Redistribution and Government Commitment*

Youngsoo Jang†

January 2024

Abstract

How and to what extent does government commitment matter for the optimality of tax-transfer systems? In an incomplete-markets economy, I characterize and solve the equilibria of a dynamic game between individuals and a benevolent government according to its commitment technologies. I find that the optimality depends heavily on time inconsistency, which is possible only with commitment. Commitment facilitates substantial long-term taxes and transfers, incurring long-run welfare losses but yielding short-run welfare gains through front-loaded factor price adjustments and income redistribution. Without commitment, the time-consistent government neglects these long-term policy impacts on the short-run economy, attaining considerably smaller short-run welfare gains.

JEL classification: E61, H21

Keywords: Taxes and Transfers, Commitment, Time-Consistency, Incomplete Markets, Transition Dynamics.

*I have benefited from helpful comments by Juan Carlos Conesa, Begoña Domínguez, Timothy Kam, Tatyana Koreshkova, Dirk Krueger, Virgiliu Midrigan, Taisuke Nakata, Yena Park, Michelle Rendall, José-Víctor Ríos-Rull, Yongseok Shin, Gianluca Violante, Minchul Yum, and the seminar participants at Monash University, the University of Southampton, Australian National University, the University of Queensland, the University of Connecticut, Korea University, Korea Labor Institute, and the University of Melbourne. I am also grateful to the session participants at Shanghai Macro Workshop, Sogang Summer Research Workshop, T2M conference in Paris, VAMS workshop in Sydney, and Jinan IESR Quantitative Macro Workshop. Discussions with Ji-Woong Moon were constructive at all stages of this project. I also thank Yunho Cho, Pantelis Kazakis, Qian Li, and Hoonsuk Park for their generous help. A previous version of the paper circulated with the title "On the Time Consistency of Universal Basic Income". All remaining errors are mine.

†School of Economics, University of Queensland, Colin Clark Building (#39), Blair Dr, St Lucia, QLD 4072, Australia. E-mail: youngsoo.jang@uq.edu.au

1 Introduction

Over the past four decades, many developed countries have increased public social spending and introduced broader and more generous social programs.[1] A large body of literature responds by examining various redistributive policies using model economies with a rich level of heterogeneity (Conesa et al., 2009; Heathcote et al., 2017; De Nardi et al., 2017; Boar and Midrigan, 2022; Guner et al., 2023). However, relatively few studies have analyzed redistributive policies in terms of *time inconsistency*. This disconnection needs addressing, given that a strand of long-lasting literature has demonstrated the importance of time inconsistency in designing government policies (Kydland and Prescott, 1977; Calvo, 1978; Fischer, 1980; Barro and Gordon, 1983; Lucas and Stokey, 1983; Persson et al., 1987; Chari and Kehoe, 1990; Klein and Ríos-Rull, 2003; Bassetto, 2005).

This paper contributes to filling this void by exploring how and to what extent government commitment affects optimal tax-transfer schemes in the standard incomplete-markets model of Bewley (1986), Huggett (1993), and Aiyagari (1994). To this end, I undertake the following steps. First, I analytically characterize the equilibria of a dynamic game between heterogeneous individuals and a benevolent government according to its commitment technologies, employing the generalized Euler equation (GEE) approach in Klein, Krusell and Ríos-Rull (2008)—the first-order condition (FOC) of the government's policy decision. Second, to quantify my theoretical findings, I develop a numerical solution algorithm that can solve the stationary equilibrium and transition dynamics of a general class of heterogeneous-agent models where the government lacks commitment. Finally, utilizing this numerical method, I quantitatively explore the variations in optimal tax-transfer schemes along the transition path based on government commitment technologies.

Throughout the above exercises, a utilitarian government levies taxes on individual income, covers a predetermined level of government spending from the tax revenue, and redistributes the rest to individuals through lump-sum transfers. The government's commitment technology distinguishes scenarios. In the case with commitment, the economy is in Ramsey equilibrium, where the government—equivalent to a Ramsey planner—is able to commit to future tax and transfer policies, choosing a sequence of income taxes and transfers along the transition path. In contrast, in the case without commitment, the economy is in Markov-perfect equilibrium (MPE), where the government can only set a tax rate and transfers for the next period and cannot commit to them thereafter, as in Krusell and Ríos-Rull (1999). As a result, the government sequentially chooses a tax and transfer policy, and this action continues perpetually, implying a *time-consistent policy*.

My main findings indicate that the optimality of taxes and transfers relies heavily on time inconsistency, which is possible only with commitment. Compared to a U.S. calibrated economy, welfare improvements, as measured by the consumption-equivalent variation, in the case with com-

[1] The average proportion of social spending out of GDP among the OECD countries increased from 14.4% in 1980 to 23% in 2020. For the U.S. case, Ben-Shalom et al. (2011) investigated its expansions in more detail.

mitment are much greater than those without commitment (+1.56% vs. +0.19%). This disparity is driven by the effects of long-term tax-transfer policy on the short-run economy. Commitment enables substantial long-term taxes and transfers that incur welfare losses in the long run but generate short-run welfare gains through two types of pro-poor changes early in the transition.

First, large transfers in the long run rapidly reduce income inequality during the early transition. These reductions are facilitated by upfront decreases in precautionary savings occurring before the implementation of large transfers, as the expected probability of reaching the borrowing constraint is reduced. This diminished precautionary motive causes low-income individuals to decrease their proportion of capital income in the short run, where inequality is most severe.

Second, large taxes and transfers in the long run induce favorable factor price changes for low-income individuals—a rise in the market wage and a fall in the market interest rate—during the early transition. Although increases in taxes and transfers reduce both capital and labor in the long run, slower adjustments in capital lead to a short-term increase in the capital-to-labor ratio, causing factor prices to be advantageous for low-income individuals whose income composition is biased toward labor.

Without commitment, the time-consistent government disregards the effects of long-term tax-transfer policy on the short-run economy, rendering it unable to leverage the two types of pro-poor changes during the early transition. Consequently, much smaller welfare improvements are attained. These findings suggest that legislating redistributive policies could be more effective than implementing them on a discretionary basis.

More specifically, in the theoretical analysis, using the GEE approach, I find that when making policy decisions, the government balances the trade-off between two types of economic forces that reshape individuals' disposable income: (i) income redistribution through taxes and transfers and (ii) changes in the factor composition of income via adjustments in factor prices (wage and interest rate)—*pecuniary externalities*—as explored in Davila, Hong, Krusell and Ríos-Rull (2012).[2] These two forces have opposite impacts on individuals' welfare based on their income level. When aggregate income redistribution is greater (less) than aggregate pecuniary externalities, the government increases (decreases) taxes and transfers accordingly.

The role of government commitment is to determine how to aggregate income redistribution and pecuniary externalities *across the temporal dimension*. With commitment, the government compares the two types of economic forces by aggregating each over the entire time horizon. This aggregation over the entire periods implies that the government considers the policy impacts not only during and after the policy takes effect but also in periods preceding the implementation. As a result, commitment leads the government to consider the impacts of long-term policies on the short-run economy, which can lead to significant short-run benefits at the expense of long-run

[2] They thoroughly investigated pecuniary externalities on constrained efficiency in incomplete-markets models.

welfare losses.[3]

Without commitment, the government compares income redistribution and pecuniary externalities by aggregating each force in every period in a forward-looking manner. This time-consistent approach implies that the government considers the effects of its policy only since it goes into effect, without accounting for the impacts in periods before its implementation. Therefore, without commitment, the government overlooks the effects of long-term policies on the short-run economy.[4] These findings imply that commitment makes fundamental differences in terms of what the government considers when formulating long-term tax-transfer policies.

To examine the quantitative implications of the above theoretical findings, I calibrate the model to the U.S. economy as a starting point and compare the differences in taxes and transfers based on government commitment technologies. I find that commitment brings quantitatively considerable differences to the aggregate economy, inequality, and welfare. With its time-inconsistent optimal policy, the Ramsey planner opts for more substantial income taxes than does the government with the time-consistent optimal policy over the entire transition path. Compared to the calibrated initial economy, the Ramsey planner gradually increases income taxes by 16 percentage points. However, in the absence of commitment, the optimal income tax rate increases by only 2 percentage points.

This gap in tax policies results in differences in the size of transfers. The ratio of transfers to initial GDP in the case with commitment increases by 9.2 percentage points, but that in the case without commitment increases by 1.4 percentage points. These differences in the tax-transfer scheme result in diverse dynamics in aggregate variables and distributions. The economy managed by the Ramsey planner is less efficient but more equal due to the imposition of higher income taxes and larger transfers. Aggregate consumption, capital, and output are greater in the case without commitment, but inequality is lower in the case with commitment. The welfare gain, as measured by the consumption-equivalent variation, is greater in the time-inconsistent optimal income tax-transfer scenario (+1.56%) than in the time-consistent scenario (+0.19%).

With its commitment, the Ramsey planner achieves front-loaded welfare gains through both reduced income inequality (considerable income redistribution) and changes in the factor composition of income (positive pecuniary externalities). Understanding how aggregate capital and the distribution of individual assets evolve over time is crucial for comprehending these short-run welfare consequences.

First, differences in adjustment speed between capital and labor play a critical role in achieving front-loaded welfare gains through pecuniary externalities. In the economy with commitment, the

[3] This strategy proves advantageous for a government with a discount factor less than one. Note that this time-lagged strategy could lead to a time-inconsistent outcome because the policy scheme desired at time 0 may not be optimal when evaluated in each period.

[4] While this time-consistent policy is optimal when evaluated in each period, it may not be the best when considering the overall impact at time 0.

slower adjustment of capital compared to labor results in changes in factor prices that are favorable for low-income individuals—an increase in wages and a decrease in interest rates.[5] Although substantial increases in taxes and transfers reduce both aggregate capital and labor, slower adjustments in capital lead to a higher capital-to-labor ratio early in the transition. This induces the aforementioned factor price changes in the short run, which benefit low-income individuals through positive pecuniary externalities.

Second, precautionary savings motives are important in understanding upfront welfare gains through income redistribution. The long-term provision of substantial transfers enables low-income individuals to reduce their precautionary savings early in the transition, resulting in a decrease in the proportion of their capital income. This change in the factor composition of income leads to a rapid reduction in after-tax income inequality for low-income individuals, as capital income exhibits greater inequality than labor income and transfers. This reduction in income inequality means facilitated income redistribution for low-income individuals, resulting in front-loaded welfare gains.

The above income redistribution channel operates primarily through low-income individuals. They maintain the initially reduced proportion of capital income until the long run because their savings are motivated by precautionary reasons, and considerable insurance via transfers continues. In contrast, middle- and high-income individuals exhibit different saving behaviors—which are shaped mainly by the substitution effect. Although they temporarily decrease their savings early in the transition, to a lesser extent, due to a decline in the market interest rate, the proportion of their capital income subsequently increases when the market interest rate rises again in the long run. These findings indicate that the reduction in precautionary savings plays a crucial role in achieving front-loaded welfare gains through income redistribution.

In exchange for these front-loaded welfare gains, the government with commitment must bear long-run welfare losses from stagnant income redistribution and negative pecuniary externalities caused by changes in the factor composition of income that are unfavorable for low-income individuals. These findings imply that the Ramsey planner counterbalances these two types of economic forces over the entire time horizon. It achieves upfront welfare gains from reduced income inequality and factor price changes favoring low-income individuals, while delaying welfare losses arising from unfavorable factor price changes for low-income individuals and stagnant income redistribution later on. This welfare trade-off between the short and long runs is quantitatively substantial. The difference in welfare changes between the short and long runs is significant (+4.66 pp). The significant short-run welfare gain (+1.56%) implies that this time lag balancing is effective when commitment is available.

[5] Note that the government prioritizes representing the interests of low-income individuals, as these forces are weighted by the marginal utility of consumption.

Without commitment, the time lag strategy mentioned above is not credible because the government cannot commit to providing substantial long-term insurance. If the government without commitment finds itself in the long-run equilibrium of the economy with commitment, it will ignore any upfront welfare gain and instead focus on rebalancing income redistribution with pecuniary externalities in each period, accounting for the effects of its policy only since its implementation. In this case, the government perceives the economy as one where the costs from negative pecuniary externalities—caused by a reduction in the market wage and an increase in the market interest rate—outweigh the benefits from income redistribution that are mitigated in the long run. Consequently, the government will consider a one-time reduction in taxes and transfers in the next period, as that is all it can do without commitment, in an attempt to improve welfare. Therefore, the time lag strategy in the commitment case is not credible without commitment.

This time-consistent strategy increases inequality in after-tax income, thereby impeding improvements in welfare. Households rationally anticipate the above government incentive and start to increase their precautionary savings and labor supply. These changes in individual decisions prevent a reduction in income inequality in the short run, leading to a much smaller welfare trade-off between the short and long runs. The difference in welfare changes between the short and long runs is smaller in the case without commitment (+0.55 pp vs. +4.66 pp). The quantitative findings suggest that this small trade-off leads to considerably smaller short-run welfare gains for the case without commitment (+0.19% vs. +1.55%). Additionally, I find that this welfare implication remains robust with changes in the tax base. These results suggest that the degree of government commitment when delivering redistributive policies could be crucial for ensuring their effectiveness.

Finally, it is worth discussing the global solution method developed to solve the case without commitment. Note that its usage is not limited to the examples presented in this paper but also applicable to other general games with incomplete financial markets, including economies with general optimal policies or political procedures. This game in MPEs, as Krusell et al. (1996); Krusell and Ríos-Rull (1999) demonstrate, is complex because it involves the interplay of three equilibrium objects: individual decisions, the aggregate law of motion for the household distribution, and the endogenous government policy function, all of which must be consistent with each other. Solving the game becomes more challenging with one-shot deviations required to solve sub-perfect Nash equilibria because the deviations produce a sequence of taxes/transfers that are off-the-equilibrium paths——alternative paths of the economy that are not selected in the equilibrium but still need to be defined to shape the equilibrium. To address these computational issues, this paper draws on the backward induction method of Reiter (2010) and makes modifications to suit the characteristics of the MPE, considering the existence of off-the-equilibrium paths.[6]

[6] Section 5 explains the key ideas and Appendix A demonstrates each step of the algorithm in detail, including

Related Literature: This paper is closely related to a strand of literature that examines the optimality of taxes and transfers using heterogeneous-agent models (Conesa and Krueger, 2006; Conesa et al., 2009; Heathcote et al., 2017; Heathcote and Tsujiyama, 2021; Boar and Midrigan, 2022; Ferriere et al., 2022; Dyrda and Pedroni, 2023). They focus on the trade-offs between the progressivity or size of taxes and the size of transfers when designing optimal policies. This paper complements these studies by exploring how the government balances the trade-offs between efficiency and equity differently according to the availability of government commitment technologies.

This paper also belongs to the stream of macroeconomic literature that examines the implications of time-inconsistent features for government policies, following the seminal study of Kydland and Prescott (1977). A branch of this literature conducts this task by investigating the features of time-consistent policies in MPEs that depend on the fundamental economic state variables (Cohen and Michel, 1988; Krusell et al., 1996; Krusell and Ríos-Rull, 1999; Klein et al., 2008; Corbae et al., 2009; Azzimonti, 2011; Song et al., 2012; Bassetto et al., 2020; Laczó and Rossi, 2020).[7] This paper contributes to the literature by examining optimal policies according to the availability of government commitment, a comparison that has not been previously explored under incomplete financial markets.

