Coarse Bayesian Updating

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Abstract

I introduce a model of belief updating—*Coarse Bayesian updating*—where, upon receipt of new information, an agent applies subjective criteria to select among competing theories of the world. Specifically, the agent partitions the probability simplex, assigns a representative point to each cell, and selects the representative of the cell containing the Bayesian posterior. I characterize this procedure, develop its main implications, and show that it accommodates much of the evidence on non-Bayesian updating. Finally, I apply it to a standard setting of decision under risk and develop comparative measures of cognitive sophistication and bias.

1 Introduction

Bayesian updating plays a central role in economic theory. A wide body of evidence, however, suggests that actual behavior cannot be reconciled with Bayes' rule in a variety of settings. For example, individuals often display *conservatism bias*: they under-react to new evidence, possibly ignoring it altogether. Others *over-react* to information by falsely extrapolating or, more generally, engaging in pattern-seeking behavior. Combinations of these forces may lead individuals to under-weight some

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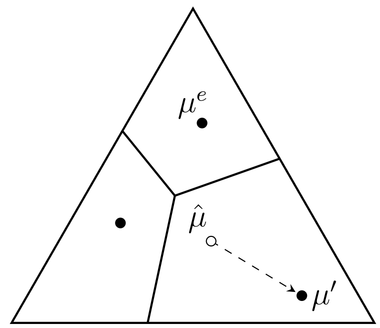


Figure 1: Coarse Bayesian Updating. In this example, there are three feasible beliefs (solid dots). The point μ^e is the prior. After observing a signal, the agent determines which cell of the partition contains the Bayesian posterior $\hat{\mu}$, then adopts the representative of that cell (in this case, μ') as his new belief.

signals while over-weighting others. In this paper, I introduce and analyze a simple generalization of Bayesian updating—*Coarse Bayesian updating*—accommodating these, and other, behavioral tendencies.

At its core, Coarse Bayesian updating is a model of bounded rationality stemming from a single key assumption: agents simplify the world by considering only a subset of the probability space. In particular, a Coarse Bayesian agent is characterized by a partition of the set of probability distributions over a state space, together with a representative distribution for each cell of the partition. One of the distributions is the prior. After observing a signal, the agent determines which cell contains the Bayesian posterior and adopts the representative of that cell as his posterior belief (see Figure 1). Since realized posteriors typically differ from their Bayesian counterparts, Coarse Bayesians may exhibit over-reaction, under-reaction, or other biases depending on the signal, the shape of the partition, and the positions of representative distributions within their cells.

The model can be interpreted in different ways. First, one might view the representative distributions as competing *theories of the world* and the partition as the agent's criteria, or *standard of proof*, for selecting among them. In this interpretation, agents correctly assess the informational content of signals and decide which, if any, competing theory ought to replace their prior. Given that the agent entertains a restricted set of theories, the procedure is a minimal deviation from Bayes' rule: the agent must have *some* criteria for choosing among feasible theories, and will adopt a given theory if it coincides with the Bayesian posterior.

Second, the procedure can be interpreted as an approximation to Bayes' rule. This captures situations where agents are cognitively constrained and cannot distinguish between beliefs within a given cell, approximating them instead by the representative point. Thus, the partition represents the agent's computational ability.

Third, one can interpret Coarse Bayesian behavior as the result of *signal distortion*, where the agent mentally transforms the observed signal before applying Bayes' rule. In this interpretation, violations of Bayes' rule are the result of imperfect perception or attention (not necessarily computational ability).

Finally, Coarse Bayesian updating can be interpreted as a form of categorical thinking. Here, each cell of the partition represents a category of beliefs, and the representative of a cell an "archetype" of that category. The agent uses available information to select a category, then adopts the archetype of that category. This interpretation can also be applied at the level of signals: the agent groups signals into categories, and updates beliefs based on the category of the realized signal.

In all cases, the parameters of the model are subjective characteristics of the individual: two Coarse Bayesians may differ in their sets of feasible beliefs, their partitions, or both. In contrast to the canonical framework of Savage (1972), then, Coarse Bayesians exhibit subjectivity not only in their prior beliefs, but also in their criteria for revising those beliefs. Some agents may tend to disregard evidence while others falsely extrapolate from it; some might be biased in favor of a particular theory, while others seek to discredit it. Even with a common prior, compelling evidence in the eyes of one agent may be completely unpersuasive for another, or result in radically different posterior beliefs. In general, Coarse Bayesians may disagree on the strength of evidence required to adopt a particular belief, or on the set of admissible beliefs to begin with, so that a given piece of evidence may yield (or magnify) disagreement among individuals.

Section 2 provides a simple characterization of the updating procedure. I take as primitive a finite, exogenous state space and an updating rule specifying an individual's beliefs at every possible signal. In my framework, signals represent messages that can be generated by stochastic information structures. Thus, a signal is a profile of numbers representing likelihoods of the associated message being generated in different states. By employing such primitives, the model is readily adaptable to any standard economic or game-theoretic setting.

The characterization involves three testable assumptions on the updating rule, each capturing some feature of "rational" Bayesian behavior. The first, *Homogeneity*, states that beliefs are invariant to scalar transformations of signals. Thus, like Bayes' rule, Coarse Bayesian updating rules only depend on the likelihood ratios of the observed signal. Second, *Cognizance* states that if two signals result in the same belief, then so does a "garbled" signal indicating that one of those signals was generated. A natural interpretation of this assumption is that the agent understands, or is cognizant of, his own updating procedure: if he is uncertain about which of two signals was generated, but recognizes that each would lead to the same posterior belief, then he adopts that belief. Finally, *Confirmation* states that if a signal exactly supports (or confirms) some feasible belief, then the updating rule associates that belief to the given signal. Theorem 1 establishes that an updating rule has a Coarse Bayesian representation if and only if it is Homogeneous, Cognizant, and Confirmatory. Moreover, the associated partition, representative elements, and prior are unique.

Section 3 explores the main implications of Coarse Bayesian updating and examines connections to related models and evidence. In section 3.1, I discuss evidence on biased belief updating and demonstrate how Coarse Bayesian models can accommodate such behavior. In addition to under- and over-reaction, I show how Coarse Bayesians may exhibit "motivated" belief updating, limited perception, extreme-belief aversion, or susceptibility to logical fallacies. Section 3.2 examines the relationship to "paradigm shifts," including the Hypothesis-Testing model of Ortoleva (2012). In particular, I examine whether Coarse Bayesians can be represented as Bayesians with second-order priors. I show that such models, dubbed *Maximum-Likelihood* updating rules, intersect the class of Coarse Bayesian rules but that neither class subsumes the other—unless there are exactly two states, in which case every Coarse Bayesian rule can be expressed as a Maximum-Likelihood rule. Finally, section 3.3 explores some basic properties of the model in dynamic settings. I show that, unlike Bayesians, most Coarse Bayesians are sensitive to the way signals are pooled and ordered.

Section 4 applies the model to a standard setting of decision under risk. In particular, I analyze how Coarse Bayesians value information (Blackwell experiments) when faced with menus of actions with state-dependent payoffs. I show that a Coarse Bayesian's ex-ante value of information can be expressed in a familiar posterior-separable form, then establish that, unlike Bayesians, Coarse Bayesians typically exhibit violations of the Blackwell (1951) information ordering—they need not assign higher ex-ante value to more informative experiments.

In section 4.2, I examine how a Coarse Bayesian's value of information changes as he becomes "more Bayesian." I consider three such orderings. First, an agent is *more sophisticated* if he employs a finer partition and a larger set of feasible posteriors. I show that more-sophisticated agents are characterized by heightened responsiveness to information, as captured by ex-ante value of information. Second, an agent is *more biased* than another if his updating rule exhibits larger distortions away from Bayesian posteriors. I show that greater bias is characterized by greater susceptibility to harmful exploitation in that worst-case losses (relative to a Bayesian) increase as bias increases. Neither greater sophistication nor lower bias imply that the agent is better off at all menus or signal realizations. The final result, therefore, shows that such welfare enhancements require the agent to be perfectly Bayesian on a larger subset of signal realizations, making the agent both more sophisticated and less biased.

Throughout the paper, my focus is on the general class of Coarse Bayesian representations and their properties. In particular, I do not take a stance on where partitions or representative elements "come from," viewing them instead as subjective (but identifiable) characteristics of an individual, much like subjective prior beliefs. There are several ways to go about endogenizing the parameters by adding assumptions about the decision problem(s) agents expect to face, the signaling structure, and costs or constraints on the fineness of the updating rule (for example, a bound on the number of cells in the partition). The results of section 4.2 suggest a slightly different approach may be valuable: rather than solving for an optimal updating rule in the context of a specific environment, one may prefer a more robust objective (characterized by the bias ordering, for example) accommodating uncertainty about the environment. I discuss this at the end of section 4.2.

1.1 Related Literature

Economists and psychologists have developed a large body of research documenting systematic violations of Bayesian updating; early contributions include Kahneman and Tversky (1972), Tversky and Kahneman (1974), and Grether (1980). As seen in the surveys of Camerer (1995), Rabin (1998), and Benjamin (2019), there is substantial variation in both the patterns of behavior displayed by subjects and the settings in which experiments are carried out. For example, studies differ in whether subjects observe individual pieces of information or larger samples (or sequences) of evidence;

whether prior beliefs are objectively induced or subjectively formed by participants; whether choices are incentivized with monetary rewards; and how problems and information are framed.

Motivated by this evidence, several authors have developed models to better understand the mechanisms behind, and consequences of, non-Bayesian updating. Models focusing on implications of biased updating are typically cast in simplified frameworks (eg, two states of the world; particular protocols or functional form assumptions), or involve non-standard elements like ambiguous signals or framing effects. See, among others, Barberis et al. (1998), Fryer et al. (2019), Gennaioli and Shleifer (2010), Rabin and Schrag (1999), and Mullainathan et al. (2008). My emphasis, particularly in sections 3 and 4, is on implications that are reasonably independent of any particular application. As such, I employ standard primitives (a finite state space; stochastic information structures; general decision problems) that can be adapted to any economic model.

Decision theorists have developed axiomatic approaches to non-Bayesian updating. Kovach (2020), for example, develops a model of conservative updating. Epstein (2006) provides a model of non-Bayesian updating accommodating under-reaction, over-reaction, and other biases; Epstein et al. (2008) extend this model to an infinite-horizon setting. Zhao (2016) axiomatizes an updating rule for signals indicating that one event is more likely than another. Like these contributions, my paper takes a general approach and characterizes behavior axiomatically. My model is not targeted toward a specific bias or application, but provides a general framework for behavior that can accommodate much of the evidence.

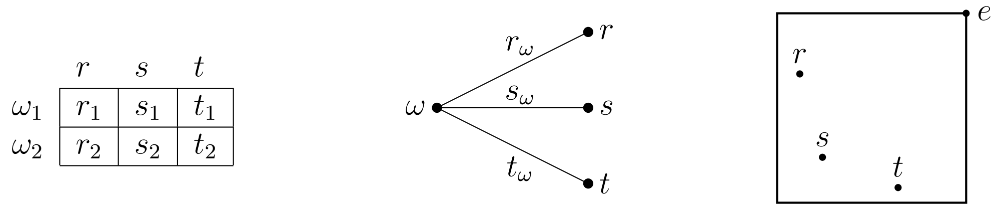
Three studies are especially relevant to Coarse Bayesian updating. First, the *hypothesis testing* model introduced and axiomatized by Ortoleva (2012) posits that agents apply standard Bayesian updating except when news is sufficiently "surprising," in which case a maximum-likelihood criterion is applied using a second-order prior. Specifically, an agent applies Bayes' rule if the prior probability of the signal exceeds a threshold $\varepsilon \geq 0$; otherwise, the agent updates a second-order prior via Bayes' rule and selects a belief of maximal probability under the revised second-order beliefs. In section 3, I show that Coarse Bayesian updating can accommodate similar behavior, and compare Coarse Bayesian updating rules to a general class of Maximum-Likelihood updating rules. I show that Coarse Bayesian rules can be expressed as Maximum-Likelihood rules if there are only two states but that, in general, neither category subsumes the other. Notably, Maximum-Likelihood rules may violate the Confirmation property—perfect evidence for a candidate belief does not guarantee that that belief is selected.

Second, Wilson (2014) studies optimal updating rules for a boundedly rational agent facing a binary decision problem and a stochastic sequence of signals. There are two states, and the agent has limited memory: only K memory states are available. In an optimal protocol, each memory state is associated with a convex set of posterior beliefs and a representative distribution for that set; if an interim Bayesian posterior belongs to some cell, then the representative of that cell is adopted as the agent's belief. Thus, the optimal protocol emerging from Wilson's model can be represented as a (dynamic) Coarse Bayesian updating procedure. Naturally, the parameters of this representation—cells and their representative points—depend on features of the environment like the signal structure, the stakes of the decision problem, and the bound K. Like standard Bayesian updating, Coarse Bayesian updating procedures do not depend on any factors other than the informational content of realized signals. I do not require Coarse Bayesian representations to be optimal in any sense, nor do I impose cognitive bounds such as a restriction on the number of cells. As illustrated in section 3, this allows the model to capture a wide range of documented behavior, including, for example, Bayesian updating except when signals are too "extreme."

Third, a working paper, Mullainathan (2002), develops a model of categorical thinking. Agents in this model follow a procedure similar to Coarse Bayesian updating where feasible posteriors represent categories and the mapping from Bayesian posteriors to categories is determined by a partition of the simplex. A key difference is that Mullainathan's partition is derived from the set of feasible posteriors: given a set of feasible posteriors, an optimality condition similar in spirit to maximization of a likelihood function is used to select a posterior. The resulting partition has convex cells, as in a Coarse Bayesian representation, but cells need not contain their representative elements. In other words, behavior in this model need not satisfy Confirmation—see appendix C for an explicit example.

2 Model

Let $\Omega = \{1, \ldots, N\}$ denote a finite set of $N \geq 2$ states and Δ the set of probability distributions over Ω. A distribution $\hat{\mu} \in \Delta$ assigns probability $\hat{\mu}_\omega$ to state $\omega \in \Omega$. Let



(a) Matrix Form (b) Tree Form (c) Signal Space

Figure 2: Three representations of an experiment $\sigma = [r, s, t]$.

Δ^0 denote the interior of Δ. If $N = 2$, the interval $[0, 1]$ of values $\hat{\mu}_1$ represents Δ.

An **experiment** is a matrix with N rows and entries in $[0, 1]$ such that each row is a probability distribution and each column has a nonzero entry. Columns represent messages that might be generated, and rows state-contingent probability distributions over messages. Let $\mathcal{E}$ denote the set of all experiments, with generic element σ.

As in Jakobsen (2020), a **signal** is a profile $s = (s_\omega)_{\omega\in\Omega} \in [0, 1]^\Omega$ such that $s_\omega \neq 0$ for at least one state ω. Let S denote the set of all signals, and endow S with the standard Euclidean metric. Intuitively, a signal s represents a column (message) of some experiment, and s_ω the likelihood of the message being generated in state ω. The notation $s \in \sigma$ indicates that s is a column of σ. I reserve e to denote the **uninformative signal**; that is, $e \in S$ and $e_\omega = 1$ for all $\omega \in \Omega$. Note that e qualifies as an experiment. Using the notation of signals, any experiment can be viewed as a collection (matrix) of signals, as a series of state-contingent distributions over signals, or as a collection of points in S that sum to e; see Figure 2.

