not only exists a probability matrix $\mathbf{p}$ and $i \in N$ such that $p_{i-1,i} > 0.5$ and $\varphi_i(t,s,\mathbf{p}) > \varphi_{i-1}(t,s,\mathbf{p})$ but it also holds that it is possible to find such a matrix $\mathbf{p}$ where $p_{i-1,i} \approx 0.5$ and $p_{ik} \approx p_{i-1,k}$ for all $k \in N \setminus \{i-1,i\}$.

Step 2 Let (t,N) be an elimination-type competition system with t being an asymmetric antler, then (t,N) violates WMS.

Proof. Clearly $N = \{1,\ldots,6\}$ holds also in this case. Assume w.l.o.g. that t's left branch has four leaves at level $h(t) = 3$, and denote them (from left to right) by λ_ℓ^{3a}, λ_ℓ^{3b}, λ_ℓ^{3c} and λ_ℓ^{3d}. Clearly, t's right branch has two leaves (λ_r^{2a} and λ_r^{2b}) at level 2. Let v_1 be the node in the left branch of t which is an immediate successor of the root of t. Note that $\{\lambda_\ell^{3a},\lambda_\ell^{3b},\lambda_\ell^{3c},\lambda_\ell^{3d}\}$ are the leaves of the balanced subtree t_1 of t whose root is v_1. We proceed again by reduction to the absurd. Assume that (t,N) satisfies WMS. By Lemma 3, the two strongest players should be seeded to the two leaves at level 2. Assume w.l.o.g. that these players are 1 and 2 and that $s(\lambda_\ell^2) = 1$ and $s(\lambda_r^2) = 2$. We then fix a seeding rule $s' : \Lambda(t_1) \to \{3,4,5,6\}$, note that $s' \in \mathcal{S}^{(t_1,N\setminus\{1,2\})}$, and consider the following two possibilities.

Case 1 ($s' \neq s'_{b4}$). Consider the matrix used in the proof of Lemma 2 and apply it to the set of players $\{3,4,5,6\}$. According to the proof, if $s \neq s_{b4}$, then we can always find a pair of players $i, i-1 \in \{3,4,5,6\}$ and a probability matrix $\mathbf{p}' \in \mathcal{P}_{R_{|\{3,4,5,6\}}}$ such that $p'_{i-1,i} > 0.5$ and $p_i'^{v_1} > p_{i-1}'^{v_1}$. Moreover $p'_{i-1,i} \approx 0.5$ and $p'_{i-1,k} \approx p'_{i,k}$ also holds for all $k \in \{3,4,5,6\}$. Let $\mathbf{p}$ be defined on N such that $p_{jk} = p'_{jk}$ for all $j,k > 2$; $p_{13} = p_{23} = p_{12} = 0.5$ and, therefore, $p_{jk} = p_{3k}$ for all $j < 3$ and all $k \in N$. By $p_{i3} \approx p_{i-1,3}$ and by $\mathbf{p}$ satisfying condition (2) we have $p_{i2} \approx p_{i-1,2}$ and $p_{i1} \approx p_{i-1,1}$. Thus, $p_{i-1,k} \approx p_{i,k}$ for all $k \in N$.

Note then that $\varphi_i(t,s,\mathbf{p}) = p_i'^{v_1} \cdot (p_{12}p_{i1} + p_{21}p_{i2})$ and $\varphi_{i-1}(t,s,\mathbf{p}) = p_{i-1}'^{v_1} \cdot (p_{12}p_{i-1,1} + p_{21}p_{i-1,2})$. By $p_i'^{v_1} > p_{i-1}'^{v_1}$, $p_{i1} \approx p_{i-1,1}$ and $p_{i2} \approx p_{i-1,2}$, we have $\varphi_i(t,s,\mathbf{p}) > \varphi_{i-1}(t,s,\mathbf{p})$ in contradiction to (t,N) satisfying WMS with respect to s.

Case 2 ($s' = s'_{b4}$). Consider the following probability matrix $\mathbf{p} \in \mathcal{P}_R$:

$$\mathbf{p} = \begin{pmatrix} 0.5 & 0.5 & 0.5+\varepsilon & 1-2\varepsilon & 1-\varepsilon & 1-\varepsilon \\ & 0.5 & 0.5+\varepsilon & 1-2\varepsilon & 1-\varepsilon & 1-\varepsilon \\ & & 0.5 & 1-3\varepsilon & 1-2\varepsilon & 1-2\varepsilon \\ & & & 0.5 & 1-3\varepsilon & 1-3\varepsilon \\ & & & & 0.5 & 0.5 \\ & & & & & 0.5 \end{pmatrix}$$

According to $\mathbf{p}$, s'_{b4} matches either 3 with 6 and 4 with 5, or it matches 3 with 5 and 4 with 6. In either case, we have $p_{23} > 0.5$ and, after making the necessary computations, $\varphi_3(t, s, \mathbf{p}) \approx 0.5 > 0.25 \approx \varphi_2(t, s, \mathbf{p})$. Thus, (t, N) violates WMS with respect to s.

As in Step 1, it is important to remark that, according to the probability matrix shown previously and the ones shown in the proofs of Lemma 2 and Lemma 3, whatever seeding $s \in \mathcal{S}^{(t,N)}$ we consider in an asymmetric antler t, not only exists a probability matrix $\mathbf{p}$ and $i \in N$ such that $p_{i-1,i} > 0.5$ and $\varphi_i(t, s, \mathbf{p}) > \varphi_{i-1}(t, s, \mathbf{p})$ but it also holds that it is possible to find such a matrix $\mathbf{p}$ where $p_{i-1,i} \approx 0.5$ and $p_{ik} \approx p_{i-1,k}$ for all $k \in N \setminus \{i-1, i\}$.

Step 3 Let (t, N) be an elimination-type competition system with t being a one-bye antler. Then (t, N) violates WMS.

Proof. We proceed again by reduction to the absurd assuming that (t, N) violates WMS. Note that $N = \{1, \ldots, 7\}$ holds in this case. Assume w.l.o.g. that t's left branch has four leaves at level $h(t) = 3$, and denote them (from left to right) by λ_ℓ^{3a}, λ_ℓ^{3b}, λ_ℓ^{3c} and λ_ℓ^{3d}. Clearly, t's right branch has two leaves (λ_r^{3a} and λ_r^{3b}) at that same level and one leaf (λ_r^2) at level 2. By Lemma 3, the best player should be seeded to λ_r^2. Assume w.l.o.g. that $s(\lambda_r^2) = 1$. We distinguish now two possibilities depending on the leaf player 7 has been seeded at.

Case 1 ($s(\lambda) = 7$ for some λ of t's right branch). There are two subcases: (i) $s(\lambda_r^{4a}) = 6$ and $s(\lambda_r^{4b}) = 7$ (or vice versa w.l.o.g.) and (ii): $s(\lambda_r^{4a}) = x$ and $s(\lambda_r^{4b}) = 7$ (or vice versa) with $x < 6$.

(i) If $s(\lambda_r^{4a}) = 6$ and $s(\lambda_r^{4b}) = 7$, then let $\mathbf{p} \in \mathcal{P}_R$ be a probability matrix such that $p_{jk} > 0.5$ for all $j, k \in N$ with $j < k$, $p_{j7} \approx 1$ for all $j \in N \setminus \{7\}$, and $p_{jk} \approx 0.5$ for all $j, k \in N \setminus \{7\}$. By making the necessary calculations, we obtain $\varphi_6(t, s, \mathbf{p}) \approx 0.25 > 0.125 \approx \varphi_5(t, s, \mathbf{p})$ while $p_{67} > 0.5$, which is a contradiction to (t, N) satisfying WMS.

(ii) If $s(\lambda_r^{4a}) = x$ and $s(\lambda_r^{4b}) = 7$, then let $\mathbf{p} \in \mathcal{P}_R$ be a probability matrix such that $p_{jk} > 0.5$ for all $j, k \in N$ with $j < k$, $p_{jk} \approx 0.5$ for all $j, k \in N \setminus \{1\}$, and $p_{1k} = 0.7$ for all $k \in N \setminus \{1\}$. By making the necessary calculations, we obtain $\varphi_6(t, s, \mathbf{p}) \approx 0.09 > 0.075 \approx \varphi_x(t, s, \mathbf{p})$ while $p_{x6} > 0.5$, reaching again a contradiction.

Case 2 ($s(\lambda) = 7$ for some λ of t's left branch). Let x be the player whose initial match is against player 7 and suppose w.l.o.g., that $s(\lambda_\ell^{4c}) = 7$ and

$s(\lambda_\ell^{4d}) = x$. Then remove from t the nodes λ_ℓ^{4c} and λ_ℓ^{4d}, and also the corresponding edges to their immediate predecessor v_λ. Note that the remaining subgraph t^A of t is a symmetric antler with v_λ being now a leaf of t^A. Consider the seeding rule $s' : \Lambda(t^A) \to \{1, \ldots, 6\}$ defined as follows: $s'(v_\lambda) = x$ and $s'(\lambda) = s(\lambda)$ for each $\lambda \in \Lambda(t^A) \setminus \{v_\lambda\}$, and note that $s' \in \mathcal{S}^{(t^A, N \setminus \{7\})}$. In other words, s' can be interpreted as a situation in which x wins his or her match against 7 and the remaining matches are not yet played.