This paper is also related to macroeconomic studies on constrained efficiency in dynamic general equilibrium models with incomplete-markets that focus on pecuniary externalities through wages and interest rates. For example, Davila et al. (2012) analyzes constrained efficiency in models presented in Aiyagari's (1994) model; Park (2018) conducts a similar analysis with human capital; and Itskhoki and Moll (2019) characterizes the optimal distribution of development policies between workers and entrepreneurs in an economy with financial constraints. Itskhoki and Moll (2019) shares a similarity with this paper in the sense that both studies show that long-term policies drive short-run welfare gains, in which factor price changes early in the transition play important roles. However, while Itskhoki and Moll (2019) focuses on the effects of the initial condition—degree of development—on the Ramsey optimal policy, this paper concentrates on the extent to which the optimality of a Ramsey policy depends on its time inconsistency.

The solution method in this paper is a non-negligible, independent contribution to the literature. Broadly, two types of methods are often used to solve macroeconomic models with MPEs. The first is Klein, Krusell and Ríos-Rull's (2008) approach, which is a solution method using the GEE. This method is accurate and efficient; however, it is not designed to handle heterogeneous-agent

outcomes related to its efficiency and accuracy.

[7] Another branch of this literature has focused on designing policies to overcome time-inconsistency issues through the use of rules (Barro and Gordon, 1983; Rogoff, 1985; Athey et al., 2005), a richer range of policy instruments (Lucas and Stokey, 1983; Debortoli et al., 2017, 2021), and reputational equilibria (Chari and Kehoe, 1990; Domínguez, 2007a,b) using representative-agent models.

models with incomplete financial markets. The method in this paper is a solution method applicable to heterogeneous-agent models. The other approach is a modification of Krusell and Smith's (1998) method, which is applicable to heterogeneous-agent models. However, this simulation-based method is computationally costly because economies without commitment would have more than one aggregate law of motion (e.g., the law of motion for distributions and the endogenous tax policy function). This structure increases the computational burden exponentially.[8] Additionally, in some cases, the endogenous policy function can be highly non-linear, which poses a challenge for capturing it accurately using the parameterized law of motion in Krusell and Smith's (1998) method. The approach used in this paper is more accurate and efficient by following the non-simulation-based solution approach that captures the non-linearity in a non-parametric way as in Reiter (2010).

The remainder of this paper proceeds as follows. Section 2 presents the model and defines the equilibria. Section 3 characterizes the equilibria by using the GEE. Section 4 explains the core ideas of the numerical solution algorithm. Section 5 describes the calibration strategy. Section 6 presents the results of the policy analysis. Section 7 concludes this paper. Finally, Appendix A demonstrates the full details of the numerical solution algorithm.

2 Model

The quantitative model here builds upon the canonical model of Aiyagari (1994), incorporating wealth effects of labor supply. In this model, given a tax policy rule, heterogeneous households make decisions on consumption, savings, and labor supply at the intensive margin, as in standard incomplete-markets models. A notable difference from the standard models is the manner in which the tax-transfer policy is set. It is determined endogenously, according to the government's commitment technology. In an equilibrium state, the tax-transfer policy, individual decisions, and the evolution of the distribution are all consistent with one another, given the government's commitment constraint.

2.1 Environment

The model economy is populated by a continuum of infinitely lived households. Their preferences follow

$$E\left[\sum_{t=0}^{\infty}\beta^t u(c_t, 1-n_t)\right] \tag{1}$$

[8] For example, Corbae et al. (2009) used this approach in their heterogeneous-agent economy with the GHH preference, which reduces the computational burden.

where c_t is consumption, $n_t \in [0, 1]$ is labor supply in period t ($(1 - n_t)$ refers to leisure), and β is the discount factor. Preferences are represented by

$$u(c_t, 1 - n_t) = \frac{{c_t}^{1-\sigma}}{1 - \sigma} + B\,\frac{(1 - n_t)^{1-1/\chi}}{1 - 1/\chi} \tag{2}$$

where σ is the coefficient of relative risk aversion, B is the utility of leisure, and χ is the Frisch elasticity of labor supply. Note that the preferences here capture the wealth effects of labor supply. Such wealth effects are crucial for a welfare analysis, closely related to efficiency loss. An increase in transfers, for example, decreases the overall labor supply, shrinking the size of the aggregate economy and playing a role in reducing welfare.

The representative firm produces output with constant returns to scale. The firm's technology is represented by

$$Y_t = zF(K_t, N_t) = zK_t^{\theta} N_t^{1-\theta} \tag{3}$$

where z is the total factor productivity (TFP), K_t is the quantity of aggregate capital, N_t is the quantity of aggregate labor, and θ is the capital income share. Capital depreciates at the rate of δ each period.

In each period, households confront an uninsurable, idiosyncratic shock ϵ_t to their wage that follows an AR-1 process:

$$\log(\epsilon_{t+1}) = \rho_\epsilon \log(\epsilon_t) + \eta^{\epsilon}_{t+1} \tag{4}$$

where $\eta^{\epsilon}_{t+1} \sim N(0, \sigma^2_\epsilon)$. Using the method in Rouwenhorst (1995), I approximate the AR-1 process as a finite-state Markov chain with transition probabilities $\pi_{\epsilon,\epsilon'}$ from state ϵ to state ϵ'. Households earn $w_t\epsilon_t n_t$ as their labor income where w_t is the market equilibrium wage. They can self insure through assets a_t. Such households have capital income of as much as $r_t a_t$ where r_t is the equilibrium risk-free interest rate.

The government obtains its tax revenue by levying taxes on household capital and labor income at proportional flat tax rate, τ_t.[9] Given its tax revenue, the government finances constant government spending G, and the rest is used for transfers $T_t \geq 0$. The government runs a balanced budget

[9] In a later section, I relax the assumption on the tax base.

each period:

$$G + \int T_t(y_t(a,\epsilon))\mu_t(\mathbf{d}(a \times \epsilon)) = \tau_t \left[r_t K_t + w_t N_t\right] \tag{5}$$

where

$$T_t = \begin{cases} T_{1,t} + T_{2,t}(y_t) & \text{before a policy reform} \\ T_{1,t} & \text{since a policy reform} \end{cases}$$

where y_t is pre-tax individual income in period t, $w_t \epsilon_t n_t + r_t a_t$, and $\mu_t(\cdot)$ is the distribution over households in period t. $T_{1,t}$ represents lump-sum transfers and $T_{2,t}(\cdot)$ denotes progressive transfers that decrease in y $(T_2'(y) < 0)$. A policy reform eliminates the progressive component of transfers and transforms the entire transfer system into lump-sum transfers, motivated by a growing number of recent studies on Universal Basic Income (UBI) (Jaimovich et al., 2022; Santos and Rauh, 2022; Conesa et al., 2023; Daruich and Fernández, 2023; Guner et al., 2023).

Note that this tax-transfer system is progressive regarding taxes net of transfers, $\tau_t y_t - T_t$, regardless of the implementation of the reform.[10] This progressivity can be captured through both a progressive tax with lump-sum transfers and a flat income tax with progressive transfers. I choose the latter because of two reasons. First, using a flat income tax provides a clearer theoretical representation when using the GEE in Section 3. Second, recently proposed UBI policies in the real world mostly feature a replacement of pre-existing social welfare systems with lump-sum transfers.

2.2 Competitive Equilibrium, Exogenous Policy

In this section, I define the competitive equilibrium, given an exogenous tax and transfer policy. I start with a setting to address the economy with commitment. To describe problems with commitment (the Ramsey problem), household dynamic problems need to be represented in a sequential manner. At the beginning of each period, households differ from one another in asset holdings a and labor productivity ϵ. $\mu_t(a,\epsilon)$ denotes the distribution of households in period t. Given a sequence of prices $\{r_t, w_t\}_{t=0}^{\infty}$, income taxes $\{\tau_t\}_{t=0}^{\infty}$, and transfers $\{T_t\}_{t=0}^{\infty}$, households in period t

[10] Before this policy reform, the ratio of the marginal rate to the average rate is $\frac{\tau_t - T'_{2,t}(y_t)}{\tau_t - (T_1/y) - (T_2(y)/y)} > \frac{\tau_t - T'_{2,t}(y_t)}{\tau_t - (T_2(y)/y)} > 1$. After a reform, the ratio is $\frac{\tau_t}{\tau_t - (T_1/y)} > 1$.

solve

$$v_t(a,\epsilon) = \max_{c_t,\, a_{t+1},\, n_t} u(c_t(a,\epsilon), 1 - n_t(a,\epsilon)) + \beta \sum_{\epsilon_{t+1}} \pi_{\epsilon_t,\epsilon_{t+1}} v_{t+1}(a_{t+1}(a,\epsilon), \epsilon_{t+1}) \tag{6}$$

such that

$$c_t + a_{t+1} = (1-\tau_t) w_t \epsilon_t n_t + (1 + r_t(1-\tau_t))a + T_t.$$

Definition 2.2.1. ***Sequential Competitive Equilibrium (SCE), given a Sequence of Taxes and Transfers***
Given G, an initial distribution $\mu_0(\cdot)$, and a sequence of income taxes and transfers $\{\tau_t, T_t\}_{t=0}^{\infty}$, an SCE is a sequence of prices $\{w_t, r_t\}_{t=0}^{\infty}$, a sequence of allocations $\{c_t, n_t, a_{t+1}, K_t, N_t\}_{t=0}^{\infty}$, a sequence of value functions $\{v_t(\cdot)\}_{t=0}^{\infty}$, and a sequence of distributions over the state space $\{\mu_t(\cdot)\}_{t=1}^{\infty}$, such that for all t

(i) Given $\{\tau_t, T_t\}_{t=0}^{\infty}$ and $\{w_t, r_t\}_{t=0}^{\infty}$, the decision rules $a_{t+1}(a,\epsilon)$ and $n_t(a,\epsilon)$ solve the household problem in (6), and $v_t(a,\epsilon)$ is the associated value function.

(ii) The representative agent firm engages in competitive pricing:

$$w_t = (1-\theta) z \left(\frac{K_t}{N_t}\right)^{\theta} \tag{7}$$

$$r_t = \theta z \left(\frac{K_t}{N_t}\right)^{\theta - 1} - \delta. \tag{8}$$

(iii) The factor markets clear:

$$K_t = \int a \, \mu_t(\mathbf{d}(a \times \epsilon)) \tag{9}$$

$$N_t = \int \epsilon \, n_t(a,\epsilon) \, \mu_t(\mathbf{d}(a \times \epsilon)) \tag{10}$$

(iv) The government budget constraint (5) is satisfied.

(v) Let $\mathcal{B}(A \times E)$ denote the Borel σ-algebra on $A \times E$. For any $B \in \mathcal{B}(A \times E)$, the sequence of distributions over individual $\{\mu_t(\cdot)\}_{t=1}^{\infty}$ satisfies

$$\mu_{t+1}(B) = \int_{\{(a,\epsilon) | (a_{t+1}(a,\epsilon), \epsilon_{t+1}) \in B\}} \sum_{\epsilon_{t+1}} \pi_{\epsilon,\epsilon_{t+1}} \mu_t(\mathbf{d}(a \times \epsilon)). \tag{11}$$

In contrast, to handle problems without commitment, it is convenient to present the household dynamic problems in a recursive manner. In addition to the individual state variables a and ϵ, there are two aggregate state variables, including the distribution of households $\mu(a, \epsilon)$ over a and ϵ and income tax τ.[11] A variable with a prime symbol denotes its value in the next period.

Let $v(a, \epsilon; \mu, \tau)$ denote the value of households associated with a state of $(a, \epsilon; \mu, \tau)$. They solve

$$v(a, \epsilon; \mu, \tau) = \max_{c>0,\, a' \geq \underline{a},\, 0 \leq n \leq 1} \left[\frac{c^{1-\sigma}}{1-\sigma} + B \frac{(1-n)^{1-1/\chi}}{1-1/\chi} + \beta \sum_{\epsilon'} \pi_{\epsilon,\epsilon'} v(a', \epsilon'; \mu', \tau') \right] \tag{12}$$

such that

$$c + a' = (1-\tau)\, w(\mu)\, \epsilon\, n + (1 + r(\mu)(1-\tau))\, a + T$$
$$\tau' = \Psi(\mu, \tau)$$
$$\mu' = \Gamma(\mu, \tau, \tau') = \Gamma(\mu, \tau, \Psi(\mu, \tau))$$

where $\underline{a} \leq 0$ is a borrowing limit, $\tau' = \Psi(\mu, \tau)$ is the perceived law of motion of taxes, and $\mu' = \Gamma(\mu, \tau, \tau')$ is the law of motion for the distribution over households. Note that households here solve the above problem given an exogenous tax policy function $\tau' = \Psi(\mu, \tau)$.

Definition 2.2.2. ***Recursive Competitive Equilibrium (RCE), given a Law of Motion for Tax.*** Given G and $\Psi(\mu, \tau)$, an RCE is a set of prices $\{w(\mu), r(\mu)\}$, a set of decision rules for households $g^a(a, \epsilon; \mu, \tau)$ and $g^n(a, \epsilon; \mu, \tau)$, a value function $v(a, \epsilon; \mu, \tau)$, a distribution of households $\mu(a, \epsilon)$ over the state space, and the law of motion for the distribution of households $\Gamma(\mu, \tau, \Psi(\mu, \tau))$ such that

(i) Given $\{w(\mu), r(\mu)\}$, the decision rules $a' = g^a(a, \epsilon; \mu, \tau)$ and $n = g^n(a, \epsilon; \mu, \tau)$ solve the household problem in (12), and $v(a, \epsilon; \mu, \tau)$ is the associated value function.

(ii) The representative agent firm engages in competitive pricing:

$$w(\mu) = (1-\theta) z \left(\frac{K}{N} \right)^{\theta} \tag{13}$$
$$r(\mu) = \theta z \left(\frac{K}{N} \right)^{\theta - 1} - \delta. \tag{14}$$

[11] Note that after a policy reform, either τ or $T = T_1$ is a state variable because once one of them chosen, the other is fixed in the balanced government budget, $G + T = \tau[rK + wN]$. Here, I choose τ as a state variable because this choice provides a better representation when GEE is applied.

(iii) The factor markets clear:

$$K = \int a\,\mu(\mathbf{d}(a \times \epsilon)) \tag{15}$$

$$N = \int \epsilon\, g^n(a, \epsilon; \mu, \tau)\,\mu(\mathbf{d}(a \times \epsilon)) \tag{16}$$

(iv) The government budget constraint (5) is satisfied.

(v) The law of motion for the distribution of households $\mu' = \Gamma(\mu, \tau, \Psi(\mu, \tau))$ is consistent with individual decision rules and the stochastic process of ϵ.

2.3 Competitive Equilibrium, Endogenous Policy

In this section, I define competitive equilibria where the income tax and transfers are endogenously determined. I model the tax-transfer choice in two ways: the optimal income tax and transfers with commitment (Ramsey problem) and the optimal income tax and transfers without commitment (time-consistent case).

In both cases, the timing of this policy reform is as follows. Until period -1, the economy is in a steady state with the progressive transfer system. At the beginning of period 0, the government unexpectedly announces this policy reform that replaces the progressive transfers with lump-sum transfers. I begin with the Ramsey problem.