For profiles $v = (v_\omega)_{\omega\in\Omega}$ and $w = (w_\omega)_{\omega\in\Omega}$ of real numbers, let $vw := (v_\omega w_\omega)_{\omega\in\Omega}$ denote the profile formed by multiplying v and w component-wise. Similarly, if $w_\omega > 0$ for all ω, let $v/w := (v_\omega/w_\omega)_{\omega\in\Omega}$. The dot product of v and w is given by $v \cdot w := \sum_{\omega\in\Omega} v_\omega w_\omega$. The notation $v \approx w$ indicates that $v = \lambda w$ for some $\lambda > 0$, where $\lambda w := (\lambda w_\omega)_{\omega\in\Omega}$ is the scalar product of λ with w. The standard Euclidean norm of v is denoted $\|v\|$.

For $\hat{\mu} \in \Delta$ and $s \in S$ where $s\hat{\mu} \neq 0$, let $B(\hat{\mu}|s) := \frac{s\hat{\mu}}{s\cdot\hat{\mu}} \in \Delta$ denote the **Bayesian posterior** of $\hat{\mu}$ at s. Finally, an **updating rule** is a function $\mu : S \to \Delta$ assigning probability distributions $\mu^s := \mu(s) \in \Delta$ to signals $s \in S$. I assume μ^e, the **prior**, has full support.

2.1 Coarse Bayesian Representations

Consider an agent whose behavior is summarized by an updating rule $\mu : S \to \Delta$ where μ^e has full support. The interpretation is that μ^s is the agent's posterior belief conditional on observing signal s.[1] In general, any updating behavior can be explained by some updating rule. The purpose of this section is to establish that Coarse Bayesian updating is characterized by three testable assumptions on μ, each capturing some aspect of "rational" information processing. The first assumption is:

Assumption 1 (Homogeneity). If $s \approx t$, then $\mu^s = \mu^t$.

Homogeneity requires the agent's analysis of a signal s to depend only on the likelihood ratios $s_\omega / s_{\omega'}$. This is a key feature of standard Bayesian updating: $B(\mu^e|s) = B(\mu^e|\lambda s)$ whenever $\lambda > 0$ and $\lambda s \in S$. An implication of this assumption is that the agent is not susceptible to certain types of framing effects. For example, whether information is stated in terms of frequencies or likelihoods has no effect on the agent's cognitive process.

By Homogeneity, the notation μ^s can be extended to all non-zero profiles $\tilde{s}$ such that $\tilde{s}_\omega \geq 0$ for all ω because such profiles can be scaled by a factor $\lambda > 0$ to yield a signal $\lambda \tilde{s} \in S$. This will be convenient for expressing the remaining assumptions.

Assumption 2 (Cognizance). If $\mu^s = \mu^t$, then $\mu^{s+t} = \mu^s$.

Cognizance states that if signals s and t result in the same posterior belief, then the agent adopts that belief if he knows that either s or t has realized. This interpretation stems from the fact that $s + t$ is a "garbled" signal indicating that either s or t was generated. Thus, an interpretation of Cognizance is that *the agent understands his own updating rule*: if he knows that one of two signals was generated and realizes that either one would lead him to the same posterior belief—that is, if he is cognizant of his own updating procedure—then he ought to adopt that belief. Although Cognizance is mainly motivated by normative considerations, it is also potentially important in applications. For example, section 4 studies how Coarse

[1] Note that updating rules condition beliefs on signal realizations s but not on experiments σ. In practice, a signal must be generated by an experiment, in which case one may wish to denote posterior beliefs by $\mu^{(\sigma,s)}$ where $s \in \sigma$. Like Bayesian updating, however, Coarse Bayesian updating depends on s but not the other columns of σ. To minimize notation, I have omitted the underlying experiment(s) σ.

Bayesians value information. This involves ex-ante rankings of information structures that rely on correct forecasts about updating behavior. For such exercises to make sense, an assumption like Cognizance is required.

Assumption 3 (Confirmation). For all s, $\mu^{\mu^s/\mu^e} = \mu^s$.

To understand Confirmation, observe that for any s, μ^s is a feasible posterior because it is in the range of the updating rule. Moreover, any signal $t \approx \mu^s/\mu^e$ satisfies $B(\mu^e|t) = \mu^s$. Thus, Confirmation states that if a signal t exactly supports (or confirms) a feasible posterior, then the agent adopts that posterior after observing t. Although quite intuitive and normatively appealing, this property is not always satisfied by some closely-related models—see section 3.2 and appendix C.

Theorem 1. *An updating rule μ is Homogeneous, Cognizant, and Confirmatory if and only if there is a partition $\mathcal{P}$ of Δ and a profile $\mu^{\mathcal{P}} = (\mu^P)_{P\in\mathcal{P}}$ of distributions such that*

(i) each cell $P \in \mathcal{P}$ is convex,

(ii) $\mu^P \in P$ for all $P \in \mathcal{P}$, and

(iii) for all $s \in S$, $B(\mu^e|s) \in P$ implies $\mu^s = \mu^P$.

Such a pair $\langle \mathcal{P}, \mu^{\mathcal{P}} \rangle$ is a ***Coarse Bayesian Representation*** *of μ. If $\langle \mathcal{Q}, \mu^{\mathcal{Q}} \rangle$ is another Coarse Bayesian Representation of μ, then $\mathcal{P} = \mathcal{Q}$ and $(\mu^P)_{P\in\mathcal{P}} = (\mu^Q)_{Q\in\mathcal{Q}}$.*

Proof. First, observe that if $\alpha, \beta \geq 0$ and $s, t, \alpha s + \beta t \in S$, then

$$\begin{aligned} B(\mu^e|\alpha s + \beta t) &= \frac{(\alpha s + \beta t)\mu^e}{(\alpha s + \beta t) \cdot \mu^e} \\ &= \frac{\alpha s \cdot \mu^e}{(\alpha s + \beta t) \cdot \mu^e} \frac{s\mu^e}{s \cdot \mu^e} + \frac{\beta t \cdot \mu^e}{(\alpha s + \beta t) \cdot \mu^e} \frac{t\mu^e}{t \cdot \mu^e} \\ &= \frac{\alpha s \cdot \mu^e}{(\alpha s + \beta t) \cdot \mu^e} B(\mu^e|s) + \frac{\beta t \cdot \mu^e}{(\alpha s + \beta t) \cdot \mu^e} B(\mu^e|t). \end{aligned} \tag{1}$$

Thus, $B(\mu^e|\alpha s + \beta t)$ is a convex combination of $B(\mu^e|s)$ and $B(\mu^e|t)$; the weight attached to $B(\mu^e|s)$ is the prior probability of signal αs given that either αs or βt is generated. It is now straightforward to verify that if μ has a Coarse Bayesian Representation, then Assumptions 1–3 are satisfied (Assumption 2 follows from equation (1) and convexity of cells $P \in \mathcal{P}$).

For the converse, we construct a Coarse Bayesian Representation as follows. First, note that Homogeneity and Cognizance imply μ is **Convex**: if $\mu^s = \mu^t$ and $\alpha \in [0,1]$, then $\mu^{\alpha s+(1-\alpha)t} = \mu^s$. It follows that μ is measurable with respect to a partition of S into convex cones. That is, there is a partition $\mathcal{C}$ of S such that (i) $\mu^s = \mu^t$ if and only if there exists $C \in \mathcal{C}$ such that $s, t \in C$, and (ii) every $C \in \mathcal{C}$ is a convex cone: if $s, t \in C$ and $\alpha, \beta \geq 0$ such that $\alpha s + \beta t \in S$, then $\alpha s + \beta t \in C$. Every $C \in \mathcal{C}$ can be identified with a subset of Δ by letting $P^C := \{B(\mu^e|s) : s \in C\}$. Each set P^C is convex by equation (1) and the fact that sets $C \in \mathcal{C}$ are convex cones. In addition, $\mathcal{P} := \{P^C : C \in \mathcal{C}\}$ is a partition of Δ because $B(\mu^e|s) = B(\mu^e|t)$ if and only if $s \approx t$, forcing s and t to belong to the same cone $C \in \mathcal{C}$. For each $P \in \mathcal{P}$, let μ^P denote the unique distribution $\hat{\mu}$ such that $\mu^s = \hat{\mu}$ for all $s \in C$, where $P = P^C$. Cognizance implies $\mu^s = \mu^P \in P$ whenever $B(\mu^e|s) \in P \in \mathcal{P}$. Uniqueness of $\langle \mathcal{P}, \mu^{\mathcal{P}} \rangle$ follows from uniqueness of $\mathcal{C}$. □

Theorem 1 formalizes the concept of a Coarse Bayesian Representation and establishes that an updating rule has such a representation if and only if it is Homogeneous, Cognizant, and Confirmatory. Each of these testable assumptions expresses some feature of "rational" information processing—indeed, each assumption is satisfied by a standard Bayesian. Nonetheless, Coarse Bayesian updating accommodates a variety of behavioral biases and other violations of Bayes' rule.

Coarse Bayesian agents partition the probability simplex, assign a representative point to each cell, and adopt the representative of a cell as posterior if the Bayesian posterior belongs to that cell. There are several interpretations of such behavior.

1. *Competing Theories.* Here, the agent simplifies the world by considering a set of feasible theories (representative points), sets criteria (the partition) for switching between them, and analyzes signals to the extent necessary to determine whether a change is justified. Thus, the partition represents the agent's subjective "standard of proof" for selecting among competing theories.

2. *Limited Computation.* Rather than careful deliberation over competing theories, Coarse Bayesian behavior may reflect limited ability to process information: unable to distinguish between posteriors in a given cell, the agent lumps them together with the representative point. Thus, the representation serves as a simplifying heuristic or approximation to Bayes' rule.

3. *Signal Distortions.* Similarly, one might interpret the procedure as the result of signal distortion: to update beliefs, the agent mentally transforms signals before applying Bayes' rule. Thus, apparent deviations from Bayes' rule are the result of imperfect perception or attention (not necessarily computational constraints). In appendix B, I formalize such a procedure and establish its equivalence to Coarse Bayesian updating.

4. *Categorical Thinking.* In this interpretation, the agent reasons about categories of beliefs, each represented by a cell of the partition. The representative point of a cell is an "archetype" of that category. When information arrives, the agent determines which category applies and adopts its archetype as posterior. Again, such agents need not fully compute Bayesian posteriors—they only glean enough information from a signal to figure out which category contains the Bayesian posterior. Signal distortion rules can also be interpreted as a form of categorical thinking: agents classify signals into categories, and update based on archetypes of those categories.

In each case, the parameters of the representation are subjective *characteristics of the individual*: agents may differ in their priors, partitions, or representative points. In the same way that standard Bayesian theories are agnostic about the source of one's prior beliefs, my model does not take a stance on how partitions or representative points are formed. Rather, Theorem 1 characterizes Coarse Bayesian behavior in terms of observable primitives (the updating rule) and establishes that all parameters can be uniquely identified from those primitives—with or without additional assumptions about how they might be derived.[2]

The next result provides a simple comparison between Bayesian and Coarse Bayesian behavior.

Proposition 1. *Suppose μ has a Coarse Bayesian Representation $\langle \mathcal{P}, \mu^{\mathcal{P}} \rangle$. Then μ is Bayesian (that is, $\mu^s = B(\mu^e|s)$ for all $s \in S$) if and only if $s \approx t$ whenever $\mu^s = \mu^t$.*

Proof. First, suppose $\mu^s = \mu^t$ implies $s \approx t$. Let $P \in \mathcal{P}$ and $\hat{\mu}, \hat{\mu}' \in P$. Choose signals s, t such that $B(\mu^e|s) = \hat{\mu}$ and $B(\mu^e|t) = \hat{\mu}'$. Then $\mu^s = \mu^t$, so that $s \approx t$ and, hence, $\hat{\mu} = B(\mu^e|s) = B(\mu^e|t) = \hat{\mu}'$. Thus, every cell $P \in \mathcal{P}$ is a singleton, making μ Bayesian. The converse follows from the fact that $B(\mu^e|s) = B(\mu^e|t) \Leftrightarrow s \approx t$. □

[2]See also the discussion at the end of section 4.2 regarding approaches to endogenizing the parameters.

Proposition 1 states that a Coarse Bayesian agent is in fact Bayesian if and only if posterior beliefs are sufficiently responsive to information: by strengthening Homogeneity to an "if and only if" form, one arrives at Bayesian updating. Thus, although the model accommodates many documented departures from Bayesian updating, the "wedge" between Bayesian and Coarse Bayesian updating is fairly small.

I conclude this section with a discussion of some of the model's limitations. First, like Bayes' rule, the procedure requires an agent's posterior belief to depend only on the realized signal s. More precisely, Homogeneity requires that only the *ratios* of entries in s can affect posterior beliefs. This rules out sensitivity to the way information is framed, as well as the possibility that extraneous features of the environment might impact beliefs.

A more technical matter is that Coarse Bayesian updating rules are typically discontinuous in s. If continuity is an essential conceptual feature of some pattern of behavior—rather than a convenient technical assumption—then Coarse Bayesian updating procedures will, at most, provide an approximation to that behavior.

Finally, requiring cells of the partition to be convex might seem limiting. This convexity is driven by Cognizance and can be discarded by dropping that assumption. However, as explained above, Cognizance is potentially important in applications because it means agents correctly forecast their own updating behavior.

3 Models, Evidence, and Implications

Coarse Bayesian updating is related to a number of other theories of non-Bayesian updating, and accommodates a variety of experimental findings. In this section, I examine these relationships and explore some of the main implications of the model. Sections 3.1–3.3 are independent of each other and can be read in any order; section 4 invokes Definition 4 (see section 3.3), but is otherwise independent of this section.

3.1 Bias, Asymmetry, and Perception

1. Asymmetric Updating. Conservative updating, or under-reaction to information, is a well-documented behavior violating Bayes' rule.[3] On the other hand, many indi-

[3] See Phillips and Edwards (1966) and Edwards (1968) for early experiments on conservative updating.

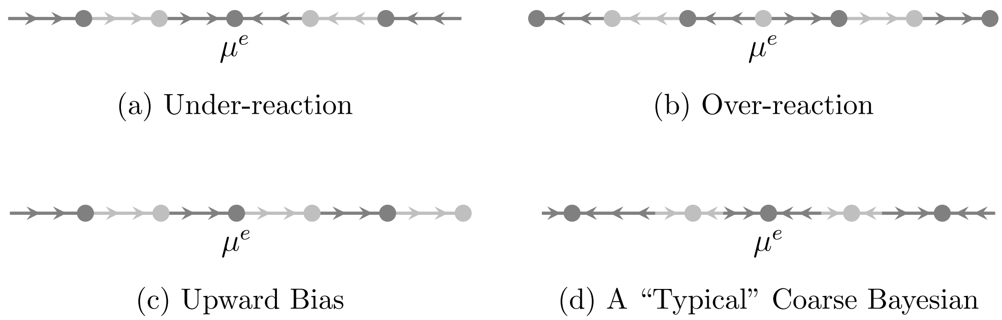
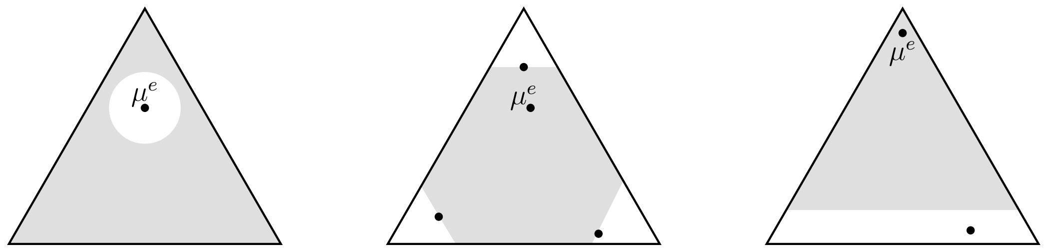


Figure 3: Four Coarse Bayesian Representations on $\Delta = [0, 1]$.

viduals also over-react to information in various settings. For example, the concept of base-rate neglect introduced and identified by Kahneman and Tversky (1973) is a form of over-reaction where individuals over-weight information relative to their priors. Rather than always under-reacting or always over-reacting, individuals may respond asymmetrically to information. Eil and Rao (2011), for example, find that when information concerns personal attributes such as attractiveness, individuals under-react to negative signals but are approximately Bayesian when processing positive signals.