By Step 1, the competition system $(t^A, N \setminus \{7\})$ violates WMS. That is, there exists a probability matrix $\mathbf{p}' \in \mathcal{P}_{R_{|N\setminus\{7\}}}$ such that, for some $i \in N \setminus \{7\}$, $p'_{i-1,i} > 0.5$ and $\varphi'_i(t^A, s', \mathbf{p}') > \varphi'_{i-1}(t^A, s', \mathbf{p}')$. Moreover, we know that $\mathbf{p}'$ can be constructed in such a way that $p'_{i-1,i} \approx 0.5$ and $p'_{ik} \approx p'_{i-1,k}$ holds for each $k \in N \setminus \{i-1, i, 7\}$.

Consider now the probability matrix $\mathbf{p} \in \mathcal{P}_R$ such that $p_{jk} = p'_{jk}$ for all $j, k \in N \setminus \{7\}$, and $p_{k7} \approx 1$ for all $k \in N \setminus \{7\}$. For the final winning probabilities of each $k < 7$ we have by construction that $\varphi_k(t, s, \mathbf{p}) \approx \varphi'_k(t^A, s', \mathbf{p}')$. By hypothesis, $p'_{i-1,i} > 0.5$ and $\varphi'_i(t^A, s', \mathbf{p}') > \varphi'_{i-1}(t^A, s', \mathbf{p}')$ holds and, thus, $p_{i-1,i} > 0.5$ and $\varphi_i(t, s, \mathbf{p}) > \varphi_{i-1}(t, s, \mathbf{p})$ holds as well. Hence, (t, N) also violates WMS in this case. Moreover, as in Step 1, it is interesting to remark for the later steps in the proof that, for any seeding in a one-bye antler t, we can always find a probability matrix $\mathbf{p} \in \mathcal{P}_R$ that makes the competition system (t, N) violate WMS and such that $p_{i-1,i} \approx 0.5$ and $p_{ik} \approx p_{i-1,k}$ holding for some $i \in N$ and all $k \in N \setminus \{i-1, i\}$.

Step 4 Let (t, N) be an elimination-type competition system with $h(t) = 3$ and t being balanced. Then (t, N) violates WMS.

Proof. Note that $N = \{1, \ldots, 8\}$ holds in this case. Let x be the player whose initial match is against player 8, and remove from t the nodes $s^{-1}(8)$ and $s^{-1}(x)$ together with the corresponding edges to their immediate predecessor, v_λ. Note that the remaining subgraph t^A of t is a one-bye antler with v_λ being now a leaf of t^A. Consider then the seeding rule $s' : \Lambda(t^A) \to \{1, \ldots, 7\}$ defined as follows: $s'(v_\lambda) = x$ and $s'(\lambda) = s(\lambda)$ for each $\lambda \in \Lambda(t_A) \setminus \{v_\lambda\}$, and notice that $s' \in \mathcal{S}^{(t^A, N \setminus \{8\})}$.

By Step 3, $(t^A, N \setminus \{8\})$ violates WMS. That is, there exists a probability matrix $\mathbf{p}' \in \mathcal{P}_{R_{|N\setminus\{8\}}}$ such that $p'_{ij} > 0.5$ and $\varphi'_j(t^A, s', \mathbf{p}') > \varphi'_i(t^A, s', \mathbf{p}')$ for the corresponding final winning probabilities of some $i, j \in N \setminus \{8\}$. Moreover, we know that $\mathbf{p}'$ can be constructed in such a way that $p'_{ij} \approx 0.5$ and $p'_{ik} \approx p'_{jk}$ holds for each $k \in N \setminus \{8\}$.

Consider now the probability matrix $\mathbf{p}$ defined on N such that $p_{jk} \approx$

p'_{jk} holds for all $j, k < 8$ and $p_{k8} \approx 1$ holds for all $k < 8$. For the final winning probabilities of each $k < 8$ we have by construction that $\varphi_k(t, s, \mathbf{p}) \approx \varphi'_k(t^A, s', \mathbf{p}')$. Therefore, $p_{ij} > 0.5$ and $\varphi_j(t, s, \mathbf{p}) > \varphi_i(t, s, \mathbf{p})$ as it is required to prove that the competition system (t, N) violates WMS. Moreover, by construction, $\mathbf{p}$ is such that $p_{ij} \approx 0.5$ and $p_{ik} \approx p_{jk}$ for each $k \in N$.

Step 5 Let (t, N) be an elimination-type competition system with t being a limited extension from the root of an antler. Then (t, N) violates WMS.

Proof. Let t^A be the (symmetric or asymmetric) antler contained in t and fix *any* $s \in \mathcal{S}^{(t,N)}$. Because s is arbitrary, to show that (t, N) violates WMS, it suffices to show that the violation holds with respect to s. Let $N'_{t_A}(s)$ be the set of players seeded by s to a leaf of t^A. For notational convenience, when $i \in N'_{t^A}(s)$ we will label this player as i'.

By Lemma 3, Step 1 in the case of symmetric antlers, and Step 2 in the case of asymmetric antlers, we know that for any seeding in t^A we can find a probability matrix $\mathbf{p} \in \mathcal{P}_{R_{|N'_{t^A}(s)}}$ that makes $\left(t^A, N\right)$ violate WMS. In particular, for $s' = s_{|\Lambda(t^A)}$, there exists a matrix $\mathbf{p}'$ and players $i', h' \in N'_{t^A}(s)$ with $p'_{h',i'} > 0.5$ and $\varphi_{i'}(t^A, s', \mathbf{p}') > \varphi_{h'}(t^A, s', \mathbf{p}')$. Moreover, we know that $\mathbf{p}'$ can be constructed in such a way that $p'_{h',i'} \approx 0.5$ and $p'_{h',k'} \approx p'_{i',k'}$ holding for each $k' \in N'_{t^A}(s)$.

Now, for all $k \in N \setminus N'_{t^A}(s)$ let $sup(k) = min\{x' \in N'_{t^A}(s)$ such that $x' > k\}$ and $inf(k) = max\{x' \in N'_{t^A}(s)$ such that $x' < k\}$.

Let us define a probability matrix $\mathbf{p}$ on N such that (1) for all $x', y' \in N'_{t^A}(s)$, $p_{x'y'} = p'_{x'y'}$; (2) for all $k \in N \setminus N'_{t^A}(s)$ such that $sup(k)$ exists, $p_{k,sup(k)} = 0.5$ (and $p_{kw} = p_{sup(k),w}$ for each $w \in N$); (3) for all $k \in N'_{t^A}(s)$ such that $sup(k)$ does not exist, $p_{k,inf(k)} = 0.5$ (and $p_{kw} = p_{inf(k),w}$ for each $w \in N$).

In other words, $\mathbf{p}$ restricted to the elements of $N'_{t^A}(s)$ is equal to $\mathbf{p}'$, and all the players that are not seeded to t^A are assimilated as equally strong as his or her immediately weaker player in $N'_{t^A}(s)$. Moreover, if for some element k not seeded to t^A there is no weaker player in $N'_{t^A}(s)$, then k is considered as equally strong as his or her immediately stronger player in $N'_{t^A}(s)$. Thus, by construction, $\mathbf{p} \in \mathcal{P}_R$.

Note that, by $p'_{h'w'} \approx p'_{i'w'}$ for each $w' \in N'_{t^A}(s)$, we have by construction that $p_{h'w} \approx p_{i'w}$ holds for each $w \in N$.

Now, let V_{v_0, v_0^A} be the set of nodes of the shortest path between the root v_0 of t and the root v_0^A of t^A. Note that, due to t being a binary tree, for each $v \in V_{v_0,v_0^A}$ with $\ell(v) \geq 1$ there always exists a unique node $v' \notin V_{v_0,v_0^A}$ with $\ell(v') = \ell(v)$ at distance 2 from v. That is, the two players having reached these two nodes play against each other to arrive at their common immediate predecessor $v'' \in V_{v_0,v_0^A}$ with $\ell(v'') = \ell(v) - 1$. By letting $N_{v'}$ be the set of all players seeded by s to some leaf of the subtree of t rooted at v', we get $\varphi_i(t, s, \mathbf{p}) = \varphi_{i'}(t^A, s', \mathbf{p}') \cdot \prod_{v \in V_{v_0,v_0^A}, \ell(v) \geq 1} \sum_{k \in N_{v'}} p_{i'k} p_k^{v'}$ and $\varphi_h(t, s, \mathbf{p}) = \varphi_{h'}(t^A, s', \mathbf{p}') \cdot \prod_{v \in V_{v_0,v_0^A}, \ell(v) \geq 1} \sum_{k \in N_{v'}} p_{h'k} p_k^{v'}$.