Definition 2.3.1. ***The Ramsey Problem:***
An SEC with the Optimal Income Tax and transfers with Commitment
Given μ_0, the government chooses $\{\tau_t, T_t\}_{t=0}^{\infty}$ such that

$$\{\tau_t, T_t\}_{t=0}^{\infty} = \underset{\{\tilde{\tau}_t, \tilde{T}_t\}_{t=0}^{\infty}}{\operatorname{argmax}} \int E_0 \sum_{\hat{t}=0}^{\infty} \beta^{\hat{t}} u\big(c_{\hat{t}}^*(a, \epsilon | \{\tilde{\tau}_{\hat{t}}, \tilde{T}_{\hat{t}}\}_{t=0}^{\infty}), 1 - n_{\hat{t}}^*(a, \epsilon | \{\tilde{\tau}_{\hat{t}}, \tilde{T}_{\hat{t}}\}_{t=0}^{\infty})\big)\mu_0(\mathbf{d}(a \times \epsilon))$$

where $\big(c_{\hat{t}}^*(a, \epsilon | \{\tau_t, T_t\}_{t=0}^{\infty}), n_{\hat{t}}^*(a, \epsilon | \{\tau_t, T_t\}_{t=0}^{\infty})\big)$ is an allocation in Definition (2.2.1) in period $\hat{t}$, given $\{\tilde{\tau}_t, \tilde{T}_{\hat{t}}\}_{t=0}^{\infty}$.

Note that the consumption and labor decisions at time t, (c_t^*, n_t^*), are affected not only by the policy in that period t but also by the entire sequence of income taxes and transfers. Therefore, the decisions at time t are influenced by taxes and transfers in periods after t, which can lead to the time-inconsistency issue.

For the case without commitment, I have employed the definition in Klein et al. (2008).

Definition 2.3.2. ***An RCE with the Optimal Income Tax and transfers without Commitment***

(i) A set of functions $\{w(\cdot), r(\cdot), g^a(\cdot), g^n(\cdot), v(\cdot), \Gamma(\cdot)\}$ satisfy Definition (2.2.2).

(ii) For each (μ, τ), the government chooses $\tau^{WO}(\mu, \tau)$ such that

$$\tau^{WO}(\mu, \tau) = \underset{\tilde{\tau}'}{\text{argmax}} \int \hat{V}(a, \epsilon; \mu, \tau, \tilde{\tau}')\mu(\mathbf{d}(a \times \epsilon)) \tag{17}$$

where

$$\hat{V}(a, \epsilon; \mu, \tau, \tilde{\tau}') = \max_{c>0,\ a' \geq \underline{a},\ 0 \leq n \leq 1} \left[\frac{c^{1-\sigma}}{1-\sigma} + B\,\frac{(1-n)^{1-1/\chi}}{1-1/\chi} + \beta \sum_{\epsilon'} \pi_{\epsilon,\epsilon'} v(a', \epsilon'; \mu', \tilde{\tau}') \right]$$

such that

$$c + a' = (1-\tau)\, w(\mu)\, \epsilon\, n + (1 + r(\mu)(1-\tau))\, a + T$$

$$\tau' = \tilde{\tau}',\ \text{and thereafter } \tau'' = \Psi(\mu', \tau' = \tilde{\tau}') \tag{18}$$

$$\mu' = \Gamma(\mu, \tau, \tilde{\tau}),\ \text{and thereafter } \mu'' = \Gamma(\mu', \tilde{\tau}, \tau'' = \Psi(\mu', \tau' = \tilde{\tau}')) \tag{19}$$

(iii) $a' = \hat{g}_a(a, \epsilon; \mu, \tilde{\tau} : \tilde{\tau}')$ and $n = \hat{g}_n(a, \epsilon; \mu, \tilde{\tau} : \tilde{\tau}')$ solve (17) at prices that clear markets and satisfy the government budget constraint, and Γ is consistent with individual decisions and the stochastic process of ϵ.

(iv) For each (μ, τ), the policy outcome function satisfies $\Psi(\mu, \tau) = \tau^{WO}(\mu, \tau)$.

Note that the government's solution to the problem is consistent with the sub-perfect Nash equilibrium obtained with the one-shot deviation principle. In the economy with the optimal income tax without commitment, the government implements a time-consistent optimal policy as in Klein et al. (2008); Corbae et al. (2009): a tax rate that is sequentially chosen only for the next period while maximizing its utilitarian welfare under this commitment constraint. The government cannot commit to the future tax rate from the period after the next period. Thus, if the chosen tax rate $\tilde{\tau}'$ deviates from the equilibrium tax policy function $\Psi(\cdot)$, future tax rates will follow the equilibrium tax policy function $\Psi(\cdot)$ because of the lack of commitment. (18) presents such a commitment constraint. The law of motion for the distribution of households $\Gamma(\cdot)$ has to capture all the changes in the evolution of distributions caused by the deviation of the income tax from the equilibrium tax function, as shown in (19). In equilibrium, for each aggregate state (μ, τ), the government's choice of the tax rate, $\tau^{WO}(\mu, \tau)$, should be equal to the equilibrium tax function $\psi(\mu, \tau)$, as presented in (iv).

3 Characterization of the Equilibria

Despite the definitions that demonstrate the government's decision-making process for taxes and transfers, the underlying economic trade-offs behind these decisions can be challenging to observe. This section employs the GEE approach, introduced by Klein et al. (2008), to characterize the equilibria, thereby illuminating the economic trade-offs considered by the government in making decisions on tax-transfer policies.

Specifically, in this section, I aim to deliver three points. First, in response to changes in taxes and transfers, two types of economic forces exist that reshape individual disposable income:

$$(1-\tau)w\epsilon n + (1 + r(1-\tau))a + T. \tag{20}$$

The first type of force is income redistribution via changes in τ and T. The second type of force is pecuniary externalities via changes in w and r. Second, the two types of economic forces differ across individuals, and their welfare impacts are opposite according to individuals' income level. Finally, the government makes policy decisions on taxes and transfers by striking a balance between the two types of economic forces, and government commitment makes differences in their balancing across the time dimension.

I will first analyze the case without commitment and then proceed to the case with commitment.

3.1 The Case without Commitment

The GEE approach provides insight into the economic forces driving the policymaker's decision through its FOC. This condition can be derived by utilizing the Benveniste-Scheinkman condition, also known as the envelope condition, to eliminate terms related to the partial derivative of the value function. To determine the FOC of the government, I take the partial derivative of the government's value, represented by $\hat{V}$, with respect to the next period's income tax rate, represented by $\tilde{\tau}'$, in its vicinity of the equilibrium value τ':

$$\begin{aligned}
0 =& \frac{d}{d\tilde{\tau}'}\Big|_{\tilde{\tau}'=\tau'} \int \hat{V}(a,\epsilon;\mu,\tau,\tilde{\tau}')\mu(d(a\times\epsilon)) \\
=& \int \frac{d}{d\tilde{\tau}'}\Big|_{\tilde{\tau}'=\tau'} \\
& \Big[u((1-\tau)\, w(\mu)\epsilon\, \tilde{g}^n(a,\epsilon;\mu,\tau,\tilde{\tau}') + (1+r(\mu)(1-\tau))a + T - \tilde{g}^a(a,\epsilon;\mu,\tau,\tilde{\tau}'), 1 - \tilde{g}^n(a,\epsilon;\mu,\tau,\tilde{\tau}')) \\
& + \beta \sum_{\epsilon'} \pi_{\epsilon,\epsilon'} v(\tilde{g}^a(a,\epsilon;\mu,\tau,\tilde{\tau}'),\epsilon';\mu' = \Gamma(\mu,\tau,\tilde{\tau}'),\tilde{\tau}')\Big]\mu(\mathbf{d}(a\times\epsilon))
\end{aligned} \tag{21}$$

Note that the tilde over g^a and g^n means that the deviation of $\tilde{\tau}'$ from its equilibrium value τ' makes the decision rules for assets and labor supply different from those in equilibrium.

An obscure part in computing the FOC (21) is the derivative of v with respect to μ'. Let m_q denote the $q-th$ moment of μ. I assume that $Q \in \mathbb{N}$ exists such that $\{m_q\}_{q=1}^{Q}$ is a sufficient statistic of μ. This assumption allows me to replace μ with $\{m_q\}_{q=1}^{Q}$ in the value function. Then, the FOC (21) is given by:

$$0 = \int \Bigg[u_c(c, 1-n) \cdot \Big((1-\tau) w(m_1) \epsilon \frac{\partial \tilde{g}^n(a, \epsilon; \{m_q\}_{q=1}^{Q}, \tau, \tau')}{\partial \tau'} - \frac{\partial \tilde{g}^a(a, \epsilon; \{m_q\}_{q=1}^{Q}, \tau, \tau')}{\partial \tau'} \Big)$$
$$- u_n(c, 1-n) \cdot \frac{\partial \tilde{g}^n(a, \epsilon; \{m_q\}_{q=1}^{Q}, \tau, \tau')}{\partial \tau'}$$
$$+ \beta \sum_{\epsilon'} \pi_{\epsilon,\epsilon'} \Big\{ \frac{\partial v(a', \epsilon', \{m'_q\}_{q=1}^{Q}, \tau')}{\partial a'} \cdot \frac{\partial \tilde{g}^a(a, \epsilon; \{m_q\}_{q=1}^{Q}, \tau, \tau')}{\partial \tau'}$$
$$+ \sum_{q=1}^{Q} \Big(\frac{\partial v(a', \epsilon', \{m'_q\}_{q=1}^{Q}, \tau')}{\partial m'_q} \cdot \frac{dm'_q}{d\tau'} \Big) + \frac{\partial v(a', \epsilon', \{m'_q\}_{q=1}^{Q}, \tau')}{\partial \tau'} \Big\} \Bigg] \mu(\mathbf{d}(a \times \epsilon)). \tag{22}$$

where $u_c(c)$ is the derivative of u in c and m_1 is the first moment of μ. Note that the first moment is sufficient to determine w and r. I will eliminate the derivative terms of the value $\frac{\partial v}{\partial a'}$, $\frac{\partial v}{\partial m'_q}$, and $\frac{\partial v}{\partial \tau'}$ by using the Benveniste-Scheinkman condition and then interpret the economic logic behind the equations. First, $\frac{\partial v}{\partial a'}$ is given by:

$$\frac{\partial v(a, \epsilon; \{m_q\}_{q=1}^{Q}, \tau)}{\partial a} = u_c(c, 1-n)(1 + r(m_1)(1-\tau)). \tag{23}$$

Similarly, with $\frac{\partial T}{\partial \tau} = rK + wN$, $\frac{\partial v}{\partial \tau}$ is given by:

$$\frac{\partial v(a, \epsilon; \{m_q\}_{q=1}^{Q}, \tau)}{\partial \tau} = u_c(c, 1-n) \Big(w(m_1)(N - \epsilon \cdot g^n(a, \epsilon; \{m_q\}_{q=1}^{Q}, \tau) + r(m_1)(K - a) \Big)$$
$$+ \omega(a, \epsilon, \{m_q\}_{q=1}^{Q}, \tau) \cdot \frac{\partial g^a(a, \epsilon; \{m_q\}_{q=1}^{Q}, \tau)}{\partial \tau}$$
$$+ \zeta(a, \epsilon; \{m_q\}_{q=1}^{Q}, \tau) \cdot \frac{\partial g^n(a, \epsilon; \{m_q\}_{q=1}^{Q}, \tau)}{\partial \tau}$$
$$+ \beta \sum_{\epsilon'} \pi_{\epsilon,\epsilon'} \Big\{ \sum_{q=1}^{Q} \Big(\frac{\partial v(a', \epsilon'; \tau', \{m'_q\}_{q=1}^{Q})}{\partial m'_q} \cdot \frac{dm'_q}{d\tau} \Big)$$
$$+ \frac{\partial v(a', \epsilon'; \tau', \{m'_q\}_{q=1}^{Q})}{\partial \tau'} \cdot \frac{\partial \Psi(\tau, \mu)}{\partial \tau} \Big\}$$
$$\text{where } \zeta(a, \epsilon; \{m_q\}_{q=1}^{Q}, \tau) = -\big(u_c(c, 1-n) \cdot (1-\tau) w(m_1) \epsilon + u_n(c, 1-n) \big)$$
$$\omega(a, \epsilon; \{m_q\}_{q=1}^{Q}, \tau) = -u_c(c, 1-n) + \beta(1 + r(m'_1)(1-\tau')) \sum_{\epsilon'} \pi_{\epsilon,\epsilon'} u_c(c', 1-n'). \tag{24}$$

ξ and ω represent wedges in the FOC for optimal leisure choice and the Euler equation for consumption, respectively.

Note that $u_c(c, 1-n)(w(N - \epsilon \cdot g^n) + r(K - a))$ implies an individual welfare change driven by income redistribution via changes in taxes and transfers. This term is one of the key economic forces in characterizing the MPE. Let χ denote this term in the subsequent discussion:

$$\chi(a, \epsilon; \{m_q\}_{q=1}^Q, \tau) = u_c(c, 1-n)\Big(w(m_1)(N - \epsilon \cdot g^n(a, \epsilon; \{m_q\}_{q=1}^Q, \tau) + r(m_1)(K - a)\Big). \tag{25}$$

χ represents *income redistribution* via taxes and transfers because it measures the difference between after- and before-tax income caused by a change in a tax rate τ, the extent of which varies across individuals. Individuals with a lower pre-tax income will have more income after receiving transfers, while those with a higher pre-tax income will have less due to heaver taxes. Thus, if an individual's effective labor ϵg^n and asset holdings a are below the average values (N and K), χ becomes positive for that individual. In contrast, if an individual's effective labor and assets are above average, χ becomes negative.