For a Coarse Bayesian, responsiveness to information depends on the set of feasible beliefs, their positions within their cells, and the "strength" of the observed signal. Thus, although under- and over-reaction are rather opposite phenomena, a Coarse Bayesian typically exhibits both behaviors: he under-reacts to some signals, but over-reacts to others. If, for example, the cell P containing the prior μ^e is not a singleton, the agent will not revise beliefs at signals yielding Bayesian posteriors in P—an under-reaction to new information. However, signals that do result in belief revision typically yield posterior beliefs that do not coincide with the Bayesian posterior, often resulting in over-reaction.

Naturally, the concepts of over- and under-reaction make the most sense in two-state settings, where the probability simplex Δ can be represented by the unit interval. Figures 3a and 3b illustrate under- and over-reaction in such a setting. In 3a, the agent never over-reacts but typically under-reacts: his posterior belief is as close as possible to μ^e given the partition of Δ into sub-intervals, resulting in conservative updating. In 3b, the agent never under-reacts but typically over-reacts: his posterior is farthest away from μ^e given the partition, resulting in a form of base-rate neglect. Figure 3c exhibits a biased agent who favors one state: posteriors typically assign



(a) Limited Perception (b) Extreme-Belief Aversion (c) Unexpected News

Figure 4: Limited Perception, Extreme-Belief Aversion, and Reactions to Unexpected News. Each point in the shaded regions represents a singleton cell.

higher probability to state 1 than the Bayesian posterior, but never less. Thus, it is relatively easy for this agent to revise beliefs upward, but relatively difficult to revise downward. This captures, for example, "motivated" reasoning where agents may place intrinsic value on their beliefs. Finally, Figure 3d depicts a "typical" Coarse Bayesian: representative points do not necessarily sit on the boundaries of cells, and therefore both over- and under-reaction occur.

2. Limited Perception, Extreme-Belief Aversion, and Reactions to Unexpected News. The model can also capture limited perception or attention. For example, consider Figure 4a. In this representation, the agent retains prior μ^e unless the Bayesian posterior is sufficiently far away from μ^e, in which case he applies Bayes' rule. An interpretation is that the agent only notices signals that are sufficiently strong or provocative to yield a large shift in the Bayesian posterior.[4]

Figure 4b exhibits rather the opposite behavior: the agent is Bayesian unless posterior beliefs are too "extreme"—that is, close to degenerate distributions representing certainty about the state. Ducharme (1970) argues that such behavior may explain some of the experimental evidence for under-reaction (see also Benjamin et al. (2016), who introduce the term "extreme-belief aversion"). Indeed, a Coarse Bayesian employing the representation in Figure 4b would effectively under-react to signals that strongly support a particular state.

Figure 4c illustrates an updating rule that coincides with Bayes' rule unless the

[4]This makes the most sense in the signal-distortion interpretation of the model, where the agent transforms signals before applying Bayes' rule. The underlying signal distortion function (formalized in appendix B) represents the agent's attention, avoiding the "circularity" of having the agent compute the Bayesian posterior of ignored signals—a common criticism of rational inattention models.

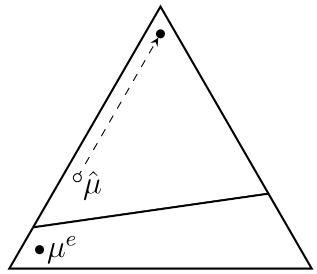


Figure 5: A straw man fallacy.

observed signal is sufficiently "surprising." In this case, the prior strongly supports a particular state, and the agent exhibits non-Bayesian behavior only if the signal has a low probability of occurrence in that state. Several studies, such as De Bondt and Thaler (1985), find that updating behavior at such unexpected signals may be inconsistent with Bayes' rule. See also Ortoleva (2012), who develops a model (discussed in the next section) to accommodate this, and related, evidence.

3. Logical Fallacies. Coarse Bayesian updating can give rise to—or make agents susceptible to—various logical or rhetorical fallacies that are common in real life. First, agents who consider only a small set of competing theories may perceive false dilemmas: they overlook plausible alternative explanations for the data, often narrowing the options down to two alternatives. For such agents, small changes to information—or even the same piece of information—can result in radically different beliefs, as there is a cutoff point for switching theories that may differ across individuals.

More generally, coarseness can lead to faulty generalizations. Larger cells indicate a greater tendency to falsely extrapolate, resulting in posteriors that need not be strongly supported by the evidence. This tendency to "jump to conclusions" can make agents susceptible to slippery-slope arguments: they may draw more extreme conclusions than are actually supported by the evidence.

Finally, Coarse Bayesian agents may be susceptible to "straw man" arguments: by providing evidence that refutes some particular theory, the agent may believe other theories are refuted as well. For example, consider Figure 5. Here, an agent's prior places high probability on state 1 (bottom-left point), and a signal strongly refutes state 2 (bottom-right point) but not state 1: the Bayesian posterior $\hat{\mu}$ is near the edge of the simplex where only states 1 and 3 have positive probability. However, the realized posterior effectively eliminates state 1. Thus, by refuting the "straw man" theory (state 2), the signal makes the agent abandon a theory that does not conflict with the evidence.

3.2 Paradigm Shifts and Maximum-Likelihood Updating

For a Coarse Bayesian, the act of updating beliefs may resemble a "paradigm shift." If, for example, different feasible beliefs μ^P represent competing theories or paradigms, then the act of revising beliefs may involve a dramatic shift in how the agent understands the world. Under this interpretation, the partition $\mathcal{P}$ represents the agent's criteria for switching among competing theories. It is natural to wonder if such behavior can be reformulated in terms of second-order beliefs. If an agent assigns a prior degree of confidence to each feasible theory μ^P, can Coarse Bayesian updating be reconciled with Bayesian updating of such second-order beliefs?

To answer this question, I take an approach similar to that of Ortoleva (2012), who proposes the Hypothesis-Testing (HT) model of belief updating. Under HT, an agent applies Bayes' rule for signals of sufficiently high prior likelihood (that is, above some threshold $\varepsilon \geq 0$, an individual parameter). For "unexpected" signals (likelihood less than ε), the agent updates beliefs by applying a maximum-likelihood criterion to a *second-order prior*, or "prior over priors." Specifically, the agent updates the second-order prior via Bayes' rule, then adopts as posterior a belief of maximal probability under the new second-order distribution. In this section, I consider a similar maximum-likelihood procedure, adapted to the domain S of noisy signals.[5]

Definition 1. A Homogeneous, Convex updating rule μ has a **Maximum-Likelihood (ML) Representation** if there exists a probability distribution Γ over Δ (with density γ) such that

$$\mu^s \in \underset{\hat{\mu}\in\Delta}{\operatorname{argmax}}\, \gamma(\hat{\mu})\hat{\mu} \cdot s$$

for all $s \in S$. The function $L : \Delta \times S \to \mathbb{R}$ given by $L(\hat{\mu}|s) = \gamma(\hat{\mu})\hat{\mu} \cdot s$ is the **likelihood function**.

In a Maximum-Likelihood Representation, the agent has a second-order prior Γ that he updates, via Bayes' rule, upon arrival of signal s. Then, he selects a belief that

[5] As Weinstein (2017) explains, the HT model allows essentially any updating to occur for unexpected news (ie, likelihood less than ε). As we shall see, extending maximum-likelihood updating procedures to the domain of noisy signals does rule out some updating behavior.

has maximal probability under the new second-order distribution. This procedure selects among beliefs $\hat{\mu}$ that maximize the likelihood function at s.[6] Intuitively, ML updating captures the behavior of an agent who assigns prior degrees of confidence to competing theories, updates these values in a Bayesian fashion, and selects the most-likely theory given available information.[7]

Proposition 2.

(i) Not every Maximum-Likelihood rule can be expressed as a Coarse Bayesian rule.

(ii) Not every Coarse Bayesian rule can be expressed as a Maximum-Likelihood rule.

(iii) If $N = 2$, then every Coarse Bayesian rule is a Maximum-Likelihood rule.

(iv) Bayesian updating is a special case of both Coarse Bayesian and Maximum-Likelihood updating. To express Bayesian updating as a Maximum-Likelihood rule, take

$$\gamma(\hat{\mu}) \propto \left\| \frac{\hat{\mu}}{\sqrt{\mu^e}} \right\|^{-1} \tag{2}$$

where $\sqrt{\mu^e} := (\sqrt{\mu^e_\omega})_{\omega \in \Omega}$.

Proposition 2 establishes that neither updating procedure subsumes the other—there exist updating rules that have Coarse Bayesian Representations but not ML Representations, and there exist updating rules that have ML Representations but not Coarse Bayesian Representations. These claims are demonstrated by Examples 1 and 2 below. Part (iii) establishes an important special case: if there are only two states, then every Coarse Bayesian rule can be expressed as a ML rule. Part (iv) asserts that Bayesian updating is a special case of both models, and provides an explicit formula for a second-order prior generating Bayesian updating in the ML procedure. For proof of claims (iii) and (iv), see the appendix.

[6]Notice that L is homogeneous (of degree 0) and convex in s. The restriction to homogeneous, convex updating rules, therefore, only takes effect when there are ties—multiple candidate beliefs that maximize L.

[7]There are other ways of reducing a second-order belief to a first-order belief. For example, one might use the second-order distribution to compute an average belief. However, such a procedure is continuous in s while Coarse Bayesian updating, in general, exhibits discontinuities in s.

Example 1. Not every ML rule can be expressed as a Coarse Bayesian rule. Suppose $N = 2$ and consider the distribution γ such that $\gamma(\mu^1) = 3/4$ and $\gamma(\mu^2) = 1/4$, where $\mu^1 = (1/3, 2/3)$ and $\mu^2 = (3/4, 1/4)$. Observe that $L(\mu^1|e) = \gamma(\mu^1)\mu^1 \cdot e = \gamma(\mu^1) > \gamma(\mu^2) = \gamma(\mu^2)\mu^2 \cdot e = L(\mu^2|e)$; thus, $\mu^e = \mu^1$. It is easy to verify that $B(\mu^e|s) = \mu^2$ if and only if $s_1/s_2 = 6$. Therefore, to be consistent with a Coarse Bayesian updating rule, we must have $L(\mu^2|s) \geq L(\mu^1|s)$ whenever $s_1/s_2 = 6$. Take $s = (1, 1/6)$. Then $L(\mu^2|s) = 19/96 < 19/72 = L(\mu^1|s)$, so that the ML rule selects μ^1 at s. This means the rule is not Confirmatory, and therefore is inconsistent with Coarse Bayesian updating. ♦

Example 2. Not every Coarse Bayesian rule can be expressed as a ML rule. Suppose $N = 3$ and consider a Coarse Bayesian representation where $\mathcal{P}$ has two cells, P and P', with $\mu^P = \mu^e$ and $\mu^{P'} = \mu' \neq \mu^e$. The boundary between P and P' corresponds to a hyperplane, H, in S. We will choose H (hence, $\mathcal{P}$) in such a way that no distribution γ on Δ (with support $\{\mu^e, \mu'\}$) can generate the same updating behavior as $\langle \mathcal{P}, (\mu^P)_{P\in\mathcal{P}} \rangle$ under the ML procedure.

Observe that if γ generates the same updating behavior, then $L(\mu^e|s) = L(\mu'|s)$ for all $s \in H$. In particular, $[\gamma(\mu^e)\mu^e - \gamma(\mu')\mu'] \cdot s = 0$ for all $s \in H$. Thus, the line $\{\lambda[\mu^e + \mu'] - \mu' : \lambda \geq 0\}$ is orthogonal to the hyperplane H. Since $\mu^e \neq \mu'$, we may assume H strictly separates μ^e and μ'. Thus, we may perturb the hyperplane H to ensure it is not orthogonal to the line. Consequently, the resulting Coarse Bayesian Representation cannot be represented by any ML rule. ♦

As demonstrated in Example 1 above, ML updating rules may be incompatible with Coarse Bayesian updating due to violations of Confirmation: ML rules are measurable with respect to some partition of Δ into convex cells, but cells need not contain their representative elements. I show in appendix C that the categorical-thinking model of Mullainathan (2002) also violates Confirmation in some cases, and for a similar reason. Rather than employing a second-order prior to compute likelihoods and select posteriors, Mullainathan's model uses a particular formula to calculate "base rates" for candidate beliefs. Thus, the categorical-thinking model is similar in spirit to a ML procedure, and the particular functional form employed can produce violations of Confirmation.

3.3 Dynamics

This section examines some basic dynamic properties of the model. Suppose an agent observes a sequence of signals $\vec{s} = (s^1, \ldots, s^n)$, where s^t is the signal generated in period t. How do properties of $\vec{s}$ affect the agent's final belief?

For standard Bayesians, the answer is quite simple: signals can be pooled and ordered in any fashion without impacting final beliefs. For example, consider a sequence $\vec{s} = (s^1, s^2, s^3)$. The terminal Bayesian belief is $B(\mu^e|s^1 s^2 s^3)$ regardless of whether the signals are arranged in a different order (eg. (s^2, s^1, s^3)), pooled differently (eg. $(s^1, s^2 s^3)$), or both.[8]

For Coarse Bayesians, the answer is more nuanced. For example, the behavior of an agent who incorporates the full history of signal realizations into current beliefs differs from one who performs signal-by-signal updating. For simplicity, I focus on "memoryless" agents who perform signal-by-signal updating.

Some additional terminology and notation is needed to proceed. A signal s is **interior** if $s_\omega > 0$ for all $\omega \in \Omega$. A **dynamic updating rule** associates a belief $\mu^{(s^1,\ldots,s^n)}$ to every finite **history** $(s^1, \ldots, s^n)$ of interior signals. Interpreting a signal s as a history of length 1, a dynamic updating rule clearly gives rise to an updating rule with prior μ^e (full support).

Definition 2. A dynamic updating rule μ is:

(i) A **Dynamic Coarse Bayesian** updating rule if there is a Coarse Bayesian Representation $\langle \mathcal{P}, \mu^{\mathcal{P}} \rangle$ for histories of length 1 such that, for every history $(s^1, \ldots, s^n)$ of length $n \geq 2$, $\mu^{(s^1,\ldots,s^n)} = \mu^P$ where $B(\mu^{(s^1,\ldots,s^{n-1})}|s^n) \in P \in \mathcal{P}$.

(ii) A **Dynamic Bayesian** updating rule if $\mu^s = B(\mu^e|s)$ for histories s of length 1 and, for all histories $(s^1, \ldots, s^n)$ of length $n \geq 2$, $\mu^{(s^1,\ldots,s^n)} = B(\mu^{(s^1,\ldots,s^{n-1})}|s^n)$.

A Dynamic Coarse Bayesian updating rule employs a fixed Coarse Bayesian Representation to perform signal-by-signal updating. Starting with prior μ^e, the agent applies $\langle \mathcal{P}, \mu^{\mathcal{P}} \rangle$ to yield some posterior μ^{s^1} after observing s^1. Then, treating μ^{s^1} as the prior, the agent applies the same representation $\langle \mathcal{P}, \mu^{\mathcal{P}} \rangle$ to reach posterior $\mu^{(s^1,s^2)}$

[8] See Cripps (2018) for a general analysis of updating rules that are invariant to how an agent partitions histories of signals.

after observing s^2, and so on. A Dynamic Bayesian updating rule is a special case where each cell of $\mathcal{P}$ is a singleton.