Recall that $p_{i',w} \approx p_{h',w}$ holds for each $w \in N$. Moreover, $p_k^{v'}$ is independent of whether i' or h' have reached node $v \in V_{v_0,v_0^A}$. Therefore, $\varphi_{i'}(t^A, s', \mathbf{p}') > \varphi_{h'}(t^A, s', \mathbf{p}')$ implies $\varphi_i(t, s, \mathbf{p}) > \varphi_h(t, s, \mathbf{p})$. Thus, the competition system (t, N) violates WMS.

Step 6 Let (t, N) be an elimination-type competition system with t being a limited extension from the root of a one-bye antler. Then (t, N) violates WMS.

The proof is analogous to the proof of Step 5.

Step 7 Let (t, N) be an elimination-type competition system with t being a limited extension from the root of a balanced tree of height 3. Then, (t, N) violates WMS.

Again, the proof is analogous to that of Step 5.

Step 8 Let t^* be a limited extension from the root of an antler t^A and (t, N) be an elimination-type competition system with t being an extension from the leaves of t^*. Then, (t, N) violates WMS.

Proof. For the proof of the statement of Step 8, we will need the following additional notation.

Let $d(v_0, v_0^A)$ stand for the geodesic distance between the root v_0 of t and the root v_0^A of t^A. For $x \in \{0, \ldots, h(t) - d(v_0, v_0^A)\}$, we denote by t_0^x the subgraph of t consisting of all nodes $v \in V(t)$ with $\ell(v) \leq d(v_0, v_0^A) + x$ and the corresponding edges of t connecting them. That is, t_0^x is just the tree t when being truncated at level $d(v_0, v_0^A) + x$. Clearly, $x = h(t^A) = 3$ implies $t_0^x = t^*$.

We denote by M^x the set of matches at level $d(v_0, v_0^A) + x$ of t_0^x (with m^x being a typical element of M^x), and by T_k^x the set of subgraphs of t_0^x that can be obtained from t_0^x by removing a number k of matches at level $d(v_0, v_0^A) + x$ (with t_k^x being a typical element of T_k^x). Clearly, $T_{|M^x|}^x = t_{|M^x|}^x = t_0^{x-1}$.

Moreover, for any tree $t_k^x \in T_k^x$ we consider a set of players $N_k^x = \{1, \ldots, n_k^x\}$ that makes competition (t_k^x, N_k^x) feasible; that is, a set of players whose cardinality is $n_k^x = |\Lambda(t_k^x)|$.

Consider now, for any $k \leq |M^4|$, any tree $t_k^4 \in T_k^4$ and the corresponding set of players N_k^4 that makes (t_k^4, N_k^4) feasible. Let R be the ordering of strength defined on N_k^4. Assume that (t_k^4, N_k^4) satisfies WMS. By Lemma 3 we know that, for t_k^4 to satisfy WMS with respect to some seeding rule $s \in \mathcal{S}^{(t_k^4, N_k^4)}$, the worst player according to R should be seeded to some leaf of t_k^4 that belongs to some match in M^4. If the worst player is not unique, then assume w.l.o.g. that the selected player is n_k^4. Let us denote by m'^4 the match to which n_k^4 is seeded, by $(\lambda_a'^4)$ and $(\lambda_b'^4)$ its two leaves, and by $(\overline{n}_k^4) \neq n_k^4$ the second player seeded to m'^4; that is, the opponent of n_k^4. Now, let $(t_{k+1}'^4, N_{k+1}^4)$ be the competition system in which $t_{k+1}'^4$ has been obtained from t_k^4 by removing the match m'^4 and N_{k+1}^4 is a set of $n_k - 1$ players. Clearly, the common immediate predecessor w of $\lambda_a'^4$ and $\lambda_b'^4$ becomes now a leaf of $t_{k+1}'^4$ to be denoted by λ_w. Hence, $\Lambda(t_{k+1}'^4) = \Lambda(t_k^4) \cup \{\lambda_w\} \setminus \{\lambda_a'^4, \lambda_b'^4\}$.

The inductive reasoning starts by proving that, roughly speaking, if the competition (t_k^4, N_k^4) satisfies WMS and the match where the worst player is seeded at is removed, then the remaining structure also satisfies WMS.

Claim Let (t_k^4, N_k^4) and $(t_{k+1}'^4, N_{k+1}^4)$ be as above. If (t_k^4, N_k^4) satisfies WMS, then $(t_{k+1}'^4, N_{k+1}^4)$ also satisfies WMS.

Proof of the Claim. Assume that (t_k^4, N_k^4) satisfies WMS but $(t_{k+1}'^4, N_{k+1}^4)$ does not. Let R' be defined on N_{k+1}^4 such that $R' = R_{|N_{k+1}^4 \setminus \{n_k^4\}}$. Consider the seeding rule $s' : \Lambda(t_{k+1}'^4) \to N_{k+1}^4$ defined as follows: for each $\lambda \in \Lambda(t_{k+1}'^4) \setminus \{\lambda_w\}$, $s'(\lambda) = s(\lambda)$ and $s(\lambda_w) = \overline{n}_k^4$ (note that $N_{k+1}^4 = N_k^4 \setminus \{n_k^4\}$ and that $\overline{n}_k^4 \in N_{k+1}^4$). That is, s' can be interpreted as a situation in which $\overline{n}_k^4$ wins his match against n_k^4 and the remaining matches are not yet played. By hypothesis $(t_{k+1}'^4, N_{k+1}^4)$ violates WMS. This implies that for the seeding rule s' there exists some probability matrix $\mathbf{p}' \in \mathcal{P}_{R'}$ defined on N_{k+1}^4 such that $p'_{ij} > 0.5$ and $\varphi_j(t_{k+1}'^4, s', \mathbf{p}') > \varphi_i(t_{k+1}'^4, s', \mathbf{p}')$ holds for some $i, j \in N_{k+1}^4$.

Let $\mathbf{p}$ be a probability matrix on N_k^4, which is defined as follows: $p_{ij} = p'_{ij}$ for all $i, j \in N_k^4 \setminus \{n_k^4\}$, and $p_{i,n_k^4} \approx 1$ for each $i \in N_k^4 \setminus \{n_k^4\}$. Note that $\mathbf{p} \in \mathcal{P}_{R_{|N_k^4}}$ by construction. Also by construction, $\varphi_i(t_{k+1}'^4, s', \mathbf{p}') \approx \varphi_i(t_k^4, s, \mathbf{p})$ holds for each $i \in N_k^4 \setminus \{n_k^4\}$. Therefore, $p_{ij} > 0.5$ and $\varphi_j(t_k^4, s, \mathbf{p}) > \varphi_i(t_k^4, s, \mathbf{p})$ holds for some $i, j \in N_k^4$. Hence, we have a contradiction to the hypothesis that (t_k^4, N_k^4) satisfies WMS, which completes the proof of the claim.

Note that this claim holds also for $k+1 = |M^4|$. In this particular case, $t_{k+1} = t^4_{|M^4|} = t^3_0 = t^*$ which leaves us with the following three possibilities:

(i) There are two leaves at distance 2 from v_0^A; that is, there is no extension from any leaf at distance 2 from v_0^A and, therefore, t^* is a limited extension from the root of a (symmetric or asymmetric) antler.

(ii) There is a unique leaf at distance 2 from v_0^A. In this case, t^* is a limited extension from the root of a one-bye antler.

(iii) There are no leaves at distance 2 from v_0^A. In this case, t^* is a limited extension from the root of a balanced tree of height 3.

For each of these three possible cases, we have proven in the previous steps that no competition system whose graph is $t^* = t^4_{|M^4|}$ does satisfy WMS.

Now, for any $k \in \{0, \dots, |M^4|-1\}$, take any competition system (t^4_k, N^4_k) with $t^4_k \in T^4_k$. Note that from (t^4_k, N^4_k) it is always possible to define a sequence (t^4_k, N^4_k), $(t^4_{k+1}, N^4_{k+1}), \dots, (t^4_{|M^4|}, N^4_{|M^4|})$ by removing the match at level $d(v_0, v_0^A)+4$ where the corresponding worst player n^4_k, n^4_{k+1}, …, $n^4_{|M^4|-1}$ has been seeded. Given that $(t^4_{|M^4|}, N^4_{|M^4|})$ violates WMS, and considering the Claim, an inductive argument also allows to prove that (t^4_k, N^4_k) violates WMS. Therefore, in particular, (t^4_0, N^4_0) violates WMS. Recalling that $t^5_{|M^5|} = t^4_0$, we can recursively replicate the inductive argument at level $d(v_0, v_0^A) + 5$ to conclude that for all $k \in \{0, 1, \dots, |M^5|-1\}$, every competition system (t^5_k, N^5_k) with $t^5_k \in T^5_k$ violates WMS. The reasoning can be successively applied when t has been truncated at higher levels, until we reach the tree $t = t_0^{h(t)-d(v_0,v_0^A)}$, which proves that the competition system (t, N) violates WMS.

Step 9 Let (t, N) be an elimination-type competition with t containing an antler. Then, (t, N) violates WMS.

Proof. The statement follows from the fact that if a tree t contains an antler, then, clearly, it is some form of extension from the leaves of a limited extension from the root of an antler and by Step 8.