The next step is to eliminate $\frac{\partial v(a', \epsilon'; \{m'_q\}_{q=1}^Q, \tau')}{\partial m'_q}$. It is difficult to determine the required value of Q to obtain sufficient statistics for μ. For the purposes of this analysis, I assume that $Q = 1$, which means that $m_1 = K$ is sufficient for capturing the evolution of the distribution, as in Krusell and Smith (1998). An alternative interpretation of this assumption is that the government only considers changes in future prices and not higher moments of the future distribution when determining income taxes and transfers. This assumption enables a further characterization of the MPE.[12]

With the Benveniste-Scheinkman condition, $\frac{\partial v}{\partial K}$ is given by:

$$\begin{aligned}\frac{\partial v(a, \epsilon; K, \tau)}{\partial K} =& u_c(c, 1-n)\Big((1-\tau)(f_{NK}(K, N)\epsilon \cdot g^n(a, \epsilon; K, \tau) + f_{KK}(K, N)a) + \frac{\partial T}{\partial K}\Big) \\ &+ \zeta(a, \epsilon; K, \tau) \cdot \frac{g^n(a, \epsilon; K, \tau)}{\partial K} + \omega(a, \epsilon; K, \tau) \cdot \frac{g^a(a, \epsilon; K, \tau)}{\partial K} \\ &+ \beta \sum_{\epsilon'} \pi_{\epsilon, \epsilon'} \Big\{ \frac{\partial v(a', \epsilon'_j; K', \tau')}{\partial K'} \cdot \frac{\partial \Gamma(K, \tau, \tau')}{\partial K} + \frac{\partial v(a', \epsilon'; K', \tau')}{\partial \tau'} \cdot \frac{\partial \Psi(K, \tau)}{\partial K} \Big\}. \end{aligned} \tag{26}$$

Note that $u_c(c, 1-n)\big((1-\tau)(f_{NK}(K, N)\epsilon + f_{KK}(K, N)a) + \frac{\partial T}{\partial K}\big)$ implies an individual welfare change driven by variations in the factor composition between capital and labor income following an increase in the current tax τ. As discussed in Davila et al. (2012), this effect differs across individuals and depends on the composition of their income. To clarify how this effect is linked to the factor composition of individual income, I proceed with further steps following Davila et al. (2012). Because f is homogeneous of degree 1, $f_{KK}(K, N)K + fKN(K, N)N = 0$. In addition,

[12] Another supportive reason is that under the assumption of $Q = 1$, the numerical outcomes are accurate. The numerical errors, measured by accuracy statistics proposed by Den Haan (2010), are quite small. Table 7 in Appendix A reports the details.

because $T = \tau(rK + wN) - G = \tau(f_K K + f_N N) - G$, $\frac{\partial T}{\partial K} = f_K(K, N)\tau$. Then, with these conditions, $\frac{\partial v}{\partial K}$ is given by:

$$\begin{aligned}\frac{\partial v(a,\epsilon;K,\tau)}{\partial K} &= u_c(c,1-n)\big((1-\tau)(-\frac{\epsilon \cdot g^n(a,\epsilon;K,\tau)}{N} + \frac{a}{K})f_{KK}(K,N)K + f_K(K,N)\tau\big) \\ &+ \zeta(a,\epsilon;K,\tau) \cdot \frac{g^n(a,\epsilon;K,\tau)}{\partial K} + \omega(a,\epsilon;K,\tau) \cdot \frac{g^a(a,\epsilon;K,\tau)}{\partial K} \\ &+ \beta \sum_{\epsilon'} \pi_{\epsilon,\epsilon'} \Big\{ \frac{\partial v(a',\epsilon'_j;K',\tau')}{\partial K'} \cdot \frac{\partial \Gamma(K,\tau,\tau')}{\partial K} + \frac{\partial v(a',\epsilon';K',\tau')}{\partial \tau'} \cdot \frac{\partial \Psi(K,\tau)}{\partial K} \Big\} \end{aligned} \tag{27}$$

The first term is another important economic force in characterizing the MPE. For further discussion, I refer to this term as ϕ:

$$\phi(a,\epsilon;K,\tau) = u_c(c,1-n)\big((1-\tau)(-\frac{\epsilon \cdot g^n(a,\epsilon;K,\tau)}{N} + \frac{a}{K})f_{KK}(K,N)K + f_K(K,N)\tau\big). \tag{28}$$

ϕ represents *pecuniary externalities* arising from changes in the factor composition of income. It measures the welfare change following a shift of the factor composition of income between r and w, which is driven by general equilibrium effects after an increase in K. ϕ is considered a type of externality because individuals take a sequence of factor prices as given in competitive equilibrium and do not consider the possibility that these prices may vary due to endogenous government policies and their impacts on welfare. Another feature of ϕ is that the extent of pecuniary externalities differs across individuals based on their factor composition of income, as noted by Davila et al. (2012). Note that because $f_{KK}(K,N)K < 0$, $f_K(K,N)\tau > 0$, the sign of this term highly depends on the value of $(-\frac{\epsilon \cdot g^n}{N} + \frac{a}{K})$. For example, suppose that there is an increase in K. In this case, if $\frac{\epsilon \cdot g^n}{N}$ is substantially larger than $\frac{a}{K}$, indicating that the factor income is biased toward labor (the factor composition of income of the low-income group), then ϕ is positive because w increases while r is reduced in general equilibrium. In contrast, high-income individuals, whose factor income is biased more toward capital, are more likely to experience a loss in welfare.

Next, I substitute (23), (24), and (27) into the derivative of the government value in $\tilde{\tau}'$, $\frac{\partial \hat{v}}{\partial \tilde{\tau}'}$, in the FOC (22) and eliminate the partial derivatives of the future values by substituting them out. Additionally, I simplify the notation by employing sequential terms. I refer to $y_{i,t}$ as the variable of y for individual i in period t.

Then, $\frac{\partial \hat{V}_{i,t}}{\partial \tau_{t+1}}$ is given by:

$$\frac{\partial \hat{V}_{i,t}}{\partial \tau_{t+1}} = E_{i,t}\left[\sum_{s=1}^{\infty} \beta^s \cdot \Big(\underbrace{\phi_{i,t+s}}_{\text{Pecuniary Externalities}} \cdot \underbrace{\frac{\Delta K_{t+s}}{\Delta \tau_{t+1}}}_{\text{Efficiency Effect}} + \underbrace{\chi_{i,t+s}}_{\text{Income Redistribution}} \cdot \underbrace{\frac{\Delta \tau_{t+s}}{\Delta \tau_{t+1}}}_{\text{Policy Scale Effect}} \Big)\right] \tag{29}$$

where $E_{i,t}$ is the conditional expectation for individual i at time t; $\frac{\Delta K_{t+s}}{\Delta \tau_{t+1}}$ is the total variation of aggregate capital K in period $t+s$ caused by a tax rate change at $t+1$; and $\frac{\Delta\tau_{t+s}}{\Delta\tau_{t+1}}$ is the total variation in the taxes in period $t+s$ caused by a tax rate change at $t+1$.[13] A notable feature is that the wedges related to individual decisions disappear above. More precisely, the product terms between these wedges and the derivative of individual decisions, such as $\omega \cdot \frac{\partial g^a}{\partial K}$ and $\zeta \cdot \frac{\partial g^n}{\partial K}$ observed in (24) and (27), are offset through the envelope theorem. This canceling out implies that the government does not consider the direct welfare impacts involved with individual decisions. The government recognizes that given a set of policies, each individual makes efficient decisions on consumption, saving, and labor supply in competitive equilibrium; therefore, there is no room for varying individual welfare via this channel.

Instead, the government considers the two types of economic forces that are heterogeneous across individuals—pecuniary externalities and income redistribution—along the transition path. Pecuniary externalities work through changes in aggregate capital. For example, if the government increases τ_{t+1}, efficiency is reduced because aggregate capital K falls below the initial level for a period of time ($\frac{\Delta K_{t+s}}{\Delta\tau_{t+1}} < 0$). These changes in K along the transition path induce variations in the factor composition of income between r and w over time. In this case, while w decreases, r increases over the transition path. Meanwhile, income redistribution works through changes in taxes and transfers. For instance, if the economy converges to the long-run equilibrium in a mean-reverting way, its tax rate τ remains above the initial level for a time once τ_{t+1} increases ($\frac{\Delta\tau_{t+s}}{\Delta\tau_{t+1}} > 0$). These increases in τ imply more transfers along the transition path, influencing income redistribution.

Finally, I substitute (29) into FOC (22). Then, the government-optimal condition is given by:

$$\underbrace{-\int E_{i,t}\left[\sum_{s=1}^{\infty}\beta^s \cdot \phi_{i,t+s} \cdot \frac{\Delta K_{t+s}}{\Delta\tau_{t+1}}\right]\mu_t(\mathbf{d}(a_{i,t}\times\epsilon_{i,t}))}_{\text{Aggregate Pecuniary Externalities}} = \underbrace{\int E_{i,t}\left[\sum_{s=1}^{\infty}\beta^s \cdot \chi_{i,t+s} \cdot \frac{\Delta\tau_{t+s}}{\Delta\tau_{t+1}}\right]\mu_t(\mathbf{d}(a_{i,t}\times\epsilon_{i,t}))}_{\text{Aggregate Income Redistribution}} \tag{30}$$

where

$$\phi_{i,\tilde{t}} = u_c(c_{i,\tilde{t}}, 1-n_{i,\tilde{t}})\big((1-\tau_{\tilde{t}})(-\frac{\epsilon \cdot g^n_{i,\tilde{t}}}{N_{\tilde{t}}} + \frac{a_{i,\tilde{t}}}{K_{\tilde{t}}})f_{KK}(K_{\tilde{t}})K_{\tilde{t}} + f_K(K_{\tilde{t}})\tau_{\tilde{t}}\big)$$
$$\chi_{i,\tilde{t}} = u_c(c_{\tilde{t}}, 1-n_{\tilde{t}})\Big(w_{\tilde{t}}(N_{\tilde{t}} - \epsilon_{\tilde{t}} \cdot g^n_{i,\tilde{t}} + r_{\tilde{t}}(K_{\tilde{t}} - a_{i,\tilde{t}})\Big).$$

The above government-optimal condition implies that for each period t, when deciding τ_{t+1}, the government without commitment strikes a balance between the two types of economic forces at the aggregate level. For further discussion, I refer to the left-hand side as *aggregate pecuniary externalities* because the absolute value of this term indicates the sum of the present discounted

[13]The precise definitions are presented in Appendix C.

value of pecuniary externalities caused by variations in the factor composition of income over all individuals. Analogously, I refer to the right-hand side as *aggregate income redistribution* because this term means the sum of the present discounted value of income redistribution caused by changes in taxes and transfers over all individuals.

The above optimal condition (30) reveals what the government takes into account in its policy decisions. First, the government places greater weight on the interests of the low-income group because the two types of economic forces ϕ and χ are weighted by the marginal utility of consumption. Additionally, the distribution of consumption is right-skewed, leading the government to further consider the interests of the low-income group. This government attention to the low-income group allows me to regard its policy decisions as a cost-benefit analysis of the low-income group for taxes and transfers. Second, the low-income group experience opposite welfare changes from the two types of economic forces, as mentioned previously with (25) and (28) . Suppose that the government decides to increase τ_{t+1}. While income redistribution χ improves welfare for the low-income group by increasing their after-tax income via transfers, pecuniary externalities ϕ are negative for the low-income group—whose income is more biased toward labor—because the tax change reduces w and increases r in general equilibrium. Third, for the low-income group, pecuniary externalities ϕ are less than or equal to $f_k(K,N)\tau$ ($\phi \geq f_k(K,N)\tau$) and income redistribution is non negative ($\chi \geq 0$). Finally, as incomes for the low-income group become more equally distributed $(a, \epsilon \cdot g^n) \to (K, N)$, pecuniary externalities ϕ converge to $-u'(c)f_K(K,N)\tau$, and their income redistribution χ converges to 0 from a positive value. These findings leads to the following proposition.

Proposition 1. *For each period t, the government without commitment makes a policy decision as follows:*

1. *When aggregate pecuniary externalities are **greater** than aggregate income redistribution, the government **reduces** τ_{t+1}.*
2. *When aggregate pecuniary externalities are **less** than aggregate income redistribution, the government **raises** τ_{t+1}.*
3. *As the income of the low-income group approaches the average, the government **reduces** τ_{t+1}.*

The first and second statements posit that pecuniary externalities can be seen as the low-income group's marginal cost of increasing τ_{t+1} and income redistribution can be regarded as their marginal benefit. Accordingly, the government decides to increase (decrease) its tax rate when the marginal cost (i.e., aggregate pecuniary externalities) is greater (less) than the marginal benefit (i.e., aggregate income redistribution). The third statement is closely related to how χ and ϕ

change as the income of the low-income group approaches the average. Specifically, the marginal benefit of increasing τ_{t+1} for the low-income group, ϕ, converges to 0 from a positive value as their income approaches the average. Additionally, as their income approaches the average, in their marginal cost of raising τ_{t+1}, χ, the impact of $-u'(c)f_k(K,N)\tau < 0$ becomes more pronounced. Therefore, the most effective way to minimize losses from χ is to reduce τ when the income of the low-income group approaches the average. This insight is distilled into the third statement.

3.2 Comparison with Constrained Efficiency

Another interesting investigation for the optimal conditions (30) is to compare this to the planner's optimal condition in Davila et al. (2012) because pecuniary externalities are thoroughly investigated in their study. The planner's optimal condition in Davila et al. (2012) is given by:

$$\omega_{i,t} + \beta \int E_{i,t}\big[\phi_{t+1}\big]\mu(\mathbf{d}(a \times \epsilon)) = 0 \tag{31}$$

where

$$\omega_{i,t} = -u'(c_{i,t}) + \beta(1 + r(K_{t+1})) \sum_{\epsilon'} \pi_{\epsilon,\epsilon'} u'(c_{t+1}) \tag{32}$$

$$\phi_{i,t} = u'(c_{i,t})\big((-\frac{\epsilon_{i,t}}{L_t} + \frac{a_{i,t}}{K_t})f_{KK}(K_t)K_t\big) \tag{33}$$

In contrast with the government's optimal conditions (30) in this paper, the consumption Euler equation part in Davila et al. (2012) does not need to be null. What matters for the social planner is to satisfy this optimal condition considering the pecuniary externality and distortions embedded in the consumption Euler equation. This distinction arises because of different assumptions between the two economies. The Davila, Hong, Krusell and Ríos-Rull's (2012) economy is centralized: the social planner can manipulate individual saving decisions while preserving the constraints caused by incomplete-markets and uninsurable idiosyncratic income risk. This centralized economy assumption makes the consumption Euler equation non-zero. However, the economy in my paper is decentralized. Although the government exists and endogenously determines taxes, individuals optimally choose consumption and saving; therefore, the government has no room for improvements in welfare via this channel.

3.3 Comparison with the Case with Commitment

Regarding the case with commitment (the Ramsey problem), I assume that the government has already implemented an optimal sequence of taxes and transfers and now considers a change in its tax rate at time $t + 1$. Then, repeating the previous calculations enables me to obtain the government's optimal condition as follows:

$$\underbrace{-\int E_{i,0}\left[\sum_{s=1}^{\infty}\beta^s\cdot\phi_{i,s}\cdot\frac{\Delta K_s}{\Delta\tau_{t+1}}\right]\mu_0(\mathbf{d}(a_{i,0}\times\epsilon_{i,0}))}_{\text{Aggregate Pecuniary Externalities at } t=0}=\underbrace{\int E_{i,0}\left[\sum_{s=1}^{\infty}\beta^s\cdot\chi_{i,s}\cdot\frac{\Delta\tau_s}{\Delta\tau_{t+1}}\right]\mu_0(\mathbf{d}(a_{i,0}\times\epsilon_{i,0}))}_{\text{Aggregate Income Redistribution at } t=0}. \tag{34}$$

In comparison with (30), the government's optimal condition in (34) highlights the role of commitment technologies. Although the government always balances the two types of economic forces in both cases, commitment makes a difference in how the balance is struck. With commitment, the government (the Ramsey planner) balances the two types of economic forces by taking conditional expectations at time 0. In contrast, as (30) shows, a lack of commitment causes the government to take a conditional expectation in each period. Because of this difference, commitment leads the government to consider the effect of taxes and transfers in a backward manner. The optimal condition in (34) suggests that the government considers how a tax change in period $t+1$ affects the economy not only in subsequent periods but also in periods before $t+1$. In contrast, as (30) shows, a lack of commitment causes the government to consider the effects of its policy only in periods after the implementation, without taking into account its influence in periods before it takes effect. These findings suggest that the government with commitment makes policy decisions that are optimal at time 0 but not necessarily desirable when evaluated in each period, resulting in a time-inconsistent outcome.

The comparison above clearly illustrates the qualitative differences in the government's policy decisions with and without commitment. However, it does not provide quantitative assessments of the magnitude of these differences. Additionally, it does not explain how individuals' decisions are shaped by the contrasting balancing methods that arise due to the presence or absence of commitment. In a later section, I will conduct a quantitative analysis to address these questions. Before doing so, in the following section, I propose a numerical solution method that allows me to conduct this quantitative analysis.

4 Numerical Solution Algorithm

Here, I focus on conveying the key ideas of the numerical solution algorithm. Appendix A demonstrates each step of the algorithm with details, including outcomes related to its efficiency and accuracy.