Definition 3. A dynamic updating rule μ is:

(i) **Invariant to signal ordering** if $\mu^{\vec{s}} = \mu^{\pi(\vec{s})}$ for all histories $\vec{s}$ and permutations $\pi(\vec{s})$ of $\vec{s}$.

(ii) **Invariant to signal pooling** if, for all histories $\vec{s} = (s^1, \ldots, s^n)$ of length $n \geq 2$ and all $k < n$, $\mu^{\vec{s}} = \mu^{(s^1,\ldots,s^{k-1},s^k s^{k+1},s^{k+2},\ldots,s^n)}$.

Definition 3 formalizes two different notions of history independence. If a dynamic updating rule is invariant to signal ordering, then any history $\vec{s}$ can be reordered without affecting the final belief.[9] Invariance to signal pooling, by contrast, requires that any signal in a history can be pooled with its successor without affecting the final belief. Clearly, invariance to signal pooling implies invariance to signal ordering.

Proposition 3. *Let μ^e have full support. Then:*

(i) The Dynamic Bayesian updating rule is invariant to signal ordering and pooling.

(ii) Dynamic Coarse Bayesian updating rules need not be invariant to signal ordering nor to signal pooling.

Proof. For (i), observe that if μ is a Dynamic Bayesian updating rule, then $\mu^{(s^1,\ldots,s^n)} = B(\mu^e | s^1 s^2 \cdots s^n)$; that is, terminal beliefs depend only on the fully-merged signal $s^1 s^2 \cdots s^n$. For (ii), see Example 3 and Proposition 4. □

Proposition 3 confirms that standard Bayesians are not affected by signal ordering or pooling, but that Coarse Bayesians potentially are. The next example illustrates that some (non-Bayesian) Dynamic Coarse Bayesian rules are invariant to signal ordering. However, such rules are somewhat rare: Proposition 4 establishes that a broad class of Dynamic Coarse Bayesian rules are not invariant to signal ordering and, thus, fail to be invariant to signal pooling as well.

[9] Rabin and Schrag (1999) analyze a model of history-dependent updating where, at each time period, information is distorted to support the agent's current belief. Such a procedure would not be invariant to signal ordering.

Example 3. Some (non-Bayesian) Dynamic Coarse Bayesian updating rules are invariant to signal ordering. For example, if $\mathcal{P}$ consists of a single cell, then $\mu^s = \mu^e$ for all $s \in S$. Less trivially, suppose $N = 2$ (so that $\Delta = [0,1]$) and consider $\mathcal{P} = \{[0,1), \{1\}\}$ with $\mu^{[0,1)} = 1/2$ and $\mu^{\{1\}} = 1$. It is straightforward to verify that this representation induces invariance to signal ordering. ♦

The next definition introduces a well-behaved class of representations needed to state Proposition 4; this definition is also invoked in section 4.

Definition 4. Given $\langle \mathcal{P}, \mu^{\mathcal{P}} \rangle$, a cell $P \in \mathcal{P}$ is **regular** if it has full dimension in Δ and its representative μ^P belongs to the relative interior of P. A Coarse Bayesian Representation $\langle \mathcal{P}, \mu^{\mathcal{P}} \rangle$ is **regular** if each of its cells is regular.

Proposition 4. *If $\langle \mathcal{P}, \mu^{\mathcal{P}} \rangle$ is regular and $\mathcal{P}$ has at least two cells, then the associated Dynamic Coarse Bayesian updating rule is not invariant to signal ordering.*

Proof. Let $P \in \mathcal{P}$ be a cell containing δ_1, where $\delta_1 \in \Delta$ assigns probability one to state ω_1. Let r be any signal such that $r\mu^e \approx \mu^P$. Choose any $P' \in \mathcal{P}$ such that $\mu^P \neq \mu^{P'}$. By regularity, μ^P and $\mu^{P'}$ have full support. Thus, there is a signal s such that $s\mu^P \approx \mu^{P'}$. It follows that $\mu^r = \mu^P$ and $\mu^{(r,s)} = \mu^{P'}$. Let t be a signal such that $t_1 = 1$ and $t_\omega = \varepsilon$ for all $\omega \neq 1$. By regularity, and the fact that $\delta_1 \in P$, there is an ε (sufficiently small) such that both $B(\mu^P|t) \in P$ and $B(\mu^{P'}|t) \in P$. Thus, $\mu^{(r,s,t)} = \mu^P$. However, $\mu^{(r,t,s)} = \mu^{P'}$ because $\mu^{(r,t)} = \mu^P$ and $B(\mu^P|s) = \mu^{P'}$. Thus, the updating rule is not invariant to signal ordering. □

4 Application: The Value of Information

Assessing the value of information is a fundamental part of decision making in many economic models. The classic characterization of Blackwell (1951) develops an informativeness ordering where one experiment is more informative than another if and only if it grants a Bayesian agent higher expected utility in all decision problems. In this section, I study the value of information for Coarse Bayesians, including its relationship to the Bayesian value of information, the Blackwell ordering, and notions of cognitive sophistication and bias. Some results invoke the concept of regularity (see Definition 4 in section 3.3).

Throughout this section, suppose μ is an updating rule with Coarse Bayesian Representation $\langle \mathcal{P}, \mu^{\mathcal{P}} \rangle$. Let $\mathcal{A}$ denote the set of all nonempty, compact subsets of $\mathbb{R}^{\Omega}$. Each $A \in \mathcal{A}$ is a **menu**, and elements $x = (x_{\omega})_{\omega \in \Omega} \in A$ represent feasible **actions** the agent may take. An agent who chooses action $x \in A$ attains payoff x_{ω} in state ω. For each $A \in \mathcal{A}$ and $s \in S$, let $c^s(A) := \text{argmax}_{x \in A}\, x \cdot \mu^s$; these are the actions in A that maximize expected utility under beliefs μ^s.

Definition 5. Let $A \in \mathcal{A}$.

(i) The **value of information** at A is given by the function $V^A : \mathcal{E} \to \mathbb{R}$ where

$$V^A(\sigma) := \max \sum_{\omega} \mu^e_{\omega} \sum_{s \in \sigma} s_{\omega} x^s_{\omega} \;\text{ subject to }\; x^s \in c^s(A). \tag{3}$$

(ii) The **Bayesian value of information** at A is given by the function $\overline{V}^A : \mathcal{E} \to \mathbb{R}$ where

$$\overline{V}^A(\sigma) := \max \sum_{\omega} \mu^e_{\omega} \sum_{s \in \sigma} s_{\omega} x^s_{\omega} \;\text{ subject to }\; x^s \in \operatorname*{argmax}_{x \in A} x \cdot \frac{s\mu^e}{s \cdot \mu^e}. \tag{4}$$

Equation (3) expresses ex-ante expected utility for a Coarse Bayesian agent. Faced with a menu A and experiment σ, the agent calculates expected utility by applying weight μ^e_{ω} to the average payoff in state ω given that signals—and subsequent choices—are generated by σ. Consistent with the Cognizance assumption, the agent correctly forecasts his own signal-contingent beliefs and, hence, signal-contingent choices. Equation (4) expresses a similar formula for an agent with the same prior μ^e but who applies Bayes' rule: signal-contingent choices maximize expected utility at beliefs $B(\mu^e|s)$ instead of beliefs μ^s.

It will be convenient to express V^A in a slightly different form. For any $\hat{\mu} \in \Delta$ and $A \in \mathcal{A}$, let

$$c^{\hat{\mu}}(A) := \operatorname*{argmax}_{x \in A} x \cdot \mu^P \text{ subject to } \hat{\mu} \in P \tag{5}$$

and

$$v^A(\hat{\mu}) := \max_{x \in c^{\hat{\mu}}(A)} x \cdot \hat{\mu}. \tag{6}$$

That is, $c^{\hat{\mu}}(A)$ consists of the actions in A that maximize expected utility for the Coarse Bayesian if the *Bayesian* posterior is $\hat{\mu}$ because the agent replaces $\hat{\mu}$ with

μ^P if $\hat{\mu} \in P$. Similarly, $v^A(\hat{\mu})$ represents expected utility at A conditional on the Bayesian posterior being $\hat{\mu}$. These mappings are well-defined because $\mathcal{P}$ partitions Δ and each cell $P \in \mathcal{P}$ has a unique representative μ^P. For a standard Bayesian, analogous mappings are given by

$$\overline{c}^{\hat{\mu}}(A) := \operatorname*{argmax}_{x \in A} x \cdot \hat{\mu} \qquad \text{and} \qquad \overline{v}^A(\hat{\mu}) := \max_{x \in \overline{c}^{\hat{\mu}}(A)} x \cdot \hat{\mu}.$$

If $\sigma \in \mathcal{E}$ and $\hat{\mu} \in \Delta$, let $\tau^\sigma(\hat{\mu}) := \sum_{s \in \sigma : B(\mu^e|s) = \hat{\mu}} s \cdot \mu^e$; this is the total probability of generating Bayesian posterior $\hat{\mu}$ under information σ and prior μ^e. That is, given μ^e, σ generates a distribution of Bayesian posteriors where $\tau^\sigma(\hat{\mu})$ is the probability of posterior $\hat{\mu}$.

Proposition 5. *For all $A \in \mathcal{A}$ and $\sigma \in \mathcal{E}$, $V^A(\sigma) = \sum_{\hat{\mu} \in \Delta} \tau^\sigma(\hat{\mu}) v^A(\hat{\mu})$.*

Proof. Using the notation defined above, (3) can be rewritten as

$$\begin{aligned} V^A(\sigma) &= \max \sum_{s \in \sigma} (s\mu^e) \cdot x^s \ \text{ subject to } \ x^s \in c^s(A) \\ &= \max \sum_{s \in \sigma} (s \cdot \mu^e) \frac{s\mu^e}{s \cdot \mu^e} \cdot x^s \ \text{ subject to } \ x^s \in c^s(A) \\ &= \max \sum_{s \in \sigma} (s \cdot \mu^e) B(\mu^e|s) \cdot x^s \ \text{ subject to } \ x^s \in c^s(A) \\ &= \sum_{\hat{\mu} \in \Delta} \tau^\sigma(\hat{\mu}) v^A(\hat{\mu}). \end{aligned}$$

□

Proposition 5 establishes that V^A can be written in posterior-separable form. In particular, it is as if the agent associates value $v^A(\hat{\mu})$ to Bayesian posteriors $\hat{\mu}$, so that the distribution of Bayesian posteriors can be used to calculate expected utility. This also facilitates comparisons between Bayesian and Coarse Bayesian payoffs (see Figure 6); clearly, $v^A(\hat{\mu}) \leq \overline{v}^A(\hat{\mu})$ for all $\hat{\mu}$ and, hence, $V^A(\sigma) \leq \overline{V}^A(\sigma)$ for all σ—the Bayesian always does better. Intuitively, Proposition 5 holds because a Coarse Bayesian updating rule is Homogeneous and, hence, a function of the Bayesian posterior: if one knows which Bayesian posterior has realized, then one knows which belief the Coarse Bayesian adopts.[10]

[10] This is the fundamental assumption of de Clippel and Zhang (2019), who study persuasion with non-Bayesian agents. A similar argument also appears in Galperti (2019). In general, posterior

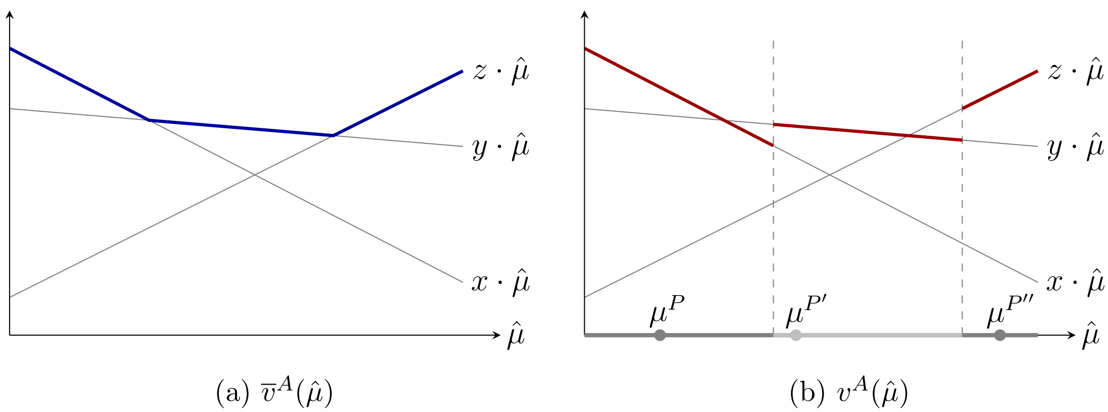


Figure 6: Bayesian vs. Coarse Bayesian value of information for $A = \{x, y, z\}$.

4.1 The Blackwell Ordering

This section examines whether and when Coarse Bayesians benefit from improvements to information. For experiments σ, σ', the relation $\sigma \sqsupseteq \sigma'$ indicates that σ is more informative than σ' in the sense of Blackwell (1951). An experiment σ' is a **garbling** of σ if there is a matrix M with entries in $[0, 1]$ such that every row is a probability distribution and $\sigma' = \sigma M$. For the purposes of this paper, the ordering $\sqsupseteq$ is defined by: $\sigma \sqsupseteq \sigma'$ if and only if σ' is a garbling of σ.

The function V^A **satisfies the Blackwell ordering** if $\sigma \sqsupseteq \sigma'$ implies $V^A(\sigma) \geq V^A(\sigma')$; if there exists $\sigma \sqsupseteq \sigma'$ such that $V^A(\sigma) < V^A(\sigma')$, then V^A **violates the Blackwell ordering**. An important part of Blackwell's characterization is that a Bayesian's value of information satisfies the Blackwell ordering in all menus A—in fact, $\sigma \sqsupseteq \sigma'$ *if and only if* $\overline{V}^A(\sigma) \geq \overline{V}^A(\sigma')$ for all $A \in \mathcal{A}$. For Coarse Bayesians, this need not be the case.

For every menu A and signal s, let $b^s(A) \subseteq A$ denote the Bayesian-optimal actions in A conditional on s. Formally, $b^s(A) := \{x \in A : x \cdot \frac{s\mu^e}{s \cdot \mu^e} \geq y \cdot \frac{s\mu^e}{s \cdot \mu^e} \; \forall y \in X\}$. Let $c(A) = \bigcup_{s \in S} c^s(A)$ and $b(A) = \bigcup_{s \in S} b^s(A)$. Thus, $c(A)$ is the set of actions in A that are chosen by the Coarse Bayesian—and $b(A)$ the set of actions chosen by the Bayesian—for at least one s. Observe that, by Confirmation, $c(A) \subseteq b(A)$.

Proposition 6. *Let $\langle \mathcal{P}, \mu^{\mathcal{P}} \rangle$ be a regular Coarse Bayesian Representation and $A \in \mathcal{A}$. The following are equivalent:*

separability means the model is well-suited for the analysis of persuasion models, as in Kamenica and Gentzkow (2011).

(i) V^A *satisfies the Blackwell ordering.*

(ii) v^A *is convex.*

(iii) $c^s(A) \cap b^s(c(A)) \neq \emptyset$ *for all* s*.*

Proposition 6 characterizes, for regular Coarse Bayesians, the class of menus A such that V^A satisfies the Blackwell ordering. The key property is (iii), asserting that Coarse Bayesian choices from A agree with Bayesian choices from menu $c(A) \subseteq A$ (that is, the submenu of actions that are actually chosen at some signal realization). When (iii) is satisfied, Coarse Bayesian behavior coincides with Bayesian behavior, making $v^A = \bar{v}^A$ convex and $V^A = \overline{V}^A$ satisfy the Blackwell ordering.