Step 10 Let (t, N) be an elimination-type competition with t being antler-free. Then, (t, N) satisfies WMS with respect to $s \in \mathcal{S}^{(t,N)}$ only if $s = s_{ib}$.

Proof. We proceed by reduction to the absurd. We assume that (t, N) satisfies WMS, $s \neq s_{ib}$, and show that this leads to a contradiction. In other words, we show that, given a strength ordering R defined on N, it is possible to find a probability matrix $\mathbf{p} \in \mathcal{P}_R$ such that if $s \neq s_{ib}$ then there exist $i, j \in N$ such that $p_{ij} > 0.5$ but $\varphi_i(t, s, \mathbf{p}) < \varphi_j(t, s, \mathbf{p})$.

We know from Lemma 3 that for (t, N) to satisfy WMS, it should be the case that for any probability matrix $\mathbf{p}$, and any $\lambda, \lambda' \in \Lambda(t)$, $\ell(\lambda) < \ell(\lambda')$ implies $p_{s(\lambda),s(\lambda')} \geq 0.5$. Moreover, by Lemma 1, $t \in ES_2^*$. Take now a maximal root-to-leaf path $\gamma(t)$ and note that for any probability matrix $\mathbf{p}$, $s \neq s_{ib}$ implies either that

(i) There exist leaves $\lambda_a, \lambda_b \in \Lambda(t)$ with $\ell(\lambda_a) = \ell(\lambda_b) < h(t)$ such that: (1) only λ_a has an immediate predecessor belonging to $V_{\gamma(t)}$ and (2) $p_{s(\lambda_b),s(\lambda_a)} > 0.5$, or that

(ii) $\left|\Lambda^{h(t)}(t)\right| = 4$ with the players in $\left\{s(\lambda) : \lambda \in \Lambda^{h(t)}(t)\right\}$ not being seeded in a balanced way.

We proceed by showing that in both cases we reach a contradiction.

Case (i) Let k be the number of players seeded by s to leaves at higher level than $\ell(\lambda_a) = \ell(\lambda_b)$. We construct the desired $\mathbf{p}$ in three steps.

First, we set $p_{n-k,n-k+1} > 0.5$ and $p_{n-k+1,z} \approx 1$ to hold for each $z > n-k+1$. By Lemma 3, the set of players seeded to the leaves at higher level than $\ell(\lambda_a)$ is $\{n, n-1, \ldots, n-k+1\}$. Thus, the probability $p_{n-k+1}^{v^{s(\lambda_a)}}$ with which player $n-k+1$ reaches node $v^{s(\lambda_a)} \in V_{\gamma(t)}$ is arbitrarily close to 1.

Second, let x_1, x_2, and x_3 be the three players seeded to the three leaves at level $\ell(\lambda_a)$ and set $p_{x_1x_2} \geq 0.5$ and $p_{x_2x_3} \geq 0.5$. By construction, $p_{x_3,(n-k+1)} > 0.5$. Note also that, with a probability arbitrary close to 1, the players $(n-k+1)$, x_1, x_2, and x_3 play a balanced sub-competition at level $\ell(\lambda_a)$ with the root of the sub-competition being $v \in V_{\gamma(t)}$ with $\ell(v) = \ell(\lambda_a) - 2$. Moreover, by hypothesis, $n-k+1$ plays a match against some player x_i ($i \in \{2, 3\}$) such that $p_{x_1x_i} > 0.5$. Therefore, by Lemma 2, it is possible to define a probability matrix $\mathbf{p}'$ on the player set $\{n-k+1, x_1, x_2, x_3\}$ such that there are players $i, j \in \{n-k+1, x_1, x_2, x_3\}$ with $p'_{ij} > 0.5$ and $p_i'^v < p_j'^v$. Moreover, we know by the proof of Lemma 2 that $\mathbf{p}'$ can always be constructed in such a way that $p'_{iw} \approx p'_{jw}$ holds for each $w \in \{n-k+1, x_1, x_2, x_3\}$. We then take $p_{xy} = p'_{xy}$ to hold for all $x, y \in \{n-k+1, x_1, x_2, x_3\}$. This implies that, according to $\mathbf{p}$, $p_i^v < p_j^v$. It also implies $p_{iw} \approx p_{jw}$ for each $w \in \{n-k+1, x_1, x_2, x_3\}$.

Third, we take $p_{zx_1} = 0.5$ to hold for each $z \in N$ who is seeded at a lower level than $\ell(\lambda_a)$. That is, every player who is seeded at a lower level than $\ell(\lambda_a)$ is considered as being equally strong as the strongest player at level $\ell(\lambda_a)$. Note that the latter fact together with $p_{n-k+1,z} \approx 1$ for each

$z > n-k+1$ implies $p_{iw} \approx p_{jw}$ for each $w \in N$.

In summary, the constructed probability matrix $\mathbf{p}$ is as follows: the restriction of p on the player set $\{n-k+1, x_1, x_2, x_3\}$ is $\mathbf{p}'$; x_1 is equally strong as every player who is seeded at lower levels; and, each player in $\{n-k+1, x_1, x_2, x_3\}$ wins with a probability arbitrarily close to 1 the match against any player being seeded at higher levels and is different from $n-k+1$. Now, by using the notation of Step 1 and recalling that $\ell(v) = \ell(\lambda_a) - 2$, we have

$$\varphi_i(t, s, \mathbf{p}) = p_i^v(s) \cdot \prod_{x \in S^1_{v'}(s), \ell(v') < \ell(v)} p_{ix} \cdot \prod_{y,z \in S^2_{v'}(s), \ell(v') < \ell(v)} (p_{iy}p_{yz} + p_{iz}p_{zy})$$

and

$$\varphi_j(t, s, \mathbf{p}) = p_j^v(s) \cdot \prod_{x \in S^1_{v'}(s), \ell(v') < \ell(v)} p_{jx} \cdot \prod_{y,z \in S^2_{v'}(s), \ell(v') < \ell(v)} (p_{jy}p_{yz} + p_{jz}p_{zy}).$$

Then, we have that $p_j^v > p_i^v$, $p_{ix} \approx p_{jx}$ for each $x \in N$, and because p_{yz} (p_{zy}) is independent of any other parameter in the respective formulae for i and j, we conclude that $\varphi_j(t, s, \mathbf{p}) > \varphi_i(t, s, \mathbf{p})$ should also hold, which is in contradiction to (t, N) satisfying WMS with respect to s in this case.

Case (ii) Note that in this case $\left|\Lambda^{h(t)}\right| = 4$ implies that the node $v \in V_{\gamma(t)}$ with $\ell(v) = h(t) - 2$ is the root of a balanced subtree of t. We construct the desired $\mathbf{p}$ in two steps.

First, let $\{a, b, c, d\} \subseteq N$ be the set of players seeded to the leaves in $\Lambda^{h(t)}$. It follows then from Lemma 2 that, for each of the two possible non-balanced seedings of the players in $\{a, b, c, d\}$ to the leaves in $\Lambda^{h(t)}$, there exists a probability matrix $\mathbf{p}'$ on $\{a, b, c, d\}$ such that $p'_{ij} > 0.5$ and $p'^v_j > p'^v_i$ holds for some $i, j \in \{a, b, c, d\}$. Moreover, it follows from the proof of Lemma 2 that $\mathbf{p}'$ can be constructed in such a way that $p'_{ij} \approx 0.5$ and $p'_{iw} \approx p'_{jw}$ for each $w \in \{a, b, c, d\}$. Thus, we take $\mathbf{p}$ to be such that $p_{xy} = p'_{xy}$ for all $x, y \in \{a, b, c, d\}$. Let a be a strongest player and d a weakest player among those in $\{a, b, c, d\}$. By Lemma 3, for all players $x \in N \setminus \{a, b, c, d\}$ and $i \in \{a, b, c, d\}$, $p_{xi} \geq 0.5$.

Second, we set $p_{zw} = 0.5$ to hold for all $z, w \in N \setminus \{b, c, d\}$.

Thus, $\mathbf{p}$ is as follows: the restriction of $\mathbf{p}$ on the player set $\{a, b, c, d\}$ is $\mathbf{p}'$, while each of the remaining players (who are seeded to leaves at lower levels that $h(t)$ in the tree) is considered as equally strong as the strongest player in $\{a, b, c, d\}$. Moreover, $p'_{iw} \approx p'_{jw}$ holding for each $w \in \{a, b, c, d\}$ implies by construction that $p_{iw} \approx p_{jw}$ is also true for each $w \in N$.