Although the characterization of the MPE in the previous section helps us better understand the government's decisions on policies, it is not very useful in numerically computing the equilibrium because of its sequential feature. Basically, solving the model entails a substantial computational burden. The law of motion for the distribution of households $\Gamma(\cdot)$ has to be consistent with individual decisions. Additionally, because the labor supply is endogenous with wealth effects, the two factor markets—K and N—must clear. Furthermore, perhaps the most challenging part is finding the equilibrium policy function $\Psi(\cdot)$ that must be consistent with individual decisions and the law of motion for the distribution of households. That is, three equilibrium objects—specifically, individual decisions, $g^n(\cdot)$ and $g^a(\cdot)$, the law of motion for the distribution, $\Gamma(\cdot)$, and the policy function, $\Psi(\cdot)$—interact and have to be consistent with one another in an MPE.

I address the above computational issues by taking ideas from the backward induction method of Reiter (2010). This study introduced a non-simulation-based solution method to solve an incomplete-markets economy with aggregate uncertainty. As in Krusell and Smith's (1998), the Reiter's (2010) approach also reduces the dimension of distributions in the law of motion $\Gamma(\cdot)$ to some finite moments of the distribution, and they are defined across the aggregate finite grid points. However, the way of finding $\Gamma(\cdot)$ differs substantially between the two methods. In Krusell and Smith (1998), their algorithm repeatedly simulates the model economy through the inner and outer loops. In the inner loops, the value is solved given a perceived law of motion for the distribution of households, and the law of motion is updated after a simulation in the outer loop. This procedure is repeated until the perceived law of motion is equal to the updated law of motion.

By contrast, the backward induction method of Reiter (2010) does not simulate the economy to update the law of motion for the distribution of households $\Gamma(\cdot)$; rather, this is updated while solving for the value given a set of proxy distributions across the aggregate finite grid points. Given a proxy distribution, finding the law of motion for the distribution of households $\Gamma(\cdot)$ is feasible by using the moment-consistent conditions. For example, individual decision rules for assets allow me to obtain information (e.g., the mean or variance) on the aggregate capital in the next period. A simulation step is followed not to update the law of motion for the distribution of households $\Gamma(\cdot)$ but to update a set of proxy distributions across the finite nodes in the aggregate state. Simulations are required much less in Reiter (2010) than in Krusell and Smith (1998), which improves the computational efficiency of the backward induction method. Additionally, with these proxy distributions, the backward induction method allows me to approximate not only the aggregate law of motion for the distribution $\Gamma(\cdot)$ but also the tax policy function $\Psi(\cdot)$. This is feasible because, with the value function, these endogenous tax functions can be directly obtained by solving the government problem (17).

However, I wish to clarify that I cannot directly apply the Reiter's (2010) method to the model in this paper because of the presence of off-equilibrium paths after one-shot deviations that are required to find the sub-perfect Nash equilibrium. In the incomplete-markets economy with aggregate uncertainty, for which the Reiter's (2010) method is originally designed, the distribution of aggregate shocks (TFP) Z is ergodic. Thus, all the aggregate states Z are not measure zero. With a positive probability, all the states in Z are realized on the equilibrium path. However, cases in the MPE do not have this property. For example, in the economy without commitment, the government chooses a tax rate by comparing one-time deviation policies, as in (17). Some tax paths will not be reached on the equilibrium path but the corresponding value needs defining to solve the government problem (17).

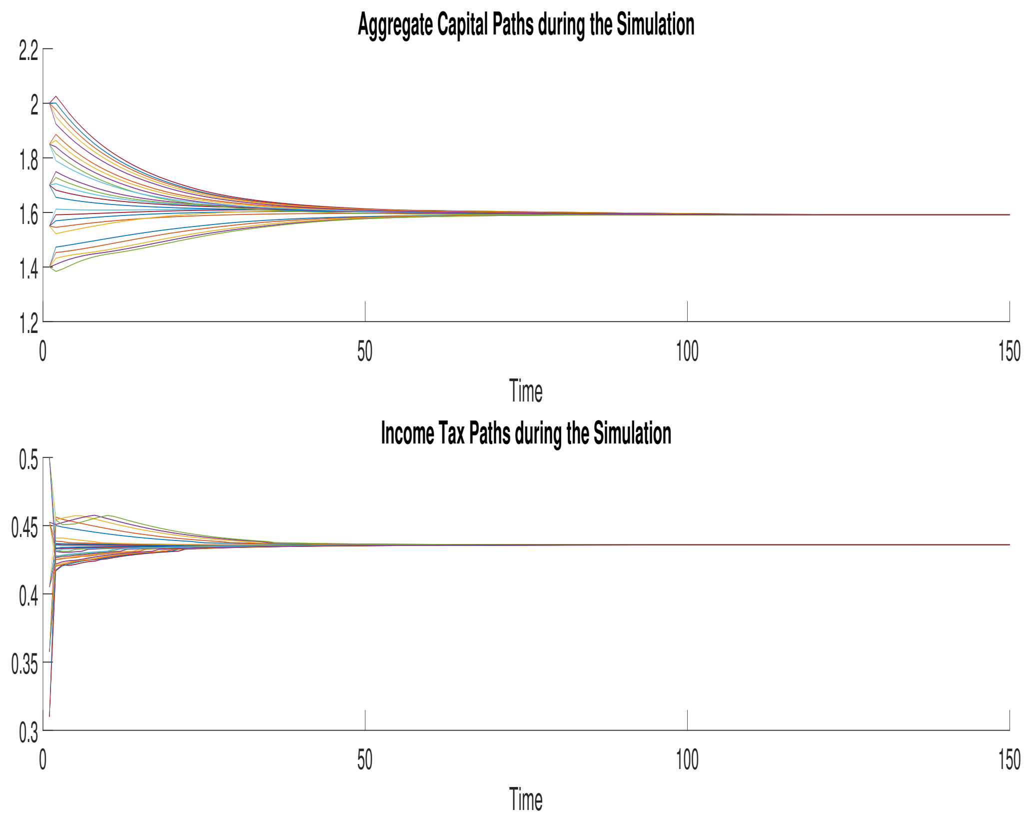

Figure 1: Transitions from off the Equilibrium to the Equilibrium

To cope with this issue, I make three changes to the original backward induction method of Reiter (2010). First, as mentioned above, I approximate not only the aggregate law of motion for the distribution of households but also the endogenous tax policy function. I find these mappings in a nonparametric way as in Reiter (2010). Second, I arrange distributions for all types of off-the-equilibrium paths by taking the initial distribution of the simulations as the previous proxy distribution for each finite grid point of the aggregate state. Figure 1 shows various transitions from off the equilibrium to the steady-state equilibrium in the case without commitment. Finally, I modify the way of constructing the reference distributions, which is required to update the proxy distributions in Reiter (2010), by reflecting the features of the MPE, in which how many times a tax rate off the equilibrium takes place is unknown before simulation. Appendix A demonstrates the full details of the solution method, in addition to its performance in terms of efficiency and accuracy.

Because of these somewhat complex variations in the Reiter's (2010) method, one might consider simply using the Krusell and Smith's (1998) method to solve this model. However, their approach is less efficient in addressing this class of models in MPE. First, finding the two aggregate laws of motion–Γ and Ψ—is computationally very costly when using this simulation-based solution method. When this method is employed to solve the economy in this paper, this process is the same as adding another outer loop to the outer loop in the Krusell and Smith's (1998) original algorithm, thereby exponentially increasing the computational burden. Second, the parametric assumption of the Krusell and Smith's (1998) approach acts as a barrier because the equilibrium tax function $\Psi(\cdot)$ could be severely non-linear in the aggregate state. The parametric assumption works well when the law of motion for household distributions $\Gamma(\cdot)$ is close to linear. However, it is possible for $\Psi(\cdot)$ to be severely non-linear and the method in this paper can cope with this non-linearity.

Regarding the case with commitment (the Ramsey problem), I employ the approach in Dyrda and Pedroni (2023) by parameterizing the transition path of income taxes as follows:

$$\tau_t = \left(\sum_{i=0}^{m_{x0}} \alpha_i^x P_i(t)\right) \exp(\lambda t) + (1 - \exp(-\lambda t)) \left(\sum_{j=0}^{m_{xF}} \beta_j^x P_j(t)\right), \quad t \leq t_F \tag{35}$$

where $\{P_i(t)\}_{i=1}^{m_{x0}}$ and $\{P_j(t)\}_{j=0}^{m_{xF}}$ are families of the Chebyshev polynomial; m_{x0} and m_{xF} are the orders of the polynomial approximation for the short- and long run dynamics, respectively; $\{\alpha_i^x\}_{i=0}^{m_{x0}}$ and $\{\beta_j^x\}_{j=0}^{m_{xF}}$ are weights on the consecutive elements of the family; and λ controls the convergence rate of the fiscal instrument. This setting assumes that the economy has the long-run steady state at the latest in period t_F. I first choose $m_{x0} = m_{xF} = 2$ and $t_F = 250$. Then, I seek $\{\alpha_0^x, \alpha_1^x, \alpha_2^x, \beta_0^x, \beta_1^x, \beta_2^x, \lambda\}$ that maximize the welfare function of the utilitarian government at time 0.[14]

5 Calibration

I calibrate the model to capture the features of the U.S. economy. I divide the parameters into two groups. The first set of the parameters is determined outside the model. I take the values of these parameters from the macroeconomic literature and policies. The other set of parameters requires solving the stationary distribution of the model to match the moments generated by the model with their empirical counterparts.

[14] The inclusion of lump-sum transfers prevents the non-existence of a Ramsey steady state, which is examined in

Table 1: Parameter Values of the Baseline Economy

Parameters	Description	Target	Data Value	Model Value
Externally chosen parameters				
$\sigma = 2$	Relative risk aversion			
$\chi = 0.75$	Frisch elasticity of labor supply			
$\rho_\epsilon = 0.955$	Persistence of wage shocks			
$\underline{a} = 0$	Borrowing constraint			
$z = 1$	TFP			
$\theta = 0.36$	Capital income share			
$\delta = 0.08$	Depreciation rate			
$\tau = 0.31$	AVG income tax			
Internally chosen parameters				
$\beta = 0.951$	Discount factor	K/Y	3	3
$B = 3.143$	Utility of leisure	AVG Wrk Hrs	1/3	1/3
$\sigma_\epsilon = 0.2$	STD of wage shocks	Wage Gini Coefficient	0.359	0.360
$T_1 = 0.039$	Flat transfers	T_1/Y	0.056	0.056
$\omega_s = 0.047$	Scale of progressive transfers	T_2/Y	0.031	0.031
$\omega_p = 0.3$	Progressivity of transfers	$\frac{var(\log(\text{disposable income}))}{var(\log(\text{pre-tax income}))}$	0.68	0.67

Table 1 displays the parameters. Eight parameters are determined outside the model. The coefficient of relative risk aversion is set to 2. The Frisch elasticity of labor supply χ is taken to be 0.75. I set the borrowing constraint as $\underline{a} = 0$. The TFP z is set as 1, and the capital income share θ is chosen to reproduce the empirical finding that the share of capital income is 0.36. The annual depreciation rate δ is 8 percent. The persistence of wage shocks ρ_ϵ is set to be 0.955, the value of which lies in the range frequently used in the literature. The flat income tax rate is chosen as 0.31 in the baseline economy.

Internally, I calibrate six parameters. The discount factor β is selected to match a capital-to-output ratio of 3, and the utility of leisure B is chosen to reproduce an average of hours worked of 8 hours a day. The standard deviation of wage shocks σ_ϵ is chosen to match the Gini coefficient of hourly wages in the Panel Study of Income Dynamics (PSID), which is 0.359.

Regarding the transfer system, I assume that the progressive component of transfers takes the following functional form:

$$T_2(y) = \frac{\omega_s}{1 + \exp(\omega_p \cdot y)} \quad \text{where} \quad y = w\,\epsilon\,n + r\,a. \tag{36}$$

ω_s captures the overall size of progressive transfers, and ω_p controls the extent of progressivity in transfers, as in Jang et al. (2023). To determine these transfer-related parameters, I take three steps.

Straub and Werning (2020). Further details are provided in Dyrda and Pedroni (2023).

First, using data from the U.S. Bureau of Economic Analysis, I calculate the average ratio of government social benefits to GDP between 1972 and 2019, which is 8.7 percent. Second, following Krusell and Ríos-Rull (1999), I divide transfers into two components—progressive transfers and flat transfers—and use the relative ratios provided in their study. This calculation yields flat and progressive transfers accounting for 5.6 and 3.1 percent of GDP, respectively. With this information, I pin down the values of T_1 and ω_s. Finally, I set ω_p to match the ratio of the variance of the log of disposable income to the variance of the log of pre-tax income in the PSID, which is 0.68. Recall that the progressive component of transfers $T_2(y)$ exists only in the baseline economy, and a policy reform eliminates it by setting ω_s to zero.

Government spending G is determined as the difference between the implied total tax revenue and the total size of transfers, which amounts to 14.6 percent of the baseline GDP. Throughout the subsequent counterfactual exercises, G remains fixed at this level.

6 Results

In this section, I quantitatively explore how commitment technologies affect the aggregate economy, inequality, and welfare. To do so, I compare the economy without commitment to the economy with commitment. I assume that all the economies are at the calibrated steady-state until period -1, and each government unexpectedly announces a policy reform at the beginning of period 0. I compare their equilibrium results along the transition path. I begin with the exercise based on proportional income taxes, and later, I will change the tax base.

6.1 Time-Consistent versus Time-Inconsistent Policy: Income Tax

Figure 2 shows the time-consistent and time-inconsistent optimal taxes and the implied ratio of transfers to the initial output. The top panel of Figure 2 implies that the government with commitment chooses more substantial income taxes than the government without commitment over the entire transition path. The Ramsey planner initially increases income taxes by 16 percentage points and then reduces them thereafter. However, without commitment, the government with the optimal income tax raises taxes by only 2 percentage points. This gap in tax policies results in differences in the size of transfers. The time-inconsistent optimal income tax-transfer economy generates larger transfers than the time-consistent optimal income tax-transfer economy. The ratio of transfers to the initial GDP in the case with commitment gradually increases by 9.2 percentage points, but in the case without commitment, it only increases by 1.4 percentage points.

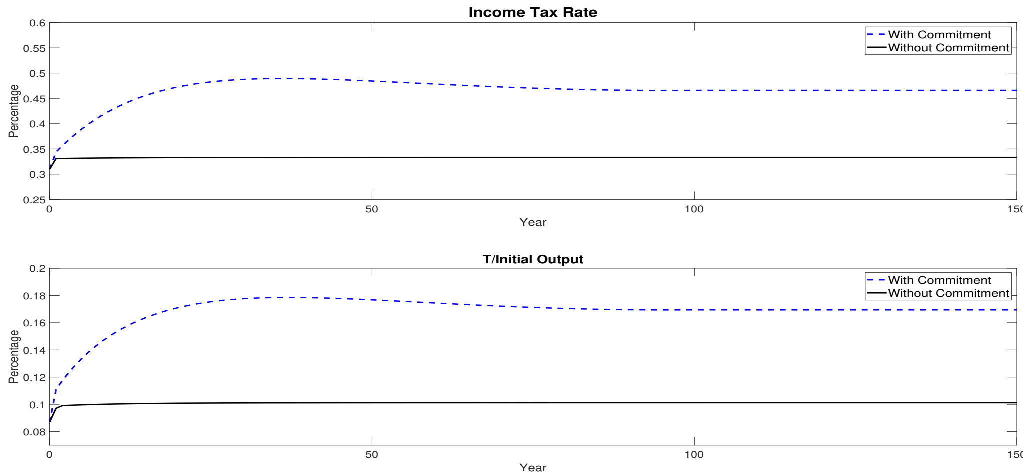

Figure 2: Time-Consistent and Time-Inconsistent Policies: Tax-transfer Transition Paths

Table 2: Welfare Outcomes According to Commitment (Income Taxes)

Welfare (CEV)	With Commitment	Without Commitment
OPT INC TAX	+1.56%	+0.19%

This distinction in income taxes and transfers creates different welfare consequences. Table 2 shows that welfare, measured by the consumption equivalent variation (CEV) of the utilitarian welfare function, is significantly higher in the case with commitment. Compared to the baseline economy, the time-inconsistent optimal policy improves welfare by 1.56 percent, while the time-consistent optimal policy improves welfare by 0.19 percent. To understand this disparity in welfare consequences, I examine how differently the inputs of the social welfare function vary over time according to the availability of commitment. Note that welfare increases when the overall level of consumption and leisure increases, and their inequalities are reduced.