Note that the regularity requirement only serves to establish (i) $\Rightarrow$ (iii). In particular, the implication (iii) $\Rightarrow$ (i) holds for all Coarse Bayesian Representations, as does the equivalence of (i) and (ii). The implication (ii) $\Rightarrow$ (i) is part of Blackwell's characterization, but the converse implication is not, and relies on the assumption that μ^e has full support (see Lemma 1 in the appendix).

Example 4. Some non-Bayesians satisfy the Blackwell ordering in all menus. Suppose $N = 2$, so that $\Delta = [0, 1]$. First, consider $\langle \mathcal{P}, \mu^{\mathcal{P}} \rangle$ where $\mathcal{P}$ contains two cells: $P = \{0\}$ and $P' = (0, 1]$. Assume $\mu^{P'} < 1$. Then, for every A, v^A is convex; as noted above, this implies V^A satisfies the Blackwell ordering, even though choices generated by $\langle \mathcal{P}, \mu^{\mathcal{P}} \rangle$ violate condition (iii) of Proposition 6. Next, let $\langle \mathcal{Q}, \mu^{\mathcal{Q}} \rangle$ consist of a cell $Q = [0, \mu^*]$ where $0 < \mu^* < 1$ and, for every $\hat{\mu} > \mu^*$, a singleton cell $\{\hat{\mu}\}$. Let $\mu^Q = \mu^*$. Choices generated by $\langle \mathcal{Q}, \mu^{\mathcal{Q}} \rangle$ satisfy condition (iii) of Proposition 6 for all A; as noted above, this implies the corresponding value of information functions satisfies the Blackwell ordering in all menus—even though $\langle \mathcal{Q}, \mu^{\mathcal{Q}} \rangle$ violates the regularity assumption. ♦

Example 4 shows that it is possible for non-Bayesian representations to generate functions V^A satisfying the Blackwell ordering for all A with or without condition (iii) of Proposition 6. Such representations are quite rare, however, in that small perturbations of the cells or representative points guarantee that V^A violates both the Blackwell ordering and condition (iii) for some A. Intuitively, violations of the Blackwell ordering arise through discontinuities in v^A because such discontinuities, except possibly on the boundary of Δ, make v^A non-convex. Most non-Bayesian representations have the property that any violation of (iii) introduces a non-convexity in v^A for some A because the "wedge" between Bayesian and non-Bayesian choices creates points of discontinuity. For regular representations, violations of (iii) are both necessary and sufficient for the existence of such discontinuities.

The next result provides a simple condition on representations $\langle \mathcal{P}, \mu^{\mathcal{P}} \rangle$ guaranteeing violations of the Blackwell ordering. For each $P \in \mathcal{P}$, let ∂P denote the relative boundary of P.[11]

Proposition 7. *If there is a cell $P \in \mathcal{P}$ and a point $\mu^* \in \partial P \cap \Delta^0$ such that $\mu^* \neq \mu^P$, then there exists $A \in \mathcal{A}$ such that V^A violates the Blackwell ordering.*

The condition highlighted by Proposition 7 is satisfied by most Coarse Bayesian Representations. For example, regular representations that have at least two cells satisfy the condition because their representative points μ^P belong to the interiors of their cells. In fact, every Coarse Bayesian Representation depicted in this paper satisfies the condition and, therefore, generates violations of the Blackwell ordering in some menus.[12]

4.2 Measures of Sophistication and Bias

In this section, I explore different notions of cognitive ability and how they relate to a Coarse Bayesian's value of information. In addition to providing basic comparative static results for the model, the findings are potentially relevant for approaches to endogenizing non-Bayesian updating rules and, hence, developing theories of where they "come from" (see the discussion at the end of the section). For any updating rule μ and signal $s \in S$, let

$$D_\mu(s) := \left\| \frac{s\mu^e}{\|s\mu^e\|} - \frac{\mu^s}{\|\mu^s\|} \right\|.$$

[11] That is, ∂P is the set of points belonging to the closure of P but not the relative interior of P.

[12] In an experiment, Ambuehl and Li (2018) find that subjects tend to undervalue improvements to instrumentally valuable information, and argue that this is due to non-Bayesian belief updating. In addition, subjects differ in their responsiveness to information. These findings are consistent with the results in this section, as well as the idea that the parameters (cells and representative points) of Coarse Bayesian updating rules may differ across individuals.

(a) More biased (b) Less biased

Figure 7: An illustration of the bias ordering. Each updating rule employs the same pair of feasible beliefs, but rule (b) is less biased than rule (a) because it exhibits smaller distortions away from Bayesian posteriors, making the cutoff between cells more "centered."

This is the Euclidean distance between μ^s and the Bayesian posterior $\frac{s\mu^e}{s \cdot \mu^e}$ after normalizing each vector to length 1. Thus, $D_\mu(s)$ provides a measure of how distorted the agent's beliefs are at signal s.

Definition 6. Suppose μ and $\dot{\mu}$ have full-support priors $\mu^e = \dot{\mu}^e$ and Coarse Bayesian Representations $\langle \mathcal{P}, \mu^{\mathcal{P}} \rangle$ and $\langle \mathcal{Q}, \dot{\mu}^{\mathcal{Q}} \rangle$, respectively. Then:

(i) $\dot{\mu}$ is **more sophisticated** than μ if $\mu^{\mathcal{P}} \subseteq \dot{\mu}^{\mathcal{Q}}$ and every cell in $\mathcal{P}$ is a union of cells in $\mathcal{Q}$.

(ii) $\dot{\mu}$ is **less biased** than μ if $D_{\dot{\mu}}(s) \leq D_\mu(s)$ for all $s \in S$.

Definition 6 provides two comparative notions of cognitive ability. Part (i) states that a Coarse Bayesian is more sophisticated if he employs both an expanded set of feasible beliefs and a finer partition, while part (ii) states that an agent is less biased if, for every signal, posterior beliefs are closer to the Bayesian posterior. Each ordering captures some aspect of what it means to be "more Bayesian," but the two concepts are quite different: higher sophistication entails higher responsiveness to information, while lower bias entails less "skewness" in the updating rule (see Figure 7). Notice that if $\dot{\mu}$ is less biased than μ, then $\mu^{\mathcal{P}} \subseteq \dot{\mu}^{\mathcal{Q}}$; thus, increasing sophistication and decreasing bias each imply an expanding set of feasible beliefs.

The goal of this section is to characterize these orderings in terms of the welfare of the agent. A natural conjecture, for example, is that a more sophisticated agent always benefits from more information, or enjoys a higher value of information than a less sophisticated agent. As the next example shows, this conjecture is false.

Example 5. Consider a two-state setting, so that $\Delta = [0, 1]$. Let $\mathcal{P} = \{P, P'\}$ where $P = \{0\}$ and $P' = (0, 1]$ and $\mathcal{Q} = \{Q, Q', Q''\}$ where $Q = \{0\}$, $Q' = [\frac{3}{4}, 1]$, and $Q'' = (0, \frac{3}{4})$. Finally, let $\mu^P = \dot{\mu}^Q = 0$, $\mu^e = \mu^{P'} = \frac{4}{5} = \dot{\mu}^{Q'} = \dot{\mu}^e$, and $\dot{\mu}^{Q''} = \frac{2}{3}$. Clearly, $\dot{\mu}$ is more sophisticated than μ. Let $A = \{x, y\}$ where $x = (1, 0)$ and $y = (0, 1)$. Then

$$v^A(\hat{\mu}_1) = \begin{cases} 1 & \text{if } \hat{\mu}_1 = 0 \\ \hat{\mu}_1 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \dot{v}^A(\hat{\mu}_1) = \begin{cases} 1 - \hat{\mu}_1 & \text{if } \hat{\mu}_1 < \frac{3}{4} \\ \hat{\mu}_1 & \text{otherwise} \end{cases},$$

so that $\dot{v}^A(\hat{\mu}_1) < v^A(\hat{\mu}_1)$ for $\frac{1}{2} < \hat{\mu}_1 < \frac{3}{4}$. Thus, $\dot{V}^A(\sigma) < V^A(\sigma)$ for some σ (for example, any σ such that $\tau^\sigma(\frac{2}{3}) = \frac{3}{5}$ and $\tau^\sigma(1) = \frac{2}{5}$). Moreover, v^A is convex but $\dot{v}^A$ is not; thus, V^A satisfies the Blackwell ordering but $\dot{V}^A$ does not. ♦

In general, higher sophistication does not guarantee welfare improvements because, without additional assumptions, there may exist menus that widen the gap between Bayesian and Coarse Bayesian choices at some signal realizations. Similarly, lower bias need not imply welfare improvements. At the end of this section, I return to this question and examine the conditions under which one Coarse Bayesian is better off than another in all decision problems (Proposition 10).

To characterize the sophistication ordering, an additional definition is required. Given $\langle \mathcal{P}, \mu^{\mathcal{P}} \rangle$, a menu A is **$\mu^{\mathcal{P}}$-decisive** if $c^s(A)$ is a singleton for all $s \in S$; that is, if no feasible posterior μ^P makes the agent indifferent between two or more options in A.

Proposition 8. *Suppose $\langle \mathcal{P}, \mu^{\mathcal{P}} \rangle$ and $\langle \mathcal{Q}, \dot{\mu}^{\mathcal{Q}} \rangle$ are regular Coarse Bayesian Representations of μ and $\dot{\mu}$, respectively, and that $\mu^e = \dot{\mu}^e$. The following are equivalent:*

(i) $\dot{\mu}$ is more sophisticated than μ.

(ii) If $\dot{V}^A(\sigma) = \dot{V}^A(\sigma')$ for all $\dot{\mu}^{\mathcal{Q}}$-decisive menus A, then $V^A(\sigma) = V^A(\sigma')$ for all $\dot{\mu}^{\mathcal{Q}}$-decisive menus A.

This result states that for regular Coarse Bayesians, higher sophistication means welfare is more responsive to information: as sophistication increases, fewer pairs σ, σ' yield identical ex-ante expected utility for (almost) all menus A. The proof of Proposition 8 shows that the characterization holds even if one restricts attention to experiments σ, σ' that are Blackwell comparable. Thus, higher sophistication means greater responsiveness to *improvements* to information. More responsive welfare, of course, does not imply higher welfare.

The characterization of the bias ordering does not involve the responsiveness of welfare, but rather a comparison to that of a Bayesian. Suppose μ is a Coarse Bayesian updating rule. For each $s \in S$ and $A \in \mathcal{A}$, let $\overline{V}^A(s) := \max_{x \in A} x \cdot B(\mu^e|s)$ and $V^A(s) := \max_{x \in A} x \cdot \mu^s$ denote the Bayesian and Coarse Bayesian payoffs at menu A conditional on signal s. Let

$$L_\mu(s) := \sup_{A \in \mathcal{A}^*} \overline{V}^A(s) - V^A(s)$$

where $\mathcal{A}^*$ denotes the set of menus A such that $\|x\| \leq 1$ for all $x \in A$. Intuitively, $L_\mu(s)$ is the maximum loss, relative to a Bayesian, that the Coarse Bayesian can incur under any decision problem A.[13] Alternatively, $L_\mu(s)$ may be interpreted as the maximum rate at which a Bayesian agent can "money pump" the Coarse Bayesian agent under public information s. So, if actions x represent bets or gambles, and a Bayesian agent is free to specify a set $A \in \mathcal{A}^*$ after both agents have observed s, then $L_\mu(s)$ is the amount of money the Bayesian can extract from the Coarse Bayesian.[14]

Proposition 9. *Suppose μ and $\dot{\mu}$ are Coarse Bayesian and $\mu^e = \dot{\mu}^e$. Then $L_{\dot{\mu}}(s) \leq L_\mu(s)$ if and only if $D_{\dot{\mu}}(s) \leq D_\mu(s)$. Thus, $\dot{\mu}$ is less biased than μ if and only if $L_{\dot{\mu}}(s) \leq L_\mu(s)$ for all $s \in S$.*

Proposition 9 establishes that $\dot{\mu}$ is less biased than μ if and only if $\dot{\mu}$ is less exploitable than μ; in particular, worst-case losses for $\dot{\mu}$, relative to a Bayesian, are smaller than those for μ.

As indicated above, neither higher sophistication nor lower bias guarantee higher payoffs in all decision problems. The next result establishes that, under mild regularity conditions, a particular refinement of these orderings is needed to improve payoffs in all decision problems.

Proposition 10. *Suppose $\langle \mathcal{P}, \mu^{\mathcal{P}} \rangle$ and $\langle \mathcal{Q}, \dot{\mu}^{\mathcal{Q}} \rangle$ are Coarse Bayesian Representations*

[13] The restriction to normalized menus $A \in \mathcal{A}^*$ is needed because $V^{\lambda A} = \lambda V^A$ for all $\lambda > 0$.

[14] Indeed, as shown in the appendix, one may restrict attention to menus of the form $A = \{0, x\}$ where, conditional on s, the Bayesian prefers the safe option 0 but the Coarse Bayesian strictly prefers x. On average, the Bayesian profits by $-x \cdot B(\mu^e|s) > 0$.

of μ and $\dot{\mu}$ such that $\mu^e = \dot{\mu}^e$ and non-singleton cells are regular. The following are equivalent:

(i) $\dot{v}^A(\hat{\mu}) \geq v^A(\hat{\mu})$ *for all* $A \in \mathcal{A}$ *and* $\hat{\mu} \in \Delta$.

(ii) $\dot{\mu}$ *is less biased, more sophisticated and, for every* $\dot{\mu}^Q \in \dot{\mu}^{\mathcal{Q}} \backslash \mu^{\mathcal{P}}$*, the cell* Q *is a singleton.*

Proposition 10 states that payoffs increase at all menu-signal pairs if and only if the agent becomes more sophisticated and all "new" feasible posteriors $\dot{\mu}^Q$ are contained in singleton cells Q. This means the agent becomes perfectly Bayesian in some subset of Δ, preventing the introduction of new or different distortions that can yield lower payoffs in some menu-signal pair. It follows immediately that the agent is less biased and that $\dot{V}^A(\sigma) \geq V^A(\sigma)$ for all $A \in \mathcal{A}$ and $\sigma \in \mathcal{E}$.

I conclude this section with a brief discussion of how my results might motivate alternative approaches for selecting, endogenizing, or rationalizing different Coarse Bayesian updating rules. One approach is to solve for an optimal updating rule in a given *environment*—a menu and signaling structure—under some constraint (for example, a fixed number of cells or a cost per additional cell). Pioneered by Wilson (2014) and Brunnermeier and Parker (2005), versions of this approach can provide a theory of where the updating rule "comes from." A drawback is that an updating rule adapted to one environment may be ill-suited for another; we have seen, for example, that more-sophisticated agents may experience lower payoffs at some menus and/or signal realizations than less-sophisticated agents. Only the "robust" ordering given by statement (ii) of Proposition 10 ensures weakly greater payoffs at all menu-signal pairs. Thus, rather than considering updating rules adapted to specific environments, one might instead endogenize them by selecting rules that are unimprovable (given constraints or costs) under the robust ordering. Alternatively, one might consider the weaker objective of minimizing worst-case losses (Proposition 9). These approaches are suitable if agents are unable to form probabilistic beliefs about their environment and, consequently, seek rules or heuristics robust to such uncertainty. Naturally, different criteria yield different predictions about updating rules; minimization of worst-case losses, for example, leads to representations exhibiting less skewness. Analysis of endogenous updating rules is beyond the scope of this paper, but—as illustrated by the characterizations in this section—the general framework of

Coarse Bayesian updating provides a natural and tractable setting in which to carry it out.