By using the notation of Step 1 and recalling that $\ell(v) = h(t) - 2$, we have

$$\varphi_i(t, s, \mathbf{p}) = p_i^v(s) \cdot \prod_{x \in S^1_{v'}(s): \ell(v') < \ell(v)} p_{ix} \cdot \prod_{y,z \in S^2_{v'}(s): \ell(v') < \ell(v)} (p_{iy}p_{yz} + p_{iz}p_{zy})$$

and

$$\varphi_j(t, s, \mathbf{p}) = p_j^v(s) \cdot \prod_{x \in S^1_{v'}(s): \ell(v') < \ell(v)} p_{jx} \cdot \prod_{y,z \in S^2_{v'}(s): \ell(v') < \ell(v)} (p_{jy}p_{yz} + p_{jz}p_{zy}).$$

Again we have that $p_j^v > p_i^v$, $p_{ix} \approx p_{jx}$ for each $x \in N$, and because p_{yz} (p_{zy}) is independent of any other parameter in the respective formulae for i and j, we conclude that $\varphi_j(t, s, \mathbf{p}) > \varphi_i(t, s, \mathbf{p})$ holds. Thus, we have again a contradiction to (t, N) satisfying WMS with respect to s. This completes the proof of Theorem 3. ■

B Appendix: Omitted proofs from Section 5

We start by introducing some additional notation and remarks. Recall that, given a competition (G, N), w_t denotes the player who reaches the root of the binary tree $t \in G$. Moreover, $i \in N$ is a winner of the competition (G, N) if $|\{t \in G : w_t = i\}| \geq |\{t \in G : w_t = j\}|$ holds for all $j \in N$. For every $t \in G$, λ_t and λ'_t denote the two leaves of t. Then, given $G = \{t_1, t_2, \ldots, t_{\#G}\}$ and $s \in \mathcal{S}^{(G,N)}$, $w = \left(w_{t_1}, w_{t_2}, \ldots, w_{t_{\#G}}\right)$ stands for the corresponding vector or configuration of tree winners, where $w_t \in \{s(\lambda_t), s(\lambda'_t)\}$ holds for each $t \in G$. Given $s \in \mathcal{S}^{(G,N)}$ and a probability matrix $\mathbf{p}$, the probability $pr(w)$ for the occurrence of a configuration w is given by $pr(w) = \prod\limits_{t \in G} \varphi_{w_t}(t, s, \mathbf{p})$. Then, for each $i \in N$, $\varphi_i(G, s, \mathbf{p})$ can be expressed by $\varphi_i(G, s, \mathbf{p}) = \sum\limits_{w \in W_{[i]}} pr(w) = \sum\limits_{w \in W_{[i]}} \prod\limits_{t \in G} \varphi_{w_t}(t, s, \mathbf{p})$, where $W_{[i]}$ stands for the set of all configurations for which i is the final winner of the competition. Additionally, when the probability matrix $\mathbf{p}$ is such that $p_{ij} = 0.5$ holds for all $i, j \in N$, we have $pr(w) = (0.5)^{\#G}$ for each configuration of tree winners, and $\varphi_i(G, s, \mathbf{p}) = (0.5)^{\#G} \cdot \left|W_{[i]}\right|$ holding for each $i \in N$.

Proof of Theorem 4. Let (G, N) be a league-type competition system with either $|\Lambda(G)| = |N|$ or $|\Lambda(G)| > |N| = 2$. We show first that (G, N) satisfies ET. Let $s \in \mathcal{S}^{(G,N)}$ be an arbitrary but fixed seeding rule and $\mathbf{p}$ the probability matrix with $p_{ij} = 0.5$ for all $i, j \in N$. Notice then that each player from N participates either in exactly one match (when $|\Lambda(G)| = |N|$) or there are only two players who participate in all matches (when $|\Lambda(G)| > |N| = 2$). Let us consider the following cases separately.

Case 1 ($|\Lambda(G)| > |N| = 2$ and $\#G$ is odd). Let $N = \{1, 2\}$ and notice that in this case we have $\varphi_1(G, s, \mathbf{p}) = (0.5)^{\#G} \cdot \left|W_{[1]}\right|$ and $\varphi_2(G, s, \mathbf{p}) = (0.5)^{\#G} \cdot \left|W_{[2]}\right|$. By $\#G$ being odd, $w \in W_{[1]}$ implies $w \notin W_{[2]}$, and $w \in W_{[2]}$ implies $w \notin W_{[1]}$. Thus, $W_{[1]} \cap W_{[2]} = \emptyset$. Finally, by $N = \{1, 2\}$ and $W_{[1]} \cap W_{[2]} = \emptyset$, we can define a bijection $f : W_{[1]} \to W_{[2]}$ by just replacing 1 by 2 and 2 by 1 as tree winners at each $w \in W_{[1]}$ as to get $f(w) \in W_{[2]}$. Thus $\left|W_{[1]}\right| = \left|W_{[2]}\right|$ and $\varphi_1(G, s, \mathbf{p}) = \varphi_2(G, s, \mathbf{p})$ holds. We conclude then that (G, N) satisfies ET.

Case 2 ($|\Lambda(G)| > |N| = 2$ and $\#G$ is even). Notice that in this case $W_{[1]} \cap W_{[2]} \neq \emptyset$. We have then that $w \in W_{[1]} \setminus W_{[2]}$ implies $w \notin W_{[2]} \setminus W_{[1]}$, and $w \in W_{[2]} \setminus W_{[1]}$ implies $w \notin W_{[1]} \setminus W_{[2]}$. Finally, we can define a bijection $f : W_{[1]} \setminus W_{[2]} \to W_{[2]} \setminus W_{[1]}$ in the same way as in Case 1 and conclude that $\left|W_{[1]} \setminus W_{[2]}\right| = \left|W_{[2]} \setminus W_{[1]}\right|$ should follow and thus, $\left|W_{[1]}\right| = \left|W_{[2]}\right|$ holds as well. We conclude that $\varphi_1(G, s, \mathbf{p}) = \varphi_2(G, s, \mathbf{p})$ and thus, (G, N) satisfies ET also in this case.

Case 3 ($|\Lambda(G)| = |N|$). Because each player participates in exactly one match in this case, then for any configuration w of tree winners, a player is a final winner of the competition only if he or she is the winner of the match displayed by the unique tree he or she is seeded at by s. Thus, $\left|W_{[i]}\right| = \left|W_{[j]}\right|$ holds for all $i, j \in N$. By $\varphi_i(G, s, \mathbf{p}) = (0.5)^{\#G} \cdot \left|W_{[i]}\right|$ holding for each $i \in N$, (G, N) satisfies ET.

Suppose now that (G, N) is a league-type competition system satisfying ET. To show that either $|\Lambda(G)| = |N|$ or $|\Lambda(G)| > |N| = 2$ holds, let $\mathbf{p}$ be the probability matrix with $p_{ij} = 0.5$ for all $i, j \in N$. Because each binary tree of G is a balanced elementary tree, it suffices to show that, for some feasible seeding rule, $|\Lambda(G)| > |N| > 2$ leads to a contradiction (as $|N| \leq |\Lambda(G)|$ follows from (G, N) satisfying ET and the definition of a feasible seeding rule). Consider then the following possible cases.

Case 1 ($|N|$ is even and $|\Lambda(G)| = |N| + 2$). Take the seeding rule $s_1 : \Lambda(G) \to N$ defined in such a way that there are exactly $\frac{|\Lambda(G)| - |N|}{2} + 1 = 2$ matches between player 1 and player 2 (with the corresponding elementary binary trees being t' and t''), while the remaining $\frac{|N|-2}{2}$ matches are only played between players from $N \setminus \{1, 2\}$ with each of these players being assigned to exactly one leaf of G. Notice that $s_1 \in \mathcal{S}^{(G,N)}$.

Consider the competition system $(G\setminus\{t'\}, N)$ and let $s_1' : \Lambda(G\setminus\{t'\}) \to N$ be such that $s_1'(\lambda) = s_1(\lambda)$ for each $\lambda \in \Lambda(G \setminus \{t'\})$. Notice that $s_1' \in \mathcal{S}^{(G\setminus\{t'\},N)}$. Denote by $W_{[i]}^{-t'}$ the set of configurations of tree winners in the subcompetition $(G \setminus \{t'\}, N)$ for $i \in N$ and observe that, by $|\Lambda(G \setminus \{t'\})| = |N|$ and as argued in Case 3 above, $\left|W_{[1]}^{-t'}\right| = \left|W_{[3]}^{-t'}\right|$. Take $w^* \in W_{[1]}^{-t'}$ and note that this implies that player 1 wins the match t''. Thus, for each $w^* \in W_{[1]}^{-t'}$ there are exactly two configurations $w', w'' \in W_{[1]}$ of tree winners for (G, N) where, everything else being the same, either player 1 wins at t' or player 2 wins at t'. On the other hand, for w^{**} to belong to $W_{[3]}^{-t'}$ necessarily 3 wins his unique match. Thus, for each $w^{**} \in W_{[3]}^{-t'}$ there is a unique configuration $w''' \in W_{[3]}$ of tree winners for (G, N) where, everything else being the same, $w_{t'} \neq w_{t''}$. We conclude then that $\left|W_{[1]}\right| > \left|W_{[3]}\right|$ holds and thus, by $\varphi_1(G, s_1, \mathbf{p}) = (0.5)^{\#G} \cdot \left|W_{[1]}\right| > (0.5)^{\#G} \cdot \left|W_{[3]}\right| = \varphi_3(G, s_1, \mathbf{p})$, we reach a contradiction to (G, N) satisfying ET.