Figure 3 illustrates changes in the levels of the aggregate variables. The figure suggests that the case without commitment generates more efficient outcomes. All the aggregate variables are larger in the economy without commitment than that with commitment. This result may seem obvious because lower taxes in the time-consistent case cause fewer distortions. However, this finding might be unclear when examining the welfare consequences. Despite the fact that consumption is substantially larger in the case with the time-consistent policy, the gaps in hours worked/leisure are not significant and do not offer a complete understanding of the welfare implications.

Figure 4 illustrates inequalities in consumption, hours worked, wealth, and after-tax income. The figure suggests that the more significant welfare improvements in the case with commitment are mainly driven by larger reductions in inequalities in consumption and leisure. While con-

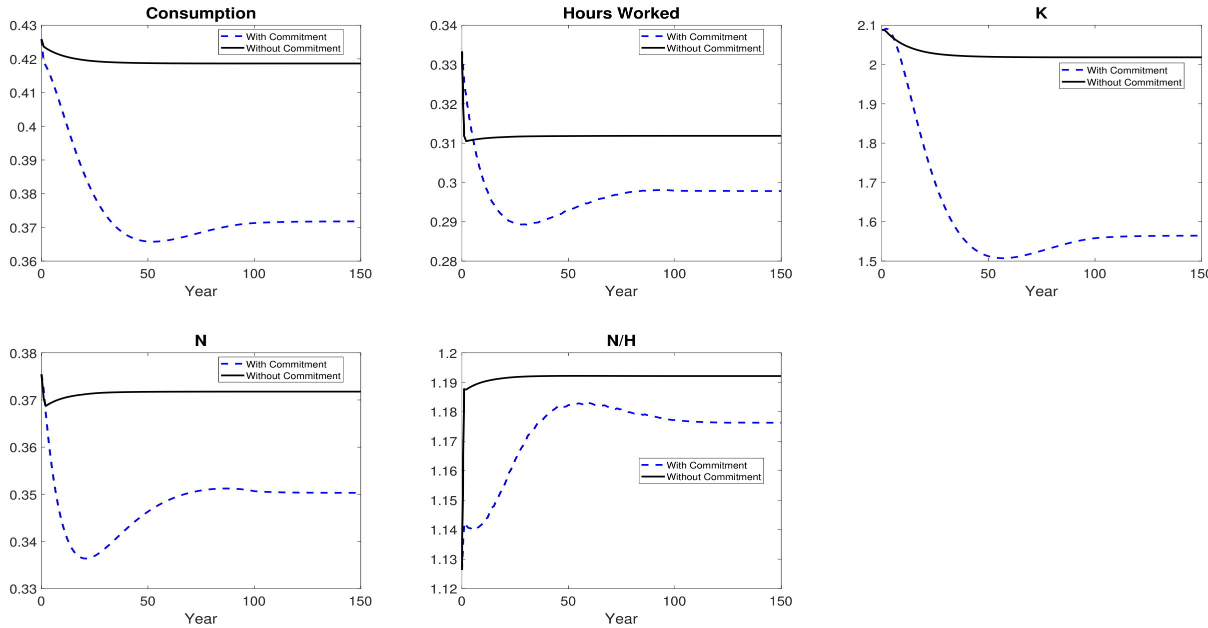

Figure 3: Time-Consistent and Time-Inconsistent Income Tax-Transfer: Aggregate Outcomes

sumption inequality, measured by the Gini coefficient, decreases by less than 1 percent with the time-consistent optimal policy, it is reduced by approximately 10 percent with the time-inconsistent optimal tax. Similarly, inequality in hours worked is also reduced more in the case with commitment. These findings imply that the commitment instrument allows the Ramsey planner to better manage the evolution of inequality, leading to a more favorable welfare outcome. However, this explanation does not provide a clear economic logic as to how the availability of government's commitment plays a role in shaping these quantitative disparities. To better understand the economic logic behind these differences, I employ the theoretical findings in the previous section to interpret these quantitative outcomes.

Figure 5 shows the dynamics of the factor prices w and r depending on the availability of commitment. This figure suggests that the government with commitment (the Ramsey planner) benefits from pecuniary externalities in the early phase of the transition by exploiting differences in the speed of adjustments between K and N. Of course, in the long run, the increased tax rate in the case with commitment reduces the ratio of K to N, leading to a decrease in w and an increase in r. However, during their adjustment, the ratio of K to N increases in the early phase of the transition because the adjustment of K is slower than that of N, leading to a rise in w and a reduction in r. These price changes help improve the welfare of the low-income group, who are more represented by the government, through pecuniary externalities during the early transition.

Figure 6 illustrates the proportion of capital income out of total disposable income across in-

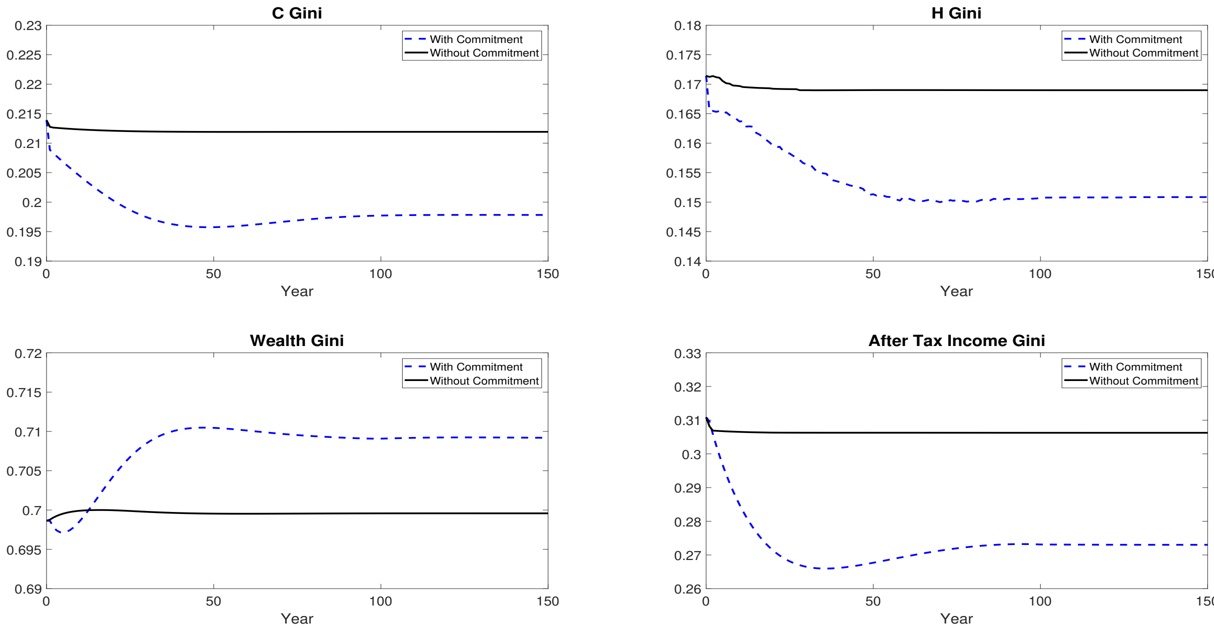

Figure 4: Time-Consistent and Time-Inconsistent Income Tax-Transfer: Distributional Outcomes

come groups, according to the government's commitment technologies. The left panel of Figure 6 indicates that in the economy with commitment, low-income individuals reduce their precautionary savings, which is not observed for the middle- and high-income groups. Initially, all income groups experience a decrease in their capital income proportion as the market interest rate declines during the early transition. However, while middle- and high-income individuals increase their capital income proportion as the interest rate rises afterward, low-income individuals maintain their reduced savings in the long run. This decrease in precautionary savings facilitates a reduction in after-tax income inequality among low-income individuals because inequality in capital income is more severe than inequality in other types of income, such as labor income and transfers. Given the government's inclination toward the interests of low-income individuals, the reduction in their income inequality in the early transition leads to front-loaded welfare gains resulting from income redistribution.

The availability of government commitment technologies is the key to driving the difference in savings across income groups. The right panel of Figure 6 shows that without commitment, the economy does not have such permanent decline in savings for low-income individuals. Low-income individuals in the economy without commitment have larger precautionary savings because the government does not provide as substantial transfers as the government with commitment in the long run. These findings imply that the front-loaded welfare gains from both types of economic forces are driven by the government's ability to commit to future policies in the long run.

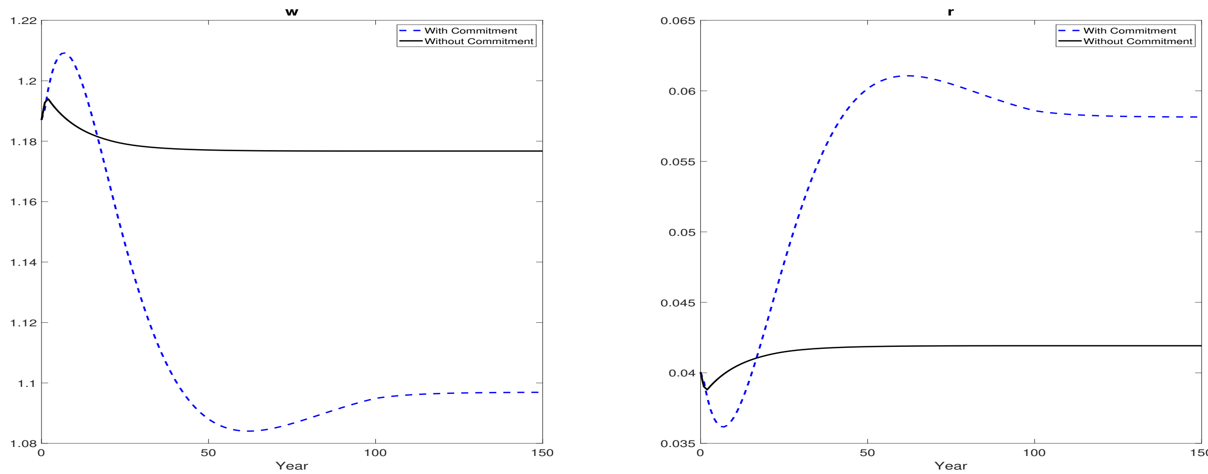

Figure 5: Time-Consistent and Time-Inconsistent Income Tax-Transfer: Dynamics of w and r

Table 3: Difference in Welfare Changes between the Short and Long Runs (Income Taxes)

Welfare (CEV)	With Commitment	Without Commitment
OPT INC TAX	+4.66 pp	+0.51 pp

In exchange for these upfront welfare gains, the government with commitment endures welfare losses in the long run. Table 3 shows the difference in welfare changes between the short and long runs is substantial (+4.66 pp). In the long run, w is lower and r is higher than their initial values, and these changes become negative pecuniary externalities for the low-income group, as observed in Figure 5. Additionally, in the long run, as Figure (34) shows, there is no further reduction in after-tax income inequality, leading to minuscule welfare gains from income redistribution. Recall that, as (34) shows, the sum of all pecuniary externalities is equal to that of all income redistribution. These findings imply that the government with commitment takes front-loaded welfare gains via income redistribution and positive pecuniary externalities while enduring welfare losses from mitigated income redistribution and negative pecuniary externalities.

Note that the above time-lagged strategy is not credible without commitment. Suppose that the government without commitment is in the long-run equilibrium of the economy with commitment. As shown in (30), a lack of commitment causes the government to measure the costs and benefits of changing a tax rate considering its impacts only in periods after the implementation. Since any front-loaded welfare gain is disregarded, the marginal cost of raising the tax rate, which equals the aggregate pecuniary externalities, is greater than that of the initial economy, and the marginal benefit, which equals the aggregate income redistribution , is close to zero. Proposition 1 indicates that the government, in this case, will decide a one-time reduction in its tax rate and transfers,

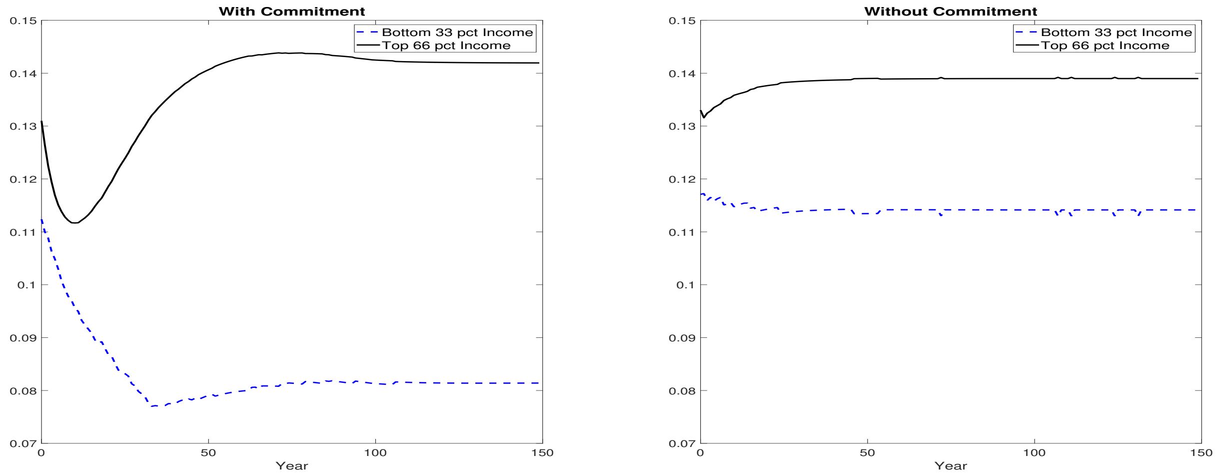

Figure 6: Time-Consistent and Time-Inconsistent Income Tax-Transfer: Proportion of Capital Income

which is all it can do due to the lack of commitment, in the next period. Therefore, the time-lagged strategy is not credible in the case without commitment.

This time-consistent policy leads to more severe economic inequalities, thereby aggravating overall welfare. Individuals will rationally understand the government's motive to reduce its tax rate and transfers in the next period. Changes in their expectations regarding the policy will cause individuals to increase precautionary savings and labor supply. This increase in precautionary motives prevents short-run welfare gains at the expense of long-run welfare losses, which is pronounced in the case with commitment. As Table 3 shows, the difference in welfare changes between the short and long runs is much smaller in the case without commitment (+0.51 pp) than the case with commitment (+4.66 pp). As shown in Table 8, this time-consistent policy is quantitatively worse than the time-inconsistent policy in terms of managing welfare outcomes, resulting in much smaller improvements in welfare (+0.19%).