5 Conclusion

In this paper, I have introduced a new model of non-Bayesian updating, *Coarse Bayesian updating*, accommodating many documented violations of standard Bayesian behavior. Three testable and normatively appealing assumptions—*Homogeneity*, *Cognizance*, and *Confirmation*—characterize the updating procedure, the parameters of which are a partition of the probability simplex and a representative belief for each cell of the partition. A Coarse Bayesian agent can be interpreted as one who applies subjective criteria to select among competing theories, crudely approximates Bayes' rule, selectively distorts signals before applying Bayes' rule, or engages in categorical thinking. I have analyzed how the model relates to existing models and evidence on non-Bayesian updating, examined its implications in dynamic settings, and applied it to standard problems of decision under risk, leading to comparative notions of cognitive sophistication and bias.

An advantage of my framework is that it employs standard primitives that frequently appear in applications. The use of noisy signals over an exogenous state space, for example, allows one to directly import Coarse Bayesian updating into familiar settings in economics and game theory. Applying the framework and results of this paper to such settings may be an interesting avenue for future research.

A Omitted Proofs

A.1 Proof of Proposition 2

Proof of part (iii). If every cell of $\langle \mathcal{P}, \mu^{\mathcal{P}} \rangle$ is a singleton, then the agent is Bayesian and the ML representation is established independently by the proof of part (iv) below. So, let $P^* \in \mathcal{P}$ be a non-singleton cell. Let I denote the set of all Coarse Bayesian Representations $i = \langle \mathcal{Q}(i), \dot{\mu}^{\mathcal{Q}(i)} \rangle$ such that $\mathcal{Q}(i)$ is finite, $\mathcal{P}$ is finer than $\mathcal{Q}(i)$, $\dot{\mu}^{\mathcal{Q}(i)} \subseteq \mu^{\mathcal{P}}$, and $P^* \in \mathcal{Q}(i)$. Define a partial order $\geq_I$ on I by $i \geq_I i'$ if and only if $\mathcal{Q}(i)$ is finer than $\mathcal{Q}(i')$ and $\dot{\mu}^{\mathcal{Q}(i)} \supseteq \dot{\mu}^{\mathcal{Q}(i')}$ (that is, i is more sophisticated than i' in the sense of Definition 6). It is straightforward to verify that $\geq_I$ is a partial order and that for all $i, i' \in I$, there exists $i^* \in I$ such that $i^* \geq_I i$ and $i^* \geq_I i'$. Thus, $(I, \geq_I)$ is a directed set.

For each $\langle \mathcal{Q}, \dot{\mu}^{\mathcal{Q}} \rangle \in I$, define a function $\gamma : \Delta \to [0, \infty)$ as follows. Since $N = 2$, the (finite) set $\dot{\mu}^{\mathcal{Q}}$ can be arranged in decreasing order of state 1: $\dot{\mu}^{\mathcal{Q}} = \{\dot{\mu}^{Q_1}, \ldots, \dot{\mu}^{Q_M}\}$, where $\dot{\mu}_1^{Q_1} > \dot{\mu}_1^{Q_2} > \ldots > \dot{\mu}_1^{Q_M}$. Since $P^* \in \mathcal{Q}$, there exists m^* such that $\dot{\mu}^{Q_{m^*}} = \mu^{P^*}$. For $1 \leq m < M$, let $\dot{\mu}^m$ denote the (unique) belief belonging to $\partial Q_m \cap \partial Q_{m+1}$ (the boundaries of Q_m and Q_{m+1}) and choose a signal s^m such that $B(\mu^e|s^m) = \dot{\mu}^m$. Now choose scalars $\alpha_m > 0$ such that, for all $1 \leq m < M$, $\alpha_m \mu^{Q_m} \cdot s^m = \alpha_{m+1} \mu^{Q_{m+1}} \cdot s^m$; taking $\alpha_{m^*} = 1$ pins down the α_m uniquely. Now define γ by

$$\gamma(\hat{\mu}) = \begin{cases} \alpha_m & \text{if } \hat{\mu} = \mu^{Q_m} \\ 0 & \text{otherwise} \end{cases}.$$

By construction, $\mu^{Q_m} \in \operatorname{argmax}_{\hat{\mu}} \gamma(\hat{\mu})\hat{\mu} \cdot s$ (that is, μ^{Q_m} maximizes the likelihood function associated with γ) if and only if $B(\mu^e|s) \in Q_m$. Moreover, every point $\gamma(\hat{\mu})\hat{\mu}$, viewed as a point in $\mathbb{R}^2$, is contained in the half-space bounded above by the line with normal s^* passing through μ^{P^*}, where s^* is any signal such that $B(\mu^e|s^*) = \mu^{P^*}$. Thus, there exists a scalar $\overline{\gamma} > 0$ such that $\gamma(\hat{\mu}) \in [0, \overline{\gamma}]$ for all $\hat{\mu}$. Observe that the bound $\overline{\gamma}$ is independent of i.

Having defined a function $\gamma^i : \Delta \to [0, \overline{\gamma}]$ for every $i \in I$, the family $\{\gamma^i\}_{i \in I}$ forms a net. Each γ^i is an element of the (compact) product set $[0, \overline{\gamma}]^\Delta$, so that $\{\gamma^i\}_{i \in I}$ has a convergent subnet. This means there is a directed set $(J, \geq_J)$ and a function $\iota : J \to I$ such that (a) $j \geq_J j'$ implies $\iota(j) \geq_I \iota(j')$, (b) for every $i \in I$, there exists $j \in J$ such that $\iota(j') \geq_I i$ for all $j' \geq_J j$, and (c) the net $\{\gamma^{\iota(j)}\}_{j \in J}$ converges to some γ^*. Thus, for every $\hat{\mu} \in \Delta$, $\gamma^{\iota(j)}(\hat{\mu})$ converges to a point $\gamma^*(\hat{\mu})$.

Let $P \in \mathcal{P}$. By definition of $(I, \geq_I)$ and properties (a) and (b) of $(J, \geq_J)$, there exists $j^P \in J$ such that $P \in \mathcal{Q}(\iota(j))$ and $\mu^P \in \dot{\mu}^{\mathcal{P}(\iota(j))}$ for all $j \geq_J j^P$. Suppose s satisfies $B(\mu^e|s) \in P$. By construction, μ^P maximizes the likelihood function associated with $\gamma^{\iota(j)}$ at s if $j \geq_J j^P$: for every $\hat{\mu} \in \Delta$, $\gamma^{\iota(j)}(\mu^P)\mu^P \cdot s \geq \gamma^{\iota(j)}(\hat{\mu})\hat{\mu} \cdot s$. Taking the limit of both sides with respect to j yields $\gamma^*(\mu^P)\mu^P \cdot s \geq \gamma^*(\hat{\mu})\hat{\mu} \cdot s$; thus, μ^P maximizes the likelihood function associated with γ^* at s. □

Proof of part (iv). Notice that $B(\mu^e|s) = \mu'$ if and only if $s \approx \mu'/\mu^e := (\mu'_\omega/\mu^e_\omega)_{\omega \in \Omega}$. Thus, it will suffice to verify that $L(\cdot|s)$ is maximized at μ' for such signals s. This is done as follows. Let $s \in S$. Then, for any $\hat{\mu} \in \Delta$, we have

$$\begin{aligned} L(\hat{\mu}|s) &= \gamma(\hat{\mu})\hat{\mu} \cdot s \\ &= \frac{\hat{\mu}}{\|\hat{\mu}/\sqrt{\mu^e}\|} \cdot s \\ &= \frac{\hat{\mu}/\sqrt{\mu^e}}{\|\hat{\mu}/\sqrt{\mu^e}\|} \cdot s\sqrt{\mu^e} \\ &= \left\| \frac{\hat{\mu}/\sqrt{\mu^e}}{\|\hat{\mu}/\sqrt{\mu^e}\|} \right\| \|s\sqrt{\mu^e}\| \cos\theta \\ &= \|s\sqrt{\mu^e}\| \cos\theta \end{aligned}$$

where θ is the angle (in radians) between $\hat{\mu}/\sqrt{\mu^e}$ and $s\sqrt{\mu^e}$. Thus, $L(\cdot|s)$ is maximized by choosing $\hat{\mu}$ such that $\hat{\mu}/\sqrt{\mu^e} \approx s\sqrt{\mu^e}$ (because then $\theta = 0$), implying $\hat{\mu} \approx s\mu^e \approx \frac{\mu'}{\mu^e}\mu^e = \mu'$. □

A.2 Proofs for Section 4

A.2.1 Proof of Proposition 6

Lemma 1. *Let $\varphi : \Delta \to \mathbb{R}$ and $\Phi : \mathcal{E} \to \mathbb{R}$ such that $\Phi(\sigma) = \sum_{\hat{\mu}} \varphi(\hat{\mu})\tau^\sigma(\hat{\mu})$. Suppose Φ satisfies the Blackwell ordering: $\sigma \sqsupseteq \sigma'$ implies $\Phi(\sigma) \geq \Phi(\sigma')$. Then φ is convex.*

Proof. Let $\hat{\mu}, \hat{\mu}' \in \Delta$, $\alpha \in (0,1)$, and $\hat{\mu}^\alpha := \alpha\hat{\mu} + (1-\alpha)\hat{\mu}'$. Since μ^e has full support, there exists $\hat{\mu}^* \in \Delta$ and $\lambda \in (0,1]$ such that $\lambda\hat{\mu}^* + (1-\lambda)\hat{\mu}^\alpha = \mu^e$. Let $\sigma = [s^*, s, s']$ and $\sigma' = [s^*, s+s']$ where $s^* = \lambda\frac{\hat{\mu}^*}{\mu^e}$, $s = (1-\lambda)\alpha\frac{\hat{\mu}}{\mu^e}$, and $s' = (1-\lambda)(1-\alpha)\frac{\hat{\mu}'}{\mu^e}$. Clearly, $\sigma \sqsupseteq \sigma'$, so that $\Phi(\sigma) \geq \Phi(\sigma')$. Moreover, $\mu^e \cdot s^* = \lambda$, $\mu^e \cdot s = (1-\lambda)\alpha$, $\mu^e \cdot s' = (1-\lambda)(1-\alpha)$, and $\mu^e \cdot (s+s') = 1-\lambda$, while $B(\mu^e|s^*) = \hat{\mu}^*$, $B(\mu^e|s) = \hat{\mu}$, $B(\mu^e|s') = \hat{\mu}'$, and $B(\mu^e|s+s') = \hat{\mu}^\alpha$. Thus, $\Phi(\sigma) = \varphi(\hat{\mu}^*)\lambda + \varphi(\hat{\mu})(1-\lambda)\alpha + \varphi(\hat{\mu}')(1-\lambda)(1-\alpha)$ and $\Phi(\sigma') = \varphi(\hat{\mu}^*)\lambda + \varphi(\hat{\mu}^\alpha)(1-\lambda)$, so that $\Phi(\sigma) \geq \Phi(\sigma')$ yields $\alpha\varphi(\hat{\mu}) + (1-\alpha)\varphi(\hat{\mu}') \geq \varphi(\hat{\mu}^\alpha)$, as desired. □

To prove Proposition 6, let $A \in \mathcal{A}$ and observe that (i) ⇒ (ii) by Lemma 1 (taking $\varphi = v^A$). The converse implication, (ii) ⇒ (i), follows from Blackwell's theorem. To see that (iii) ⇒ (i), observe that if $c^s(A) \cap b^s(c(A)) \neq \emptyset$ for all s, then every Coarse Bayesian choice from A is Bayesian-optimal in the menu $A' = c(A)$. Since Coarse Bayesian choices from A are identical to those from A', it follows that

$V^A(\sigma) = V^{A'}(\sigma) = \overline{V}^{A'}(\sigma)$ for all σ. That is, V^A coincides with the Bayesian value of information in some menu, and therefore satisfies the Blackwell ordering.

Finally, we prove that (i) $\Rightarrow$ (iii). Suppose (iii) is violated; that is, there exists $s \in S$ such that $c^s(A) \cap b^s(c(A)) = \emptyset$. Let $\hat{\mu} = B(\mu^e|s)$. Then there exists $x \in c(A)$ such that $v^A(\hat{\mu}) = x \cdot \hat{\mu} < y \cdot \hat{\mu}$ for all $y \in b^s(c(A))$. Choose any $y \in b^s(c(A))$ and $P \in \mathcal{P}$ such that $y \in c^{\mu^P}(A)$. Let $t \in S$ such that $B(\mu^e|t) = \mu^P$. By regularity, P has full dimension in Δ and μ^P belongs to the interior of P; therefore, we may assume $B(\mu^e|s+t) \in P$ (if necessary, scale s and t down by some $\lambda > 0$ sufficiently small). Observe that

$$B(\mu^e|s+t) = \frac{s \cdot \mu^e}{(s+t) \cdot \mu^e}\hat{\mu} + \frac{t \cdot \mu^e}{(s+t) \cdot \mu^e}\mu^P := \hat{\mu}',$$

and that there exists $y' \in c^{\mu^P}(A)$ such that

$$v^A(\hat{\mu}') = y' \cdot \hat{\mu}' = \frac{s \cdot \mu^e}{(s+t) \cdot \mu^e}y' \cdot \hat{\mu} + \frac{t \cdot \mu^e}{(s+t) \cdot \mu^e}y' \cdot \mu^P.$$

In particular, y' maximizes the above expression, so we have $y' \cdot \hat{\mu} \geq y \cdot \hat{\mu}$ and $y' \cdot \mu^P = y \cdot \mu^P$ because $y \in c^{\mu^P}(A)$. Now let $\sigma = [s, t, e-s-t]$ and $\sigma' = [s+t, e-s-t]$. Clearly, $\sigma \sqsupseteq \sigma'$. Let $v^A(e-s-t) := v^A(B(\mu^e|e-s-t))[(e-s-t) \cdot \mu^e]$. Then

$$\begin{aligned}
V^A(\sigma') &= v^A(\hat{\mu}')[(s+t) \cdot \mu^e] + v^A(e-s-t) \\
&= (y' \cdot \hat{\mu})(s \cdot \mu^e) + (y' \cdot \mu^P)(t \cdot \mu^e) + v^A(e-s-t) \\
&\geq (y \cdot \hat{\mu})(s \cdot \mu^e) + (y \cdot \mu^P)(t \cdot \mu^e) + v^A(e-s-t) \\
&> (x \cdot \hat{\mu})(s \cdot \mu^e) + (y \cdot \mu^P)(t \cdot \mu^e) + v^A(e-s-t) \\
&= V^A(\sigma),
\end{aligned}$$

so that V^A violates the Blackwell ordering.

A.2.2 Proof of Proposition 7

Let $S^0 := \{s \in S : s_\omega < 1 \ \forall \omega \in \Omega\}$. For any signal $s \in S^0$, let $\sigma^s := [s, (1-s_1)e^1, \ldots, (1-s_N)e^N] \in \mathcal{E}$, where $e^\omega \in S$ such that $e^\omega_\omega = 1$ and $e^\omega_{\omega'} = 0$ for $\omega' \neq \omega$.