Case 2 ($|N|$ is even and $|\Lambda(G)| > |N| + 2$). Take the seeding rule s_1 as defined in Case 1, fix a configuration w, and note that, given the definition of s_1, player 3 is a final winner at w only if, for each $i \in N$, $w_t = i$ holds for at most one tree t of G. However, by $\frac{|\Lambda(G)|-|N|}{2} + 1 > 2$, either player 1, player 2 or both necessarily win more than one match. We conclude then that $W_{[3]} = \emptyset$ should hold. Meanwhile, $W_{[1]} \neq \emptyset$ because player 1 is a final winner of the competition if he or she wins all its matches. We have then $\varphi_1(G, s_1, \mathbf{p}) = (0.5)^{\#G} \cdot \left|W_{[1]}\right| > 0 = (0.5)^{\#G} \cdot |\emptyset| = (0.5)^{\#G} \cdot \left|W_{[3]}\right| = \varphi_3(G, s_1, \mathbf{p})$ in contradiction to (G, N) satisfying ET.

Case 3 ($|N|$ is odd). Let $N = \{1, \ldots, n\}$ and take in this case the seeding rule $s_2 : \Lambda(G) \to N$ defined as follows. Exactly $\frac{|\Lambda(G)|-|N|+1}{2}$ matches are played between player 1 and player 2, exactly one match between player 1 and player n, and the remaining $\frac{|N|-3}{2}$ matches are played between players from $N \setminus \{1, n\}$ with each of these players being assigned to exactly one leaf of G. Notice that $s_2 \in \mathcal{S}^{(G,N)}$. We proceed by considering the following three possible sub-cases.

Case 3.1 ($\frac{|\Lambda(G)|-|N|+1}{2} = 1$). Denote by t' the unique tree at which players 1 and 2 are seeded, and let t'' be the unique tree at which players 1 and n are seeded. These are the four possible combinations of winners in $\{t', t''\}$: (i) 1 wins in both t' and t''; (ii) 1 wins in t' and n wins in t''; (iii) 2 wins in t' and 1 wins in t''; and (iv) 2 wins in t' and n wins in t''. Notice that for every configuration of winners in $G \setminus \{t', t''\}$ the maximal number of matches that a player from $N \setminus \{1, 2, n\}$ wins is one. Therefore, for each of the configurations of winners in $G \setminus \{t', t''\}$, three out of the four possible combinations of winners in $\{t', t''\}$ give as a result a configuration $w \in W_{[1]}$ and only two give as a result a configuration $w \in W_{[n]}$. We conclude then that $\left|W_{[1]}\right| > \left|W_{[n]}\right|$ holds and thus, by $\varphi_1(G, s_2, \mathbf{p}) = (0.5)^{\#G} \cdot \left|W_{[1]}\right| > (0.5)^{\#G} \cdot \left|W_{[n]}\right| = \varphi_n(G, s_2, \mathbf{p})$, we reach a contradiction to (G, N) satisfying ET.

Case 3.2 ($\frac{|\Lambda(G)|-|N|+1}{2} = 2$). Denote by t' and t'' the two trees at which players 1 and 2 are seeded, and let t''' be the unique tree at which player n is seeded. Analogously to Case 3.1, it is easy to compute that, for each of the configurations of winners in $G \setminus \{t', t'', t'''\}$, three out of the eight possible combinations of winners in $\{t', t'', t'''\}$ give as a result a configuration $w \in W_{[1]}$ and only two give as a result a configuration $w \in W_{[n]}$. Therefore $\left|W_{[1]}\right| > \left|W_{[n]}\right|$ and thus, by $\varphi_1(G, s_2, \mathbf{p}) = (0.5)^{\#G} \cdot \left|W_{[1]}\right| > (0.5)^{\#G} \cdot \left|W_{[n]}\right| = \varphi_n(G, s_2, \mathbf{p})$, we reach again a contradiction to (G, N) satisfying ET.

Case 3.3 ($\frac{|\Lambda(G)|-|N|+1}{2} > 2$). Fix a configuration w and note that, given the definition of s_2, player n is a final winner at w only if, for any $i \in N$, $w_t = i$ holds for at most one tree t of G. However, by $\frac{|\Lambda(G)|-|N|+1}{2} > 2$, the latter condition is violated for either player 1, player 2, or for both players. We conclude then that $W_{[n]} = \emptyset$ should hold. Meanwhile, $W_{[1]} \neq \emptyset$ because player 1 is the winner of the competition if he or she wins all the matches that he or she plays. We have then $\varphi_1(G, s_2, \mathbf{p}) = (0.5)^{\#G} \cdot \left|W_{[1]}\right| > 0 = (0.5)^{\#G} \cdot |\emptyset| = (0.5)^{\#G} \cdot \left|W_{[n]}\right| = \varphi_n(G, s_2, \mathbf{p})$ in contradiction to (G, N) satisfying ET. ■

Proof of Theorem 5. Note first that if $|\Lambda(G)| = |N|$ holds for some (G, N), then the assertion follows from Theorem 4 and the fact that WET is weaker than ET. Suppose now that (G, N) is such that $|\Lambda(G)| = m \cdot (|N| - 1) \cdot |N|$ for some integer $m \geq 1$. Let $s \in \mathcal{S}^{(G,N)}$ be the rule letting each player participate in m matches against every other player from N. Moreover, let $\mathbf{p}$ be the probability matrix with $p_{ij} = 0.5$ for all $i, j \in N$. We show that (G, N) satisfies WET with respect to s. Recalling that $\varphi_k(G, s, \mathbf{p}) = (0.5)^{\#G} \cdot \left|W_{[k]}\right|$ holds for each $k \in N$, then it is enough to prove that the number of configurations of tree winners at which a particular player is a final winner is the same across all players. We proceed as follows.

For $i, k \in N$, let $t_{ik} = (t_{ik}^1, t_{ik}^2, \ldots, t_{ik}^m)$ stand for the vector of trees of G which display the m matches between player i and player k. Fix now two players i and j, and for each $w \in W_{[j]}$ define the function $f(w)$ as follows:

(1) For each $x \in \{1, 2, \ldots, m\}$: set $f(w_{t_{ji}^x}) \neq w_{t_{ji}^x}$;

(2) For each $k \in N \setminus \{i, j\}$ and each $x \in \{1, 2, \ldots, m\}$: if $w_{t_{jk}^x} = j$ and $w_{t_{ik}^x} = k$, set $f(w_{t_{jk}^x}) = k$ and $f(w_{t_{ik}^x}) = i$;

(3) For each $k \in N \setminus \{i, j\}$ and each $x \in \{1, 2, \ldots, m\}$: if $w_{t_{jk}^x} = k$ and $w_{t_{ik}^x} = i$, set $f(w_{t_{jk}^x}) = j$ and $f(w_{t_{ik}^x}) = k$;

(4) For each $k \in N \setminus \{i, j\}$ and each $x \in \{1, 2, \ldots, m\}$: if $w_{t_{jk}^x} = w_{t_{ik}^x} = k$, set $f(w_{t_{jk}^x}) = f(w_{t_{ik}^x}) = k$;

(5) For each $k \in N \setminus \{i, j\}$ and each $x \in \{1, 2, \ldots, m\}$: if $w_{t_{jk}^x} = j$ and $w_{t_{ik}^x} = i$, set $f(w_{t_{jk}^x}) = j$ and $f(w_{t_{ik}^x}) = i$;

(6) For each $q \in N \setminus \{i, j\}$, each $k \in N \setminus \{q\}$, and each $x \in \{1, 2, \ldots, m\}$: set $f(w_{t_{qk}^x}) = w_{t_{qk}^x}$.

Notice that, by construction, the number of matches won by i at $f(w)$ is the same as the number of matches won by j at w and vice versa. Moreover, also by construction, for each $k \neq i, j$, the number of matches won by k is the same in both w and $f(w)$. Hence, $w \in W_{[j]} \setminus W_{[i]}$ implies $f(w) \in W_{[i]} \setminus W_{[j]}$, while $w \in W_{[j]} \cap W_{[i]}$ implies $f(w) \in W_{[i]} \cap W_{[j]}$ and, thus, $f(w) \in W_{[i]}$ holds.

Let us now show that f is a bijection, that is, that $w \neq w'$ for $w, w' \in W_{[j]}$ implies $f(w) \neq f(w')$.

If $w_{t_{ij}^x} \neq w'_{t_{ij}^x}$ for some $x \in \{1, 2, \ldots, m\}$ or $w_{t_{qk}^x} \neq w'_{t_{qk}^x}$ for some $q \in N \setminus \{i, j\}$ and $k \in N \setminus \{q\}$, then $f(w_{t_{ij}^x}) \neq f(w'_{t_{ij}^x})$ and $f(w_{t_{qk}^x}) \neq f(w'_{t_{qk}^x})$ clearly holds due to parts (1) and (6), respectively, of the above construction.