6.2 Time-Consistent versus Time-Inconsistent Policy by Tax Base

Figure 7 displays the time-consistent and time-inconsistent optimal taxes and transfers according to the tax base. The top (bottom) panels illustrate the outcomes when labor (capital) income taxes are permitted to vary over time while the capital (labor) income tax is held at the calibrated level of 0.31. In this section, I concentrate on the cases where labor income taxes change over time instead of the cases where capital income taxes change over time. The bottom panels demonstrate that when capital income tax is the only instrument available to the government, the time-consistent government raises it to the maximum rate of 100 percent. This extremely high capital income

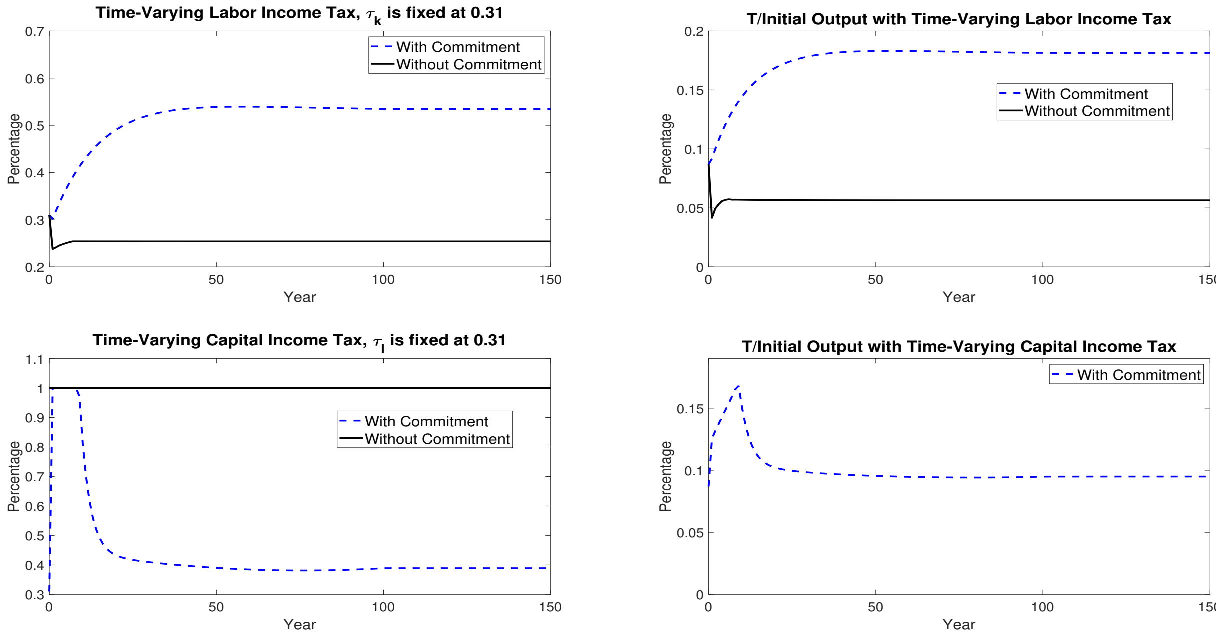

Figure 7: Time-Consistent and Time-Inconsistent Policies: Taxes/transfers by Tax Base

tax has been analyzed theoretically in previous studies, such as Chari and Kehoe (1990). These studies have shown that the time-consistent government makes the capital income tax confiscatory, resulting in no savings. In my quantitative exercise, time-consistent capital income taxes reduce the size of the economy to such an extent that the tax revenue cannot cover the exogenous government spending in its budget.[15] The results of the time-inconsistent capital income tax economy are presented in Appendix B, which are in line with those in Dyrda and Pedroni (2023).

Table 4: Welfare Outcomes According to Commitment (Labor Income Taxes)

Welfare (CEV)	With Commitment	Wihtout Commitment
OPT Labor INC TAX (τ_k is given by 0.31)	+1.50%	-1.34%

The top panels of Figure 7 indicate that the economies with labor income taxes deliver similar messages as in the cases with proportional income taxes. The government with commitment raises its labor income taxes early in the transition, resulting in more substantial transfers. However, without commitment, the government reduces its labor income taxes, resulting in lower transfers. Table 4 shows that the welfare consequences are also similar to the cases with proportional income taxes: the government with commitment brings more significant welfare improvements to the economy than does the government without commitment.

[15]To have sustainable capital income tax rates, the size of government spending might need to be endogenous by assuming households to value it, as in Klein et al. (2008). The current setting does not have this component.

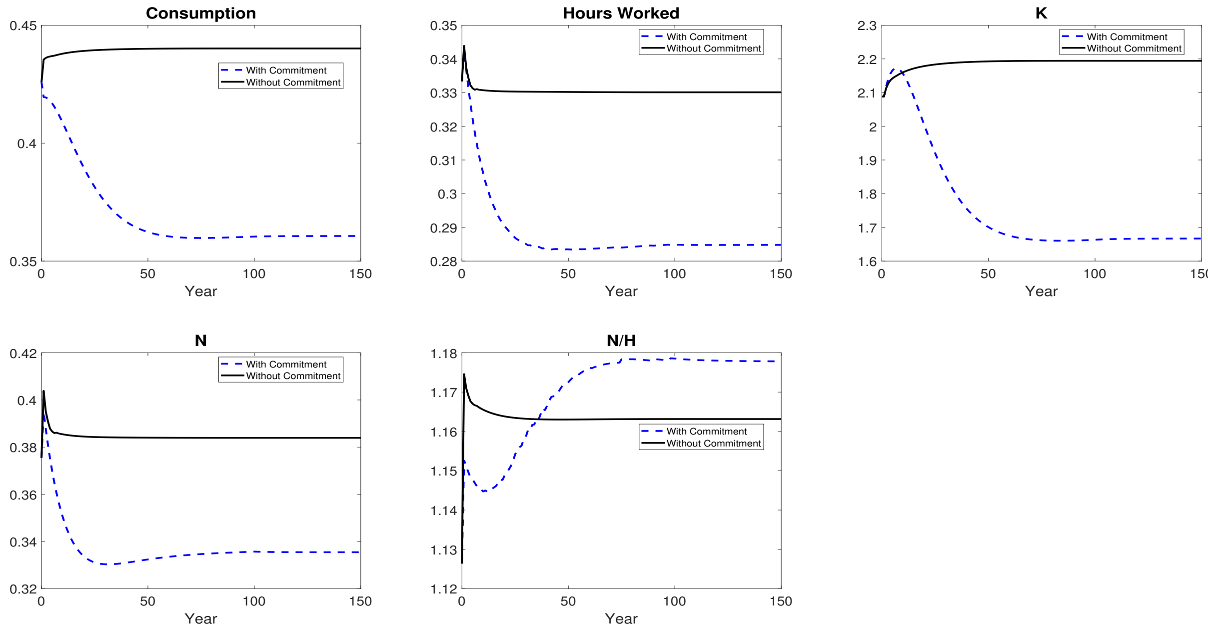

Figure 8: Time-consistent and Time-inconsistent Labor Income Tax-Transfer: Aggregate Outcomes

Figure 8 shows that heavier labor income taxes in the case with commitment lead to a less efficient economy than without commitment, which is in line with the findings in the cases with proportional income taxes. In the economy with commitment, aggregate consumption, hours worked, capital, and effective labor are lower than those in the case without commitment because heavier labor income taxes in the case with commitment lead to a greater loss in efficiency, causing the economy to shrink. Different dynamics appear in the effective labor to hours worked ratio, but this gap is caused solely by labor market selection driven by heavier labor income taxes in the case with commitment. The dynamics of overall aggregate variables are similar to those in the cases with proportional income taxes.

Figure 9 shows that the dynamics of inequality play a crucial role in the welfare outcomes in both cases. The economy with commitment experiences more substantial transfers, which results in reduced inequality in after-tax income. This reduction in inequality leads to more equally distributed consumption and hours worked, ultimately improving welfare. In contrast, without commitment, the government reduces taxes and transfers, leading to more significant inequality in after-tax income. This increase in inequality worsens welfare in the economy. Overall, these findings are consistent with those in the cases with proportional income taxes. To obtain a better understanding, I apply the theoretical findings to the quantitative results.

Note that the government with commitment obtains positive externalities from reduced income

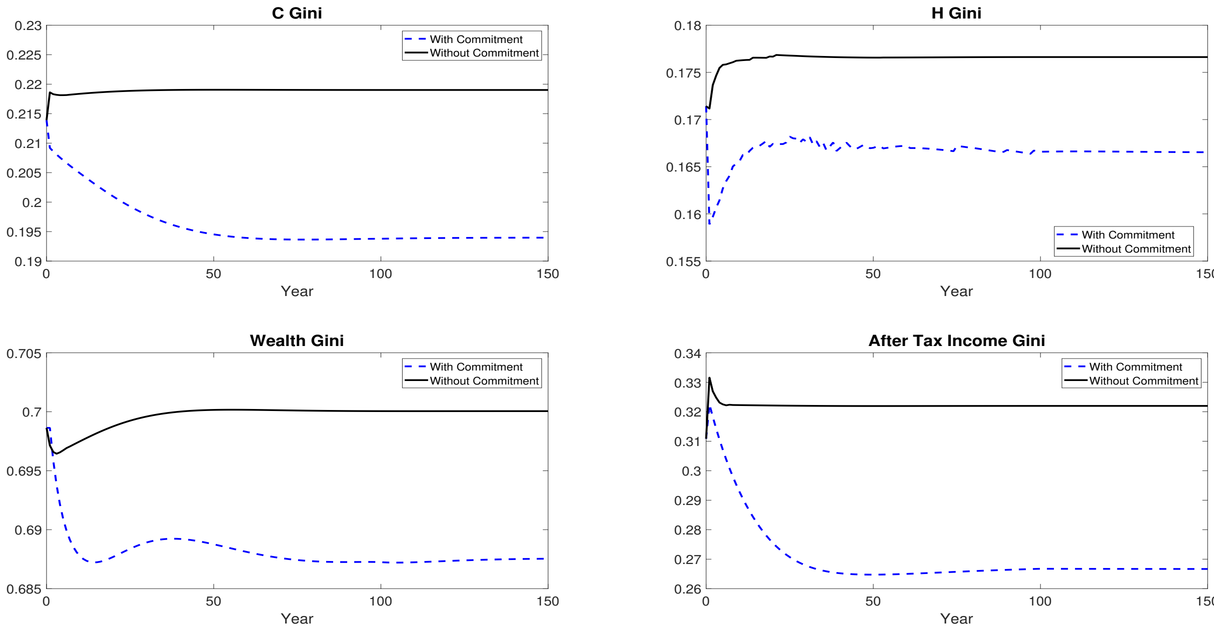

Figure 9: Time-consistent and Time-inconsistent Labor Income Taxes: Distributional Outcomes

inequality, as the bottom-right panel of Figure 9 shows. Furthermore, Figure 10 indicates that it also obtains positive pecuniary externalities from changes in the factor composition of income—increases in w and decreases in r—in the early phase of the transition. These price changes seem inconsistent with the outcomes after increasing labor income taxes, which implies the opposite changes due to an increase in the capital-to-labor ratio in the long run. Nonetheless, as Figure 8 shows, increases in the labor income tax reduce the aggregate capital and labor supply in the early phase of the transition. Additionally, because the speed of adjustment in capital is slower than that in labor, the government obtains front-loaded positive pecuniary externalities from increases in w and decreases in r, along with positive externalities from reduced income inequality.

Consistent with the results from the exercise with proportional income taxes, Figure 11 demonstrates that government commitment induces a reduction in precautionary savings for low-income individuals. With commitment, the government can provide substantial transfers, leading to a decrease in precautionary savings, as seen in the case of proportional income taxes. This reduction in precautionary savings induces overall inequality in after-tax income, as observed in Figure 9.

As mentioned previously, the reduction in precautionary savings decreases inequality in after-tax income because capital income has more severe inequality than labor income and lump-sum transfers. Therefore, as observed in Figure 9, this change contributes to redistributing income during the early transition—front-loaded income redistribution. However, for the economy without commitment, low-income individuals do not reduce their savings; instead, they increase their

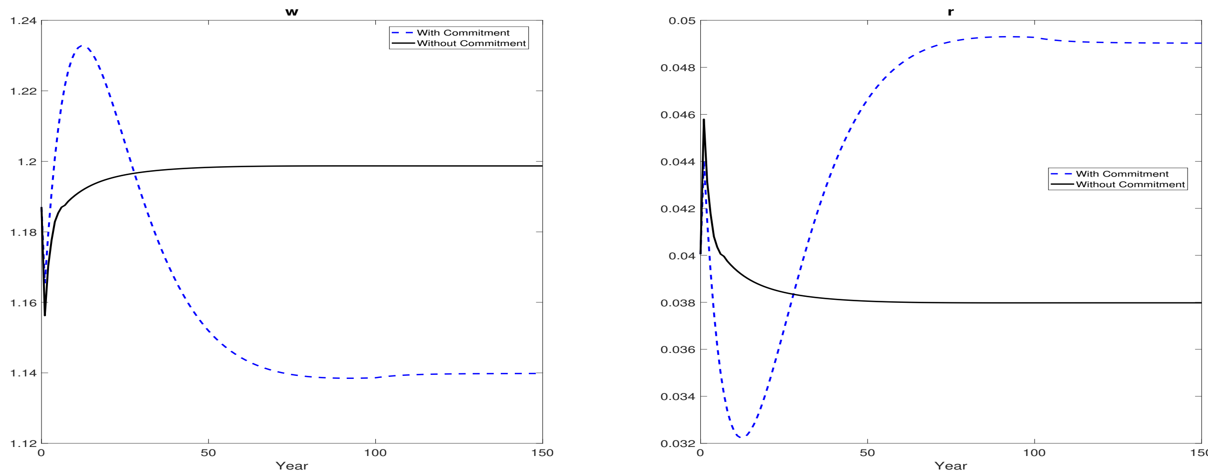

Figure 10: Time-Consistent and Time-Inconsistent Labor Income Tax-Transfer : Dynamics of w and r

precautionary savings because transfers is smaller than transfers in the baseline economy. These differences lead to disparities in the distribution of the aggregate variables. These findings imply that commitment is the main driving force behind upfront welfare gains via both types of externalities, which is in line with those in the case with proportional income taxes.

For these upfront welfare gains, in the long run, the government with commitment must endure welfare losses from negative pecuniary externalities and mitigated income redistribution, as Table 5 Implies. As Figure 10 shows, the factor prices are unfavorable for low-income individuals—an increase in r and a decrease in w. This change in the factor composition of income leads pecuniary externalities to be negative. At the same time, in the long run, because transfers increases during the initial transition and maintains its level thereafter, income redistribution becomes stagnant in the long run—mitigated income redistribution. These results imply that the government with commitment balances the two types of economic forces by allocating positive externalities in the early phase of the transition while shifting negative externalities later. More substantial welfare improvements in the case with commitment suggest that this way of balancing is better for welfare.