Lemma 2. *Let $s, t \in S^0$ such that $s_\omega \leq t_\omega$ for all ω. Then σ^t is a garbling of σ^s.*

Proof. Since $s_\omega \leq t_\omega$ for all ω, there exists a vector δ such that $t = s + \delta$ and $\delta_\omega \geq 0$

for all ω. Let $\alpha_\omega := \frac{\delta_\omega}{1-s_\omega}$. Notice that $\alpha_\omega \in [0,1]$ because $s_\omega < 1$ and $s_\omega + \delta_\omega = t_\omega < 1$. Let

$$M = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ \alpha_1 & 1-\alpha_1 & 0 & \cdots & 0 \\ \alpha_2 & 0 & 1-\alpha_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_N & 0 & 0 & \cdots & 1-\alpha_N \end{bmatrix}.$$

Clearly, each row of M is a probability distribution. Moreover,

$$\begin{aligned} \sigma^s M &= \left[s + \sum_{\omega \in \Omega} \alpha_\omega (1-s_\omega) e^\omega, (1-\alpha_1)(1-s_1)e^1, \ldots, (1-\alpha_N)(1-s_N)e^N \right] \\ &= \left[s + \sum_{\omega \in \Omega} \delta_\omega e^\omega, (1-s_1-\delta_1)e^1, \ldots, (1-s_N-\delta_N)e^N \right] \\ &= \left[s+\delta, (1-t_1)e^1, \ldots, (1-t_N)e^N \right] \\ &= \sigma^t. \end{aligned}$$

Thus, σ^t is a garbling of σ^s. □

To prove Proposition 7, suppose $\mu^* \in \partial P \cap \Delta^0$ and $\mu^* \neq \mu^P$. There are two cases.

Case 1: $\mu^* \notin P$. There is a hyperplane that strictly separates μ^P and μ^*. Therefore there is a menu $A = \{x, y\}$ such that $\mu^P \cdot (x-y) > 0$ and $\mu' \cdot (y-x) < 0$ for all μ' sufficiently close to μ^*. Therefore, there exists a sequence of signals $s^n \to s^*$ such that such that $B(\mu^e|s^*) = \mu^*$, $B(\mu^e|s^n) \in P$ for all n, and $B(\mu^e|s^n) \cdot (y-x) > 0$ for all n. Since $\mu^* \in \Delta^0$, we have $s^*_\omega > 0$ for all ω. Combined with the fact that $B(\mu^e|\tilde{s}) = B(\mu^e|\lambda\tilde{s})$ for all $\lambda > 0$ such that $\lambda\tilde{s} \in S$, we may assume that $s^n_\omega \leq s^*_\omega < 1$ for all n and ω. Thus, there is a sequence of vectors $\delta^n \to 0$ such that $s^* = s^n + \delta^n$ for all n. Let $\sigma^n = [s^n, \lambda^n_1 e^1, \ldots, \lambda^n_N e^N]$ and $\sigma^* = [s^*, \lambda^*_1 e^1, \ldots, \lambda^*_N e^N]$ as in Lemma 2. Then σ^* is a garbling of σ^n. Since $(s^n \mu^e) \cdot x \to (s^* \mu^e) \cdot x < (s^* \mu^e) \cdot y$, it follows that $\lim_{n\to\infty} V^A(\sigma^n) < V^A(\sigma^*)$. Thus, for large enough n, $V^A(\sigma^n) < V^A(\sigma^*)$, which is a reversal of the Blackwell ordering.

Case 2: $\mu^* \in P$. Choose a cell $P' \neq P$ as follows. If there exists $P'' \neq P$ such that μ^* belongs to the closure of P'', take $P' = P''$. Otherwise, there exists $P'' \neq \mu^*$ such that every point on the line connecting μ^* and $\mu^{P''}$ is the representative of a (singleton) cell. Take P' to be any such P''. There is a hyperplane strictly separating μ^P and $L := \text{co}\{\mu^*, \mu^{P'}\}$. Thus, there is a menu $A = \{x, y\}$ such that $\mu^P \cdot (x - y) > 0$ and $\mu' \cdot (y - x) > 0$ for all μ' sufficiently close to μ^*, including all $\mu' \in L$. In a similar fashion to the previous case, this means we can construct a sequence $\sigma^n \to \sigma^*$ where $B(\mu^e|s^n) \in L$, $B(\mu^e|s^*) = \mu^*$, and σ^n is a garbling of σ^*. The sequence satisfies $\lim_{n\to\infty} V^A(\sigma^n) > V^A(\sigma^*)$, so that V^A does not satisfy the Blackwell ordering.

A.2.3 Proof of Proposition 8

For any $\langle \mathcal{P}, \mu^{\mathcal{P}} \rangle$ and $P \in \mathcal{P}$, let $S^P := \{s \in S : B(\mu^e|s) \in P\}$. For any σ, let $s^{P,\sigma} := \sum_{s \in \sigma \cap S^P} s$. Then σ and σ' are **$\mathcal{P}$-equivalent** if $s^{P,\sigma} = s^{P,\sigma'}$ for all $P \in \mathcal{P}$.

Lemma 3. *Suppose $\langle \mathcal{P}, \mu^{\mathcal{P}} \rangle$ is regular and let $\sigma, \sigma' \in \mathcal{E}$. Then σ and σ' are $\mathcal{P}$-equivalent if and only if $V^A(\sigma) = V^A(\sigma')$ for every $\mu^{\mathcal{P}}$-decisive menu A.*

Proof. Suppose σ and σ' are $\mathcal{P}$-equivalent. Observe that for every $\mu^{\mathcal{P}}$-decisive menu A and experiment $\hat{\sigma}$, $V^A(\hat{\sigma}) = \sum_{P \in \mathcal{P}} (\mu^e s^{P,\hat{\sigma}}) \cdot c^{\mu^P}(A)$ because $c^{\mu^P}(A)$ is a singleton for all $P \in \mathcal{P}$. Thus, $V^A(\sigma) = V^A(\sigma')$ because $s^{P,\sigma} = s^{P,\sigma'}$ for all $P \in \mathcal{P}$.

For the converse, suppose σ and σ' are not $\mathcal{P}$-equivalent. We construct a $\mu^{\mathcal{P}}$-decisive menu A such that $V^A(\sigma) \neq V^A(\sigma')$. For each $P \in \mathcal{P}$, let $\delta^P := s^{P,\sigma} - s^{P,\sigma'}$. Since experiments consist of finitely many signals, there are finitely many (but at least two) cells P such that $\delta^P \neq 0$. Let $\mu^\delta := \{\mu^P : \delta^P \neq 0\}$ and let μ^{P^*} be an extreme point of the convex hull of μ^δ. Since μ^δ is finite, μ^{P^*} can be strictly separated from the convex hull of $\mu^\delta \backslash \{\mu^{P^*}\}$; that is, there exists x such that $x \cdot \mu^{P^*} > 0 > x \cdot \mu^{P'}$ for all $\mu^{P'} \in \mu^\delta \backslash \{\mu^{P^*}\}$. By regularity, we may assume that x is such that the menu $A = \{x, 0\}$ is $\mu^{\mathcal{P}}$-decisive. Then $V^A(\sigma) - V^A(\sigma') = \sum_{P \in \mathcal{P}} (\mu^s \delta^P) \cdot c^{\mu^P}(A) = (\mu^e \delta^{P^*}) \cdot x$ because $c^{\mu^P}(A) = 0$ for all $\mu^P \in \mu^\delta \backslash \{\mu^{P^*}\}$. Thus, $V^A(\sigma) \neq V^A(\sigma')$ provided $(\mu^e \delta^{P^*}) \cdot x \neq 0$. Since the separation is strict (and $\langle \mathcal{P}, \mu^{\mathcal{P}} \rangle$ is regular), we may perturb x if necessary to ensure $(\mu^e \delta^{P^*}) \cdot x \neq 0$. □

Proof that (i) implies (ii). Suppose $\dot{V}^A(\sigma) = \dot{V}^A(\sigma')$ for all $\mu^{\mathcal{Q}}$-decisive A. By Lemma 3, σ and σ' are $\mathcal{Q}$-equivalent. Since $\mathcal{Q}$ is finer than $\mathcal{P}$, it follows that σ and σ' are $\mathcal{P}$-equivalent. Since $\mu^{\mathcal{P}} \subseteq \dot{\mu}^{\mathcal{Q}}$, this implies $V^A(\sigma) = V^A(\sigma')$ for all $\dot{\mu}^{\mathcal{Q}}$-decisive A.

Proof that (ii) implies (i). Let $Q \in \mathcal{Q}$ and suppose $s, t \in S^Q$. Let $\sigma = [s, t, e - s - t]$ (if necessary, scale s and t down by a factor $\lambda > 0$ to make σ well-defined), and let

$\sigma' = [s+t, e-s-t]$. By Convexity, $s+t \in S^Q$ and, thus, σ and σ' are $\mathcal{Q}$-equivalent. By Lemma 3 and the hypothesis of (ii), this implies σ and σ' are $\mu^{\mathcal{P}}$-equivalent. Thus, there exists $P \in \mathcal{P}$ such that $s, t \in S^P$ (otherwise, there are distinct cells $P', P'' \in \mathcal{P}$ such that $s \in P'$ and $t \in P''$; but then σ and σ' are not $\mathcal{P}$-equivalent, as $s+t$ belongs to a single cell). We have shown that any two signals belonging to a common S^Q ($Q \in \mathcal{Q}$) belong to a common S^P ($P \in \mathcal{P}$). Thus, $\mathcal{Q}$ is finer than $\mathcal{P}$.

We now verify that for every $P \in \mathcal{P}$, there exists $Q \in \mathcal{Q}$ such that $\mu^P = \dot{\mu}^Q$. Suppose toward a contradiction that $\mu^P \neq \dot{\mu}^Q$ for all $Q \in \mathcal{Q}$. Let $A = \{x, 0\}$ be a $\dot{\mu}^{\mathcal{Q}}$-decisive menu such that $x \cdot \mu^P = 0$ (that is, the agent is indifferent between x and 0 at beliefs μ^P). By regularity, and the fact that $\mathcal{Q}$ is finer than $\mathcal{P}$, there exists $Q \in \mathcal{Q}$ and $s, t \in S^Q \subseteq S^P$ such that $(s\mu^e) \cdot x > 0 > (t\mu^e) \cdot x$. Let $\sigma = [s, t, e-s-t]$ and $\sigma' = [s+t, e-s-t]$ (again, scale s and t down if necessary). Then σ and σ' are $\mathcal{Q}$-equivalent. By definition of V^A, and by our choice of s and t, we have $V^A(\sigma) = V^A(s) + V^A(t) + V^A(e-s-t) > V^A(s+t) + V^A(e-s-t) = V^A(\sigma')$, contradicting $\mathcal{Q}$-equivalence of σ and σ'.

A.2.4 Proof of Proposition 9

Fix $s \in S$ and let $\mu^* = B(\mu^e|s)$ and $\mu^P = \mu^s$ where $\mu^s \in P \in \mathcal{P}$. If $A \in \mathcal{A}^*$, then there exist $x^*, y^* \in A$ such that $\overline{V}^A(s) = x^* \cdot \mu^*$ and $V^A(s) = y^* \cdot \mu^P$. In particular, $x^* \cdot \mu^* \geq x \cdot \mu^*$ and $y^* \cdot \mu^P \geq y \cdot \mu^P$ for all $x, y \in A$. Let $A^* = \{x^* - x^*, y^* - x^*\} = \{0, y^* - x^*\}$. Then $\overline{V}^A(s) = \overline{V}^{A^*}(s)$ and $V^A(s) = V^{A^*}(s)$. Hence, to compute $L_\mu(s)$, it is without loss of generality to consider menus of the form $\{0, y\}$ where $\|y\| \leq 1$. We therefore rewrite the $L_\mu(s)$ as

$$\begin{aligned} L_\mu(s) &= \sup_{\|y\|\leq 1} 0 \cdot \mu^* - y \cdot \mu^* \ \text{ subject to: } 0 \cdot \mu^* \geq y \cdot \mu^* \ \text{ and } \ y \cdot \mu^P > 0 \cdot \mu^P \\ &= \inf_{\|y\|\leq 1} y \cdot \mu^* \ \text{ subject to: } 0 \geq y \cdot \mu^* \ \text{ and } \ y \cdot \mu^P > 0. \end{aligned}$$

The first constraint ensures the Bayesian prefers action 0 over y at signal s while the second ensures the Coarse Bayesian prefers y over 0 at s. Hence, we seek the infimum of $y \cdot \mu^*$ over all y on the unit (hyper)sphere bounded by the planes $y \cdot \mu^* \leq 0$ and $y \cdot \mu^P > 0$. Clearly, the infimum is characterized by a point y^* on the plane $y \cdot \mu^P = 0$. Thus, we seek a point on the disc $\{y : y \cdot \mu^P = 0 \text{ and } \|y\| \leq 1\}$ tangent to a plane $y \cdot \mu^* = c$ with normal μ^*. There are two such points; one maximizes $y \cdot \mu^*$, the other minimizes it.

Restricting attention to the case $\mu^* \neq \mu^P$, the first constraint does not bind. Thus, the Lagrangian is

$$\mathscr{L} = -y \cdot \mu^* + \lambda_1(y \cdot \mu^P) + \lambda_2(y \cdot y - 1).$$

Setting $\frac{\partial \mathscr{L}}{\partial y_\omega} = 0$ gives $2\lambda_2 y = \mu^* - \lambda_1 \mu^P$. Then $y \cdot \mu^P = 0$ implies $0 = \mu^* \cdot \mu^P - \lambda_1 \|\mu^P\|^2$ and $y \cdot y = 1$ implies $2\lambda_2 = \mu^* \cdot y - \lambda_1 \mu^P \cdot y = \mu^* \cdot y$. Thus, $\lambda_1 = \frac{\mu^* \cdot \mu^P}{\|\mu^P\|^2}$, so that

$$2\lambda_2 y = \mu^* - \left(\frac{\mu^* \cdot \mu^P}{\|\mu^P\|^2} \right) \mu^P.$$

Since $2\lambda_2 = \mu^* \cdot y$, this implies $(\mu^* \cdot y)y = \mu^* - \left(\frac{\mu^* \cdot \mu^P}{\|\mu^P\|^2} \right) \mu^P$. Thus, any solution y satisfies

$$\begin{aligned}
(\mu^* \cdot y)^2 &= \|\mu^*\|^2 - \frac{(\mu^* \cdot \mu^P)^2}{\|\mu^P\|^2} \\
&= \|\mu^*\|^2 - \frac{\|\mu^*\|^2 \|\mu^P\|^2 \cos^2 \theta}{\|\mu^P\|^2} \\
&= \|\mu^*\|^2 \sin^2 \theta
\end{aligned}$$

where $\theta \in (0, \frac{\pi}{2}]$ is the angle (in radians) between μ^* and μ^P. Thus, $L_\mu(s) = |y \cdot \mu^*| = \|\mu^*\| \sin \theta$, which is increasing in θ. Observe that $D_\mu(s) = \left\| \frac{\mu^*}{\|\mu^*\|} - \frac{\mu^P}{\|\mu^P\|} \right\|$ is the length of the chord connecting the points $\frac{\mu^*}{\|\mu^*\|}$ and $\frac{\mu^P}{\|\mu^P\|}$ on the unit circle. The length of a chord with central angle θ is $2 \sin\left(\frac{\theta}{2}\right)$, which is strictly increasing on $[0, \frac{\pi}{2}]$. Thus, $D_\mu(s) = 2 \sin\left(\frac{\theta}{2}\right)$ increases if and only if θ increases, so that $D_\mu(s)$ increases if and only if $L_\mu(s)$ increases.

A.2.5 Proof of Proposition 10

To see that (ii) implies (i), observe that $\dot{v}^A(\hat{\mu}) \neq v^A(\hat{\mu})$ only if $\hat{\mu}$ belongs to a cell Q such that $\dot{\mu}^Q \notin \mu^{\mathcal{P}}$. Since every such Q is a singleton, $\dot{v}^A(\hat{\mu}) = \overline{v}^A(\hat{\mu}) \geq v^A(\hat{\mu})$.

To prove that (i) implies (ii), first apply Proposition 9 to get that $\dot{\mu}$ is less biased than μ. Therefore, $\mu^{\mathcal{P}} \subseteq \dot{\mu}^{\mathcal{Q}}$. We need to show that $\mathcal{Q}$ is finer than $\mathcal{P}$ and that every cell Q such that $\dot{\mu}^Q \in \dot{\mu}^{\mathcal{Q}} \backslash \mu^{\mathcal{P}}$ is a singleton.