If $j = w_{t_{jk}^x} \neq w'_{t_{jk}^x} = k$ for some $k \in N \setminus \{i, j\}$ and $x \in \{1, 2, \ldots, m\}$, then the following four cases are possible:

(a) $w_{t_{ik}^x} = w'_{t_{ik}^x} = i$. We have then $f(w_{t_{jk}^x}) = j$, $f(w_{t_{ik}^x}) = i$, $f(w'_{t_{jk}^x}) = j$, $f(w'_{t_{ik}^x}) = k$;

(b) $w_{t_{ik}^x} = w'_{t_{ik}^x} = k$. We have then $f(w_{t_{jk}^x}) = k$, $f(w_{t_{ik}^x}) = i$, $f(w'_{t_{jk}^x}) = k$, $f(w'_{t_{ik}^x}) = k$;

(c) $w_{t_{ik}^x} = i$ and $w'_{t_{ik}^x} = k$. We have then $f(w_{t_{jk}^x}) = j$, $f(w_{t_{ik}^x}) = i$, $f(w'_{t_{jk}^x}) = k$, $f(w'_{t_{ik}^x}) = k$;

(d) $w_{t_{ik}^x} = k$ and $w'_{t_{ik}^x} = i$. We have then $f(w_{t_{jk}^x}) = k$, $f(w_{t_{ik}^x}) = i$, $f(w'_{t_{jk}^x}) = j$, $f(w'_{t_{ik}^x}) = k$.

Thus, $f(w) \neq f(w')$ holds for each of the possible cases.

Finally, if $i = w_{t^x_{ik}} \neq w'_{t^x_{ik}} = k$ for some $k \in N \backslash \{i, j\}$ and $x \in \{1, 2, \ldots, m\}$ the situation is completely analogous to the previous one and, thus, one can also in this last case conclude that $f(w) \neq f(w')$. We conclude that f is a bijection and, therefore, $\left|W_{[i]}\right| = \left|W_{[j]}\right|$ holds. Hence, (G, N) satisfies WET with respect to s. ■

Proof of Theorem 6. We first show that if $N = \{1, 2\}$, then (G, N) satisfies MS. Let $\mathbf{p}$ be an arbitrary but fixed probability matrix. Let $s \in \mathcal{S}^{(G,N)}$ and notice that, by the definition of a feasible seeding rule and $N = \{1, 2\}$, s assigns each player from N to exactly $\frac{|\Lambda(G)|}{2}$ leaves of G. Suppose now that $p_{12} > 0.5$. We have to show that $\varphi_1(G, s, \mathbf{p}) - \varphi_2(G, s, \mathbf{p}) \geq 0$ holds. If $\#G = 1$ then the inequality follows immediately. Suppose then that $\#G \geq 2$ and consider the following two cases.

Case 1 ($|\Lambda(G)| > |N| = 2$ and $\#G$ is odd). Note that in this case we have

$$\varphi_1(G, s, \mathbf{p}) = p_{12}^{\#G} + p_{12}^{\#G-1} \cdot p_{21} + p_{12}^{\#G-2} \cdot p_{21}^2 + \ldots + p_{12}^{(\#G+1)/2} \cdot p_{21}^{(\#G+1)/2-1}$$

and

$$\varphi_2(G, s, \mathbf{p}) = p_{21}^{\#G} + p_{21}^{\#G-1} \cdot p_{12} + p_{21}^{\#G-2} \cdot p_{12}^2 + \ldots + p_{21}^{(\#G+1)/2} \cdot p_{12}^{(\#G+1)/2-1}.$$

Thus, $\varphi_1(G, s, \mathbf{p}) - \varphi_2(G, s, \mathbf{p}) = \left(p_{12}^{\#G} - p_{21}^{\#G}\right) + p_{12} \cdot p_{21} \cdot \left(p_{12}^{\#G-2} - p_{21}^{\#G-2}\right) + \ldots + p_{12}^{(\#G+1)/2-1} \cdot p_{21}^{(\#G+1)/2-1} \cdot (p_{12} - p_{21}) > 0$, where the inequality follows from $p_{12} > 0.5$. We conclude that (G, N) satisfies MS.

Case 2 ($|\Lambda(G)| > |N| = 2$ and $\#G$ is even). When expressing $\varphi_1(G, s, \mathbf{p})$ and $\varphi_2(G, s, \mathbf{p})$ for this case there are two differences in comparison to the corresponding expressions in Case 1. First, $(\#G+1)/2$ should be replaced by $\#G/2+1$, and $(\#G+1)/2-1$ by $\#G/2-1$. Second, in both expressions the probabilities of the configurations where both players, 1 and 2, are winners should be considered; that is, the term $p_{12}^{\#G/2} + p_{21}^{\#G/2}$ should be added in both expressions. Since these configurations for player 1 and player 2 do coincide (due to $\#G$ being even), the added terms are the same for both players. Thus, they cancel out when taking the difference between $\varphi_1(G, s, \mathbf{p})$ and $\varphi_2(G, s, \mathbf{p})$. Thus, by reproducing the same reasoning as in Case 1, we can conclude that (G, N) satisfies MS also in this case.

Suppose now that (G, N) satisfies MS. We show that $|N| > 2$ leads to a contradiction. Consider first the case where $N = \{1, 2, 3\}$ and the following probability matrix:

$$\mathbf{p} = \begin{pmatrix} 0.5 & 0.5+\varepsilon & 1-\varepsilon \\ & 0.5 & 1-2\varepsilon \\ & & 0.5 \end{pmatrix}$$

Take the seeding rule $s \in \mathcal{S}^{(G,N)}$ that assigns players to leaves in such a way that there is exactly one match between player 1 and player 2, and there are $\frac{|\Lambda(G)|-2}{2}$ matches between player 2 and player 3.

If $\#G = 2$, then $\varphi_1(G, s, \mathbf{p}) \approx 0.5$ and $\varphi_2(G, s, \mathbf{p}) \approx 1$ in contradiction to $p_{12} > 0.5$ and (G, N) satisfying MS.

If $\#G > 2$, given s and v, we have that player 2 wins with probability arbitrarily close to 1 all matches except one and, therefore, he or she wins with probability close to one most of the matches. Thus, $\varphi_2(G, s, \mathbf{p}) \approx 1$ and $\varphi_1(G, s, \mathbf{p}) \approx 0$, reaching again a contradiction to $p_{12} > 0.5$ and (G, N) satisfying MS.

Suppose next that $|N| > 3$ holds and consider a probability matrix $\mathbf{p}$ such that $p_{23} > 0.5$ and $p_{3k} \approx 1$ for all $k \in N \setminus \{1, 2\}$. We distinguish the following two cases.

Case 1 ($|\Lambda(G)| = |N|$). In this case each feasible seeding rule assigns each player to exactly one leaf of G. Take $s_1 \in \mathcal{S}^{(G,N)}$ to be such that there are initial matches between players 1 and 2, and 3 and 4, respectively. Note that, for any configuration of tree winners, a player is a final winner in the entire competition only if he or she wins his or her unique match according to s_1. Thus, $\varphi_3(G, s_1, \mathbf{p}) \approx 1$ and $\varphi_2(G, s_1, \mathbf{p}) \leq 0.5$ in contradiction to $p_{23} > 0.5$ and (G, N) satisfying MS.

Case 2 ($|\Lambda(G)| > |N|$). Consider the seeding rule $s_2 \in \mathcal{S}^{(G,N)}$ defined as follows: players are assigned to leaves in such a way that players 1 and 2 are seeded exactly once and they play against each other, while player 3 participates in each of the other $\frac{|\Lambda(G)|-2}{2}$ matches. Note that, according to s_2 and the probability matrix p, player 3 wins with probability arbitrarily close to one all matches he is participating in and, therefore, he or she wins with probability close to one every match of the competition except one (the one played between players 1 and 2). Hence, $\varphi_3(G, s_2, \mathbf{p}) \approx 1$ and $\varphi_2(G, s_2, \mathbf{p}) \approx 0$ in contradiction to $p_{23} > 0.5$ and (G, N) satisfying MS. ■

Proof of Theorem 7. Let us start with the case where $|\Lambda(G)| = |N|$. Let $\mathbf{p}$ be an arbitrary but fixed probability matrix and, recalling that $|N|$ is even, consider the seeding rule $s_1 : \Lambda(G) \to N$ defined by $s_1(\lambda(t)) + s_1(\lambda'(t)) = n+1$ holding for each $t \in G$. That is, s_1 matches in the elementary binary trees of

G the best player with the worst one, the second best with the second worst, and so on. Clearly, given s_1, a player is a final winner in a configuration of tree winners if and only if he is the winner of the unique match that he or she plays. Thus, for $k \in N$, we have $\varphi_k(G, s_1, \mathbf{p}) = p_{kk'}$ with $k' \in N$ being the player seeded by s_1 at the same tree as player k.

Suppose now that $p_{ij} > 0.5$ holds for some $i, j \in N$. We have to show that $\varphi_i(G, s_1, \mathbf{p}) \geq \varphi_j(G, s_1, \mathbf{p})$ follows in such a case. For this, let $t^* \in G$ be the unique binary tree at which i is seeded by s_1 together with some other player i', and $t^{**} \in G$ be the unique binary tree at which j is seeded by s_1 together with some other player j'. If $i' = j$, then i and j are seeded by s_1 to the same tree and thus, $\varphi_i(G, s_1, \mathbf{p}) = p_{ij} > p_{ji} = \varphi_j(G, s_1, \mathbf{p})$ follows. If $i' \neq j$, we have from $p_{ij} > 0.5$ that iRj holds, while $j'Ri'$ holds due to the construction of s_1. Thus, $p_{ii'} \geq p_{jj'}$ follows by (3). We have then $\varphi_i(G, s_1, \mathbf{p}) = p_{ii'} \geq p_{jj'} = \varphi_j(G, s_1, \mathbf{p})$ as required for showing that (G, N) satisfies WMS with respect to s_1.

Let us now consider a league-type competition (G, N) such that $|\Lambda(G)| \geq 2\,(|N| - 1)$ holds. Fix a probability matrix $\mathbf{p}$ and consider the seeding rule s_2 defined as follows: player 1 is seeded to each tree of G, each player $k \in \{3, \ldots, n\}$ is seeded to exactly one tree of G, and player 2 is seeded to each of the remaining trees of G. Because there are at least $(|N| - 1)$ matches in the competition, $s_2 \in \mathcal{S}^{(G,N)}$ follows. We show that (G, N) satisfies WMS with respect to s_2 in three steps. In what follows $T^{(12)}$ stands for the set of matches played between 1 and 2 according to s_2.

Step 1 If $p_{12} > 0.5$, then $\varphi_1(G, s_2, p) \geq \varphi_2(G, s_2, p)$.

Proof. Recall that $\varphi_1(G, s_2, \mathbf{p}) = \sum\limits_{w \in W_{[1]}} pr(w)$ and $\varphi_2(G, s_2, \mathbf{p}) = \sum\limits_{w \in W_{[2]}} pr(w)$. Given a configuration w of winners let $T^{(12)}_{w_t=1}$ be the set of trees t in $T^{(12)}$ such that $w_t = 1$, and let $T^{(12)}_{w_t=2}$ be the set of trees t in $T^{(12)}$ such that $w_t = 2$.

Define the mapping $f : W_{[2]} \to W_{[1]}$ by just exchanging, for each $t \in T^{(12)}$, the winner of the tree. That is, for all $t \in T^{(12)}$, $w_t = 1$ if and only if $f(w)_t = 2$ and $w_t = 2$ if and only if $f(w)_t = 1$. Notice that for $w, w' \in W_{[2]}$ with $w \neq w'$, $f(w) \neq f(w')$ follows and thus, f is a bijection between $W_{[2]}$ and a subset of $W_{[1]}$.

Note also that, for any configuration w of tree winners we have: $pr(w) = \prod\limits_{t \in G \setminus T^{(12)}} \varphi_{w_t}(t, s_2, \mathbf{p}) \prod\limits_{t \in T^{(12)}_{w_t=1}} p_{12} \prod\limits_{t \in T^{(12)}_{w_t=2}} p_{21}$.

Finally, note that, by $w \in W_{[2]}$, player 2 wins at w at least as many matches against player 1 as player 1 against player 2; that is, $\left|T^{(12)}_{w_t=2}\right| \geq \left|T^{(12)}_{w_t=1}\right|$. Thus, $\left|T^{(12)}_{f(w)_t=1}\right| \geq \left|T^{(12)}_{f(w)_t=2}\right|$. We have then from $p_{12} > p_{21}$ that $\prod_{t \in T^{(12)}_{w_t=1}} p_{12} \prod_{t \in T^{(12)}_{w_t=2}} p_{21} \leq \prod_{t \in T^{(12)}_{f(w)_t=1}} p_{12} \prod_{t \in T^{(12)}_{(w)_t=2}} p_{21}$ holds. Therefore, $pr(w) \leq pr(f(w))$ holds for each $w \in W_{[2]}$, which allows us to conclude that $\varphi_1(G, s_2, \mathbf{p}) \geq \varphi_2(G, s_2, \mathbf{p})$.

Step 2 If $p_{ij} > 0.5$ for $i \in \{1, 2\}$ and $j \in N \setminus \{1, 2\}$, then $\varphi_i(G, s_2, \mathbf{p}) \geq \varphi_j(G, s_2, \mathbf{p})$.

Proof. Note first that if $\left|T^{(12)}\right| > 2$ then $W_{[j]} = \emptyset$ and $\varphi_i(G, s_2, \mathbf{p}) \geq 0 = \varphi_j(G, s_2, \mathbf{p})$ immediately follows.

If $\left|T^{(12)}\right| = 2$, then $w \in W_{[j]}$ implies that every player wins exactly one match (including players 1 and 2) and all players are winners of the entire competition. Thus, $w \in W_{[j]}$ implies $w \in W_{[i]}$; that is, $W_{[j]} \subseteq W_{[i]}$ holds and hence, $\varphi_i(G, s_2, \mathbf{p}) \geq \varphi_j(G, s_2, \mathbf{p})$ follows.

Suppose now that $\left|T^{(12)}\right| = 1$. Recall that only configurations in $W_{[i]} \setminus W_{[j]}$ and in $W_{[j]} \setminus W_{[i]}$ do matter for the comparison of $\varphi_i(G, s_2, \mathbf{p})$ and $\varphi_j(G, s_2, \mathbf{p})$. If $i = 1$, then $W_{[j]} \setminus W_{[1]} = \{w\}$ with w consisting of player 1 losing all his matches and thus $pr(w) = p_{21} \cdot p_{31} \cdot \ldots \cdot p_{n1}$. Meanwhile, the configuration w' of tree winners where player 1 wins all the matches he or she plays does definitely belong to $W_{[1]} \setminus W_{[j]}$, with $pr(w') = p_{12} \cdot p_{13} \cdot \ldots \cdot p_{1n}$. Then, given that $p_{1k} \geq p_{k1}$ for each $k \in N \setminus \{1\}$ (with strict inequality holding for $k = j$), we have that $\varphi_1(G, s_2, \mathbf{p}) > \varphi_j(G, s_2, \mathbf{p})$.

In contrast, $i = 2$ implies $W_{[j]} \setminus W_{[2]} = \{w\}$ with w consisting of 1 only winning his match against 2 and $pr(w) = p_{12} \cdot p_{31} \cdot p_{41} \cdot \ldots \cdot p_{n1}$. Consider now the configuration w' of tree winners which differs from w only with respect to the fact that at w' player 2 wins his unique match against player 1 and player j loses his unique match against player 1. This involves that $w' \in W_{[2]} \setminus W_{[j]}$ with $pr(w') \geq pr(w)$ due to $p_{21} \geq p_{j1}$ following from $p_{2j} > 0.5$ and condition (2). We conclude then that $\varphi_2(G, s_2, \mathbf{p}) \geq \varphi_j(G, s_2, \mathbf{p})$.

Step 3 If $p_{ij} > 0.5$ for some $i, j \in N \setminus \{1, 2\}$, then $\varphi_i(G, s_2, \mathbf{p}) \geq \varphi_j(G, s_2, \mathbf{p})$.

Proof. Recall that in this situation each of the players i and j participates in exactly one match against player 1. Let us then consider the following three possible cases.

Case 1 ($\left|T^{(12)}\right| = 1$). In this case, there is a unique configuration $w \in W_{[j]} \setminus W_{[i]}$ consisting of player j winning his match against player 1 and player 1 winning only his match against player i. The probability of w is then $pr(w) = p_{j1} \cdot p_{1i} \cdot \prod_{k \neq i,j} p_{k1}$. Similarly, there is a unique configuration $w' \in W_{[i]} \setminus W_{[j]}$ with $pr(w') = p_{i1} \cdot p_{1j} \cdot \prod_{k \neq i,j} p_{k1}$. By $p_{ij} > 0.5$ and condition (2), $p_{1j} \cdot p_{i1} \geq p_{j1} \cdot p_{1i}$ follows. We have then $pr(w') \geq pr(w)$ which implies $\varphi_i(G, s_2, \mathbf{p}) \geq \varphi_j(G, s_2, \mathbf{p})$.

Case 2 ($\left|T^{(12)}\right| = 2$). In this case, each of the two configurations of tree winners in $W_{[j]}$ has the following structure: each of the players in $N \setminus \{1, 2\}$ (including i and j) wins his or her match against player 1; player 1 uniquely wins one of his or her matches against player 2, and player 2 wins the other match that players 1 and 2 are playing together. Clearly, $W_{[j]} = W_{[i]}$ and thus, $\varphi_i(G, s_2, \mathbf{p}) = \varphi_j(G, s_2, \mathbf{p})$ holds.

Case 3 ($\left|T^{(12)}\right| > 2$). In this case, given that there are more than two matches between player 1 and 2, $W_{[i]} = W_{[j]} = \emptyset$. Thus $\varphi_i(G, s_2, \mathbf{p}) = \varphi_j(G, s_2, \mathbf{p}) = 0$. ■

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