First, we verify that $\mathcal{Q}$ is finer than $\mathcal{P}$. Suppose toward a contradication that there is a cell $Q \in \mathcal{Q}$ that intersects two or more distinct cells of $\mathcal{P}$. Then there is a unique $P \in \mathcal{P}$ such that $\dot{\mu}^Q \in P$. Let $P' \neq P$ be another cell of $\mathcal{P}$ such that $Q \cap P' \neq \emptyset$. Clearly, $\dot{\mu}^Q \notin P'$. There are two cases.

Case 1: $\dot{\mu}^Q \notin \partial P'$. Then, since P' is convex, there exists $x \in \mathbb{R}^N$ that strictly separates $\dot{\mu}^Q$ and P': $x \cdot \dot{\mu}^Q > 0 > x \cdot \hat{\mu}$ for all $\hat{\mu} \in P'$. Let $A = \{x, 0\}$. Then $v^A(\hat{\mu}') = 0$ for all $\hat{\mu}' \in Q \cap P'$ because $0 > x \cdot \mu^{P'}$. However, $\dot{v}^A(\hat{\mu'}) = x \cdot \hat{\mu}'$ for all $\hat{\mu}' \in Q \cap P'$ because $x \cdot \dot{\mu}^Q > 0$. Since $0 > x \cdot \hat{\mu}'$ for all $\hat{\mu}' \in Q \cap P'$, it follows that $\dot{v}^A(\hat{\mu}') < v^A(\hat{\mu}')$ for such $\hat{\mu}'$, a contradiction.

Case 2: $\dot{\mu}^Q \in \partial P'$. Then P' is not a singleton (otherwise $\mu^{P'} = \dot{\mu}^Q \notin P'$), forcing P' to be regular. Moreover, Q is regular because it intersects the (disjoint) sets P and P'. Thus, there are disjoint open neighborhoods $N_Q \subseteq Q$ and $N_{P'} \subseteq P'$ of $\dot{\mu}^Q$ and $\mu^{P'}$. Since N_Q and $N_{P'}$ are convex, there exists $x \in \mathbb{R}^N$ that strictly separates them: $x \cdot \hat{\mu} > 0 > x \cdot \hat{\mu}'$ for all $\hat{\mu} \in N_Q$ and $\hat{\mu}' \in N_{P'}$. Moreover, $\dot{\mu}^Q \in Q \cap \partial P'$ implies $N_Q \cap P' \neq \emptyset$; by regularity, $N_Q \cap P'$ is a full-dimensional subset of $Q \cap P'$. Perturb x so that the plane $x \cdot \hat{\mu} = 0$ passes through the interior of $N_Q \cap P'$ (but not the point $\dot{\mu}^Q$); this can be done by shifting the plane toward the point $\dot{\mu}^Q$. Then x no longer separates N_Q and $N_{P'}$, but the set $C := \{\hat{\mu} \in N_Q \cap P' : 0 > x \cdot \hat{\mu}\}$ is nonempty, and we still have $x \cdot \dot{\mu}^Q > 0$ and $0 > x \cdot \hat{\mu}'$ for all $\hat{\mu}' \in N_{P'}$. Letting $A = \{x, 0\}$, it follows that $v^A(\hat{\mu}) = 0 > x \cdot \hat{\mu} = \dot{v}^A(\hat{\mu})$ for all $\hat{\mu} \in C$, a contradiction.

Next, we verify that every cell Q such that $\dot{\mu}^Q \in \dot{\mu}^{\mathcal{Q}} \backslash \mu^{\mathcal{P}}$ is a singleton. Suppose toward a contradiction that there exists $\dot{\mu}^Q \in \dot{\mu}^{\mathcal{Q}} \backslash \mu^{\mathcal{P}}$ such that Q is not a singleton. Since $\dot{\mu}$ is more sophisticated than μ, there is a unique $P \in \mathcal{P}$ such that $Q \subseteq P$. Note that $\mu^P = \dot{\mu}^P \in \dot{\mu}^{\mathcal{P}}$. Since μ^Q belongs to the relative interior of Q, there exists $\mu^* \in Q$ such that $\dot{\mu}^Q \notin \{\alpha \mu^* + (1-\alpha)\mu^P : \alpha \in [0,1]\} := L$. The set L is closed and convex, and therefore can be strictly separated from $\dot{\mu}^Q$: there exists $x \in \mathbb{R}^N$ such that $x \cdot \dot{\mu}^Q > 0 > x \cdot \hat{\mu}$ for all $\hat{\mu} \in L$. In particular, both $x \cdot \mu^* < 0$ and $x \cdot \mu^P < 0$. Let $A = \{0, x\}$. Then, at (Bayesian) posterior $\mu^* \in Q \subseteq P$, the $\langle \mathcal{P}, \mu^{\mathcal{P}} \rangle$ representation selects 0 from A: $v^A(\mu^*) = 0$. Under representation $\langle \mathcal{Q}, \mu^{\mathcal{Q}} \rangle$, however, x is selected from A at posterior μ^* because $\mu^* \in Q$ and $x \cdot \dot{\mu}^Q > 0$. Thus, $\dot{v}^A(\mu^*) = x \cdot \mu^* < 0$, so that $\dot{v}^A(\mu^*) < v^A(\mu^*)$.

B Signal Distortions

This appendix provides an alternative representation of Coarse Bayesian updating rules. The idea, formalized by the next definition, is that non-Bayesian reactions are due to errors or biases in the agent's perception of information.

Definition 7. An updating rule μ has a **Signal Distortion Representation** if there is a function $d : S \to S$ (a **signal distortion**) such that $\mu^s = B(\mu^e | d(s))$ for all $s \in S$.

In a Signal Distortion Representation, an agent who observes signal s updates beliefs by applying Bayes' rule with a modified signal $d(s)$. Thus, the function d is a behavioral parameter capturing the agent's tendency to distort information. Such distortions could be due to imperfections in the agent's perception or, in some applications, may reflect the agent's beliefs about the accuracy or reliability of the information source.[15]

Note that if μ has a Signal Distortion Representation, then $d(e) \approx e$. This is the only substantive property of d implied by such a representation. Therefore, without additional restrictions, the concept of signal distortion can explain almost any updating behavior. The next definition provides restrictions on d that make Signal Distortion Representations equivalent to Coarse Bayesian Representations.

Definition 8. A signal distortion d is

(i) **Homogeneous** if $d(s) \approx d(t)$ whenever $s \approx t$.

(ii) **Convex** if $d(s) \approx d(t)$ implies $d(\lambda s + (1-\lambda)t) \approx d(s)$ for all $\lambda \in [0, 1]$.

(iii) **Idempotent** if $d(d(s)) = d(s)$ for all s.

Homogeneity states that two distorted signals have common likelihood ratios if the original signals have common likelihood ratios. Convexity states that if two signals have common distorted likelihood ratios, then those ratios also come about from distorting mixtures of those signals. Idempotency requires the distortion process to be stable: the distortion of $d(s)$ is $d(s)$.

[15] See also Aydogan et al. (2017), who propose a model of signal distortion similar in spirit to that of Rabin and Schrag (1999).

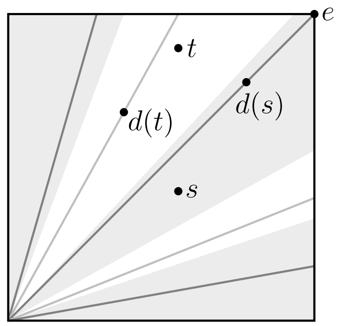


Figure 8: A Signal Distortion Representation on S for two states.

As illustrated by Figure 8, signal distortions d that are Homogeneous, Convex, and Idempotent effectively categorize signals and assign common distorted likelihood ratios to signals in the same category. In particular, d generates a partition of S into convex cones and a representative ray for each cell. Signals in a given cell are distorted to points along the representative ray, ensuring common likelihood ratios.

Theorem 2. *An updating rule has a Coarse Bayesian Representation if and only if it has a Homogeneous, Convex, Idempotent Signal Distortion Representation. If d and d' are two such representations, then $d(s) \approx d'(s)$ for all $s \in S$.*

Proof. First, suppose μ has a Coarse Bayesian Representation. Note that for every $s \in S$ the signal $\frac{\mu^s/\mu^e}{\|\mu^s/\mu^e\|}$ is well-defined because μ^e has full support. Define $d : S \to S$ by

$$d(s) = \begin{cases} s & \text{if } \mu^s = B(\mu^e|s) \\ \frac{\mu^s/\mu^e}{\|\mu^s/\mu^e\|} & \text{otherwise} \end{cases}.$$

It is straightforward to verify that $\mu^s = B(\mu^e|d(s))$ for all s and that d is Homogeneous, Convex, and Idempotent.

Conversely, suppose μ has a Signal Distortion Representation with Homogeneous, Convex, and Idempotent d. Define a binary relation $\sim$ on S by $s \sim t$ if and only if $d(s) \approx d(t)$. Clearly, $\sim$ is an equivalence relation; thus, its equivalence classes partition S. Homogeneity and Convexity of d ensure each equivalence class is a convex cone. Thus, as in the proof of Theorem 1, each equivalence class is associated with a convex subset of Δ, and these subsets form a partition $\mathcal{P}$ of Δ. For each cell $P \in \mathcal{P}$, let $\mu^P := B(\mu^e|d(s))$ such that s belongs to the equivalence class associated with P. By Idempotency, $\mu^P \in P$. □

Theorem 2 establishes the sought-after equivalence between Coarse Bayesian and Signal Distortion Representations. Thus, any updating rule satisfying Homogeneity, Convexity, and Confirmation has such a Signal Distortion Representation, and the distortion d is unique up to scalar transformation (likelihood ratios).

C Relationship to Mullainathan (2002)

In a working paper, Mullainathan (2002) develops a model of categorical thinking sharing several features of Coarse Bayesian updating. In this appendix, I show that the categorical thinking model (adapted to my framework of states and signals) satisfies Homogeneity and Cognizance but not necessarily Confirmation.

Mullainathan works with a type space T and prior p where $p(t)$ is the prior probability of type $t \in T$. The analogous components in my model are the state space Ω and prior μ^e, where μ^e_ω is the prior probability of state $\omega \in \Omega$. Data d in Mullainathan's model can be expressed by conditional probabilities $p(d|t)$ indicating the probability of observing the data given type t; in my model, data corresponds to a signal realization s, and s_ω (the probability of observing s in state ω) plays the role of $p(d|t)$.

A set C of probability distributions over T constitutes a set of "categories." These are feasible beliefs that the agent can hold in Mullainathan's model. Thus, the set C is analogous to the set $\{\mu^P : P \in \mathcal{P}\}$ in my model. For a category c and data d, $p(d|c)$ is the probability of generating data d in category c; this is analogous to $s \cdot \mu^P$, which is the probability of observing signal s if μ^P is the true probability law. Finally, Mullainathan defines $p(c) := \int_t p(t)c(t)$ to be the "base rate" of category c.[16] In my model, the analogous rate is $\mu^e \cdot \mu^P$.

Like Coarse Bayesians, agents in Mullainathan's model partition the probability simplex and assign posterior beliefs as a function of the cell containing the Bayesian posterior. Any set of C of categories (feasible posteriors) is permitted; however, the partition is derived from C using an optimality criterion resembling that of Maximum-Likelihood rules in section 3.2. In particular, let $c^*(d) \in C$ denote the agent's posterior

[16] I have modified the notation slightly; Mullainathan writes $q_c(\cdot)$ instead of $c(\cdot)$ to indicate the probability distribution over T associated with category $c \in C$.

after observing data d. Mullainathan requires that

$$c^*(d) \in \underset{c \in C}{\operatorname{argmax}}\, p(d|c)p(c). \tag{7}$$

In my framework, the analogous condition is

$$\mu^s \in \underset{\hat{\mu} \in \hat{C}}{\operatorname{argmax}} (s \cdot \hat{\mu})(\mu^e \cdot \hat{\mu}), \tag{8}$$

where $\hat{C} \subseteq \Delta$ is some set of feasible posteriors. This is very similar to maximization of the likelihood function specified in section 3.2; the main difference is that my likelihood functions use a second-order belief γ instead of the base rate $p(c)$ proposed by Mullainathan.

Thus, Mullainathan's model works by specifying a set C of categories (feasible posteriors) from which the criterion (7) selects posteriors after observing data d. Because of the functional forms employed, it is as if there is a partition of the probability simplex such that the agent's selected posterior only depends on which cell contains the Bayesian posterior.

Unlike Coarse Bayesians, categorical thinkers need not satisfy Confirmation because condition (7) does not guarantee that beliefs $c^*(d)$ belong to the cell containing the Bayesian posterior associated with data d.[17] Below, I prove these claims in my framework (in particular, employing condition (8)).

First, let $\hat{C}$ be a nonempty set of feasible posteriors. Suppose that some $\mu^* \in \hat{C}$ is a solution to the maximization problem in (8) for both s and t. That is, μ^* solves both

$$\max_{\hat{\mu} \in \hat{C}} (s \cdot \hat{\mu})(\mu^e \cdot \hat{\mu}) \qquad \text{and} \qquad \max_{\hat{\mu} \in \hat{C}} (t \cdot \hat{\mu})(\mu^e \cdot \hat{\mu}).$$

Then, if $\alpha, \beta \geq 0$, it follows at μ^* solves

$$\max_{\hat{\mu} \in \hat{C}} ((\alpha s + \beta t) \cdot \hat{\mu})(\mu^e \cdot \hat{\mu}).$$

It follows that the map $s \mapsto \operatorname{argmax}_{\hat{\mu} \in \hat{C}} (s \cdot \hat{\mu})(\mu^e \cdot \hat{\mu})$ is measurable with respect to a partition of S into convex cones. As demonstrated in the proof of Theorem 1, such

[17] Note that the partitions in Mullainathan's model will typically have convex cells. Convexity only fails if the maximization problem in (7) has more than one solution and the agent's tie-breaking criterion is not convex.

convex cones can be associated with convex subsets of Δ by mapping signals s to Bayesian posteriors $B(\mu^e|s)$.

Thus, any updating rule satisfying (8) satisfies Homogeneity and Cognizance if one restricts attention to signals that yield unique solutions to the optimization problem. For signals that involve ties, Homogeneity and/or Cognizance may be violated if the agent's tie-breaking selection is not Homogeneous or Convex.

A more substantive difference between Mullainathan's model and Coarse Bayesian updating is that condition (8) does not guarantee that the updating rule satisfies Confirmation. To see this, suppose $|\Omega| = 2$ and let $\mu^e = (\frac{1}{3}, \frac{2}{3})$. Suppose that $\hat{\mu}, \hat{\mu}' \in \hat{C}$ where $\hat{\mu} = (\frac{1}{4}, \frac{3}{4})$ and $\hat{\mu}' = (\frac{1}{5}, \frac{4}{5})$. Let $s = (\frac{3}{8}, \frac{9}{16})$. It follows that $B(\mu^e|s) = \hat{\mu}$; so, Confirmation requires $\hat{\mu}$ to solve

$$\max_{\tilde{\mu} \in \hat{C}} (s \cdot \tilde{\mu})(\mu^e \cdot \tilde{\mu}).$$

However,

$$(s \cdot \hat{\mu})(\mu^e \cdot \hat{\mu}) = \frac{77}{256} < \frac{63}{200} = (s \cdot \hat{\mu}')(\mu^e \cdot \hat{\mu}').$$

Thus, $\hat{\mu}$ is not selected at s, violating Confirmation.

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