Table 5: Difference in Welfare Changes between the Short and Long Runs (Labor Income Taxes)

Welfare (CEV)	With Commitment	Without Commitment
OPT Labor INC TAX	+4.26 pp	-0.69 pp

Without commitment, this strategy is not credible. Suppose that, as in the case with proportional income taxes, the government without commitment is in the long-run equilibrium of the economy with commitment. This government ignores the upfront welfare changes and compares

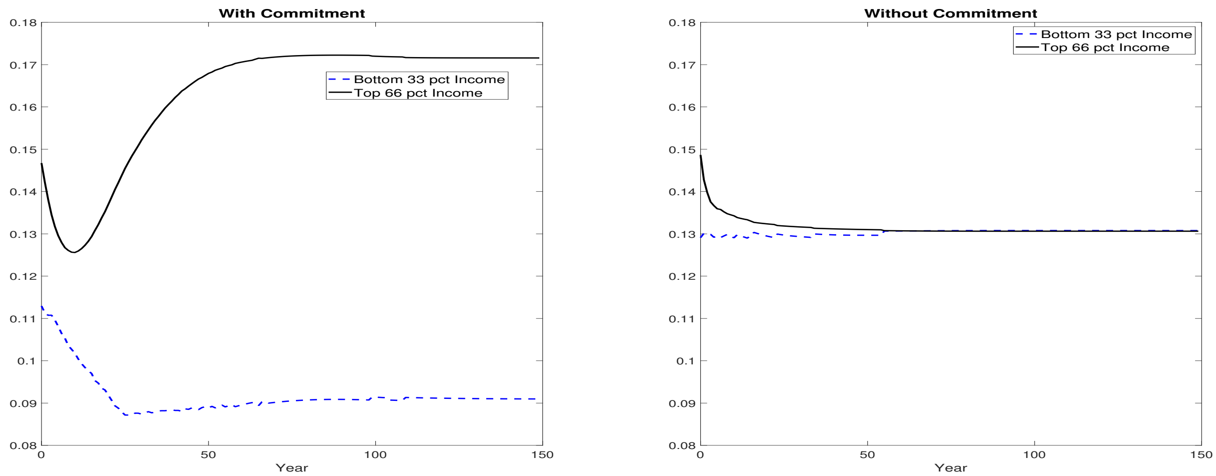

Figure 11: Time-Consistent and Time-Inconsistent Labor Income Tax-Transfer : Proportion of Capital Income

the two types of economic forces every period. From the government's perspective, negative pecuniary externalities from the factor composition of income are more significant than stagnant changes in income inequality. Therefore, the government is willing to do a one-time rebalancing between the two types of economic forces by reducing labor income taxes and transfers because doing so will alleviate negative externalities from lower wages and spare more income under the lack of commitment.

Similar to the case of proportional income tax, this time-consistent strategy leads to worse welfare outcomes, as indicated in Table 4. Individuals recognize the government's inability to commit to future policies and rationally anticipate the government's incentive, which leads to a shift in their expectations regarding future policies. Consequently, individuals exhibit stronger precautionary motives, resulting in increased labor supply and savings. Table 5 implies that these increased precautionary motives prevent short-run welfare gains at the expense of long-run welfare losses, leading to worse welfare outcomes.

7 Conclusion

This paper examines how the availability of government commitment affects the optimal design of transfers along the transition. To this end, I arrange a dynamic game between heterogeneous individuals and a benevolent government in the standard incomplete-markets model and characterize and solve for its equilibria according to its commitment technologies. I find that when making policy decisions on income taxes and transfers, commitment affects how the government balance the two types of economic forces: income redistribution through taxes and transfers and pecuniary externalities from changes in the factor composition of income. Commitment allows the government to balance throughout the entire transition; however, without commitment, the government strikes a balance in each period, considering the effect of its policy decision only in periods after the decision.

I assess the magnitude of this qualitative difference using a quantitative method in a calibrated economy. The quantitative analysis shows that commitment has significant impacts on the government's policy decisions along the equilibrium path. The key mechanism is that commitment enables the government to provide substantial transfers in the long run, resulting in upfront welfare gains while delaying welfare losses to the long run. Without commitment, the government lacks the ability to provide such long-term public insurance and cannot obtain as large welfare gains as the government with commitment.

Note that the solution method itself could provide many opportunities for studying unexplored research topics. Given the fundamental feature of Reiter (2010), this solution method can be compatible with aggregate uncertainty. This research direction makes it possible to extend the previous fiscal policy analyses with incomplete markets Bhandari et al. (2017a,b), investigating the implications of governments' political decisions and commitment. Another exciting application of the method is addressing the interactions between policies and life-cycle dimensions. The Kim's (2021) method makes this direction reachable. She extends the Reiter's (2010) backward induction method to solve an overlapping generations model. Such analyses are deferred to future work.

References

Aiyagari, S Rao, "Uninsured idiosyncratic risk and aggregate saving," *Quarterly Journal of Economics*, 1994, *109* (3), 659–684.

Athey, Susan, Andrew Atkeson, and Patrick J Kehoe, "The optimal degree of discretion in monetary policy," *Econometrica*, 2005, *73* (5), 1431–1475.

Azzimonti, Marina, "Barriers to investment in polarized societies," *American Economic Review*, 2011, *101* (5), 2182–2204.

Barro, Robert J and David B Gordon, "Rules, discretion and reputation in a model of monetary policy," *Journal of Monetary Economics*, 1983, *12* (1), 101–121.

Bassetto, Marco, "Equilibrium and government commitment," *Journal of Economic Theory*, 2005, *124* (1), 79–105.

_, Zhen Huo, and José-Víctor Ríos-Rull, "Institution Building without Commitment," *Working Paper*, 2020.

Ben-Shalom, Yonatan, Robert A Moffitt, and John Karl Scholz, "An assessment of the effectiveness of anti-poverty programs in the United States," Technical Report, National Bureau of

Economic Research 2011.

Bewley, Truman, "Stationary monetary equilibrium with a continuum of independently fluctuating consumers," *Contributions to mathematical economics in honor of Gérard Debreu*, 1986, *79.*

Bhandari, Anmol, David Evans, Mikhail Golosov, and Thomas J Sargent, "Fiscal policy and debt management with incomplete markets," *Quarterly Journal of Economics*, 2017, *132* (2), 617–663.

_ , _ , _ , and _ , "Public debt in economies with heterogeneous agents," *Journal of Monetary Economics*, 2017, *91*, 39–51.

Boar, Corina and Virgiliu Midrigan, "Efficient redistribution," *Journal of Monetary Economics*, 2022, *131*, 78–91.

Calvo, Guillermo A, "On the Time Consistency of Optimal Policy in a Monetary Economy," *Econometrica*, 1978, *46* (6), 1411–1428.

Carroll, Christopher D, "The method of endogenous gridpoints for solving dynamic stochastic optimization problems," *Economics Letters*, 2006, *91* (3), 312–320.

Chari, Varadarajan V and Patrick J Kehoe, "Sustainable plans," *Journal of Political Economy*, 1990, *98* (4), 783–802.

Cohen, Daniel and Philippe Michel, "How should control theory be used to calculate a time-consistent government policy?," *Review of Economic Studies*, 1988, *55* (2), 263–274.

Conesa, Juan Carlos and Dirk Krueger, "On the optimal progressivity of the income tax code," *Journal of Monetary Economics*, 2006, *53* (7), 1425–1450.

Conesa, Juan Carlos Conesa, Bo Li, and Qian Li, "A quantitative evaluation of universal basic income," *Journal of Public Economics, Forthcoming*, 2023.

Conesa, Juan Carlos, Sagiri Kitao, and Dirk Krueger, "Taxing capital? Not a bad idea after all!," *American Economic Review*, 2009, *99* (1), 25–48.

Corbae, Dean, Pablo D'Erasmo, and Burhanettin Kuruscu, "Politico-economic consequences of rising wage inequality," *Journal of Monetary Economics*, 2009, *56* (1), 43–61.

Daruich, Diego and Raquel Fernández, "Universal basic income: A dynamic assessment," *American Economic Review, Forthcoming*, 2023.

Davila, Julio, Jay H Hong, Per Krusell, and José-Víctor Ríos-Rull, "Constrained efficiency in the neoclassical growth model with uninsurable idiosyncratic shocks," *Econometrica*, 2012, *80* (6), 2431–2467.

Debortoli, Davide, Ricardo Nunes, and Pierre Yared, "Optimal time-consistent government debt maturity," *Quarterly Journal of Economics*, 2017, *132* (1), 55–102.

_ , _ , and _ , "Optimal fiscal policy without commitment: Revisiting Lucas-Stokey," *Journal of Political Economy*, 2021, *129* (5), 1640–1665.

Domínguez, Begoña, "On the time-consistency of optimal capital taxes," *Journal of Monetary Economics*, 2007, *54* (3), 686–705.

_ , "Public debt and optimal taxes without commitment," *Journal of Economic Theory*, 2007, *135* (1), 159–170.

Dyrda, Sebastian and Marcelo Pedroni, "Optimal Fiscal Policy in a Model with Uninsurable Idiosyncratic Income Risk," *The Review of Economic Studies*, 2023, *90* (2), 744–780.

Ferriere, Axelle, Philipp Grübener, Gaston Navarro, and Oliko Vardishvili, "On the optimal design of transfers and income-tax progressivity," *International Finance Discussion Paper*, 2022, (1350).

Fischer, Stanley, "Dynamic inconsistency, cooperation and the benevolent dissembling government," *Journal of economic dynamics and control*, 1980, *2*, 93–107.

Guner, Nezih, Remzi Kaygusuz, and Gustavo Ventura, "Rethinking the welfare state," *Econometrica*, 2023, *91* (6), 2261–2294.

Haan, Wouter J Den, "Assessing the accuracy of the aggregate law of motion in models with heterogeneous agents," *Journal of Economic Dynamics and Control*, 2010, *34* (1), 79–99.

Heathcote, Jonathan and Hitoshi Tsujiyama, "Optimal income taxation: Mirrlees meets Ramsey," *Journal of Political Economy*, 2021, *129* (11), 3141–3184.

_ , Kjetil Storesletten, and Giovanni L Violante, "Optimal tax progressivity: An analytical framework," *The Quarterly Journal of Economics*, 2017, *132* (4), 1693–1754.

Huggett, Mark, "The risk-free rate in heterogeneous-agent incomplete-insurance economies," *Journal of economic Dynamics and Control*, 1993, *17* (5-6), 953–969.

Itskhoki, Oleg and Benjamin Moll, "Optimal development policies with financial frictions," *Econometrica*, 2019, *87* (1), 139–173.

Jaimovich, Nir, Itay Saporta-Eksten, Ofer Setty, and Yaniv Yedid-Levi, "Universal basic income: Inspecting the mechanisms," 2022.

Jang, Youngsoo, Takeki Sunakawa, and Minchul Yum, "Tax-Transfer Progressivity and Business Cycles," *Quantitative Economics, Forthcoming*, 2023.

Kim, Heejeong, "Inequality, disaster risk, and the great recession," *Review of Economic Dynamics*, 2021.

Klein, Paul and José-Víctor Ríos-Rull, "Time-consistent optimal fiscal policy," *International Economic Review*, 2003, *44* (4), 1217–1245.

_ , Per Krusell, and José-Víctor Ríos-Rull, "Time-consistent public policy," *Review of Economic Studies*, 2008, *75* (3), 789–808.

Krusell, Per and Anthony A Smith Jr, "Income and wealth heterogeneity in the macroeconomy," *Journal of political Economy*, 1998, *106* (5), 867–896.

_ and José-Víctor Ríos-Rull, "On the size of US government: political economy in the neoclassical growth model," *American Economic Review*, 1999, *89* (5), 1156–1181.

_ , Vincenzo Quadrini, and José-Víctor Ríos-Rull, "Are consumption taxes really better than income taxes?," *Journal of Monetary Economics*, 1996, *37* (3), 475–503.

Kydland, Finn E and Edward C Prescott, "Rules rather than discretion: The inconsistency of optimal plans," *Journal of Political Economy*, 1977, *85* (3), 473–491.

Laczó, Sarolta and Raffaele Rossi, "Time-consistent consumption taxation," *Journal of Monetary Economics*, 2020, *114*, 194–220.

Lucas, Robert and Nancy Stokey, "Optimal fiscal and monetary policy in an economy without capital," *Journal of Monetary Economics*, 1983, *12* (1), 55–93.

Nardi, Mariacristina De, Svetlana Pashchenko, and Ponpoje Porapakkarm, "The lifetime costs of bad health," Technical Report, National Bureau of Economic Research 2017.

Park, Yena, "Constrained efficiency in a human capital model," *American Economic Journal: Macroeconomics*, 2018, *10* (3), 179–214.

Persson, Mats, Torsten Persson, and Lars EO Svensson, "Time consistency of fiscal and monetary policy," *Econometrica: Journal of the Econometric Society*, 1987, pp. 1419–1431.

Reiter, Michael, "Recursive computation of heterogeneous agent models," *manuscript, Universitat Pompeu Fabra*, 2002, pp. 28–35.

_ , "Solving the incomplete markets model with aggregate uncertainty by backward induction," *Journal of Economic Dynamics and Control*, 2010, *34* (1), 28–35.

Rogoff, Kenneth, "The optimal degree of commitment to an intermediate monetary target," *Quarterly Journal of Economics*, 1985, *100* (4), 1169–1189.

Rouwenhorst, K Geert, "Asset Pricing Implications of Equilibrium Business Cycle Models. In: Cooley, T.F. (Ed.)," in "Frontiers of business cycle research," Princeton University Press, 1995, pp. 294–330.

Santos, Marcelo Rodrigues and Christopher Rauh, "How Do Transfers and Universal Basic Income Impact the Labor Market and Inequality?," *CEPR Discussion Paper No. DP16993*, 2022.

Song, Zheng, Kjetil Storesletten, and Fabrizio Zilibotti, "Rotten parents and disciplined children: A politico-economic theory of public expenditure and debt," *Econometrica*, 2012, *80* (6), 2785–2803.

Straub, Ludwig and Iván Werning, "Positive long-run capital taxation: Chamley-Judd revisited," *American Economic Review*, 2020, *110* (1), 86–119.

Online Appendix

Appendix A Numerical Solution Algorithm

Solving the Markov-Perfect Equilibria (MPE) of consecutive governments entails heavy computational burdens with heterogeneous agents. As in standard macroeconomic heterogeneous agent models, individual decisions should be consistent with the aggregate law of motion for the distribution of agents. On top of that, the aggregate tax policy function must be compatible with individual decisions and the aggregate law of motion for the distribution of agents. In other words, these three equilibrium objects—individual decisions, the law of motion for the distribution, and the tax policy function—have to be consistent with each other in the Markov-perfect equilibrium.

I address this computational issue by taking ideas from the Backward Induction method of Reiter (2010). This method discretizes the aggregate state into finite grid points. For each aggregate grid point, the Backward Induction algorithm allows updating the aggregate law of motion while solving the decision rules thanks to the existence of the proxy distribution. This means that for each aggregate grid point, the backward induction algorithm would make it possible to approximate not only the aggregate law of motion for the distribution; but also the tax policy function consistent with the choice of government lacking commitment. With the value function, this endogenous tax policy outcome can be directly obtained when the proxy distribution is explicitly available.

Unfortunately, the original Reiter's (2010) method cannot directly be applied to the MPE models because the existence of off-the-equilibrium paths makes it challenging to arrange the proxy distribution. In the model of Krusell and Smith (1998), for which the Reiter's (2010) method is originally designed, the distribution of TFP shocks Z is stationary, thus all the aggregate states Z are not measure zero. With a positive probability, all the states Z are realized on the equilibrium path. However, the MPE economy does not have this property. Let us think about a case in a stationary distribution. In this equilibrium, this optimal policy is obtained by comparing among one-time deviation policies. Some tax paths would not be reached at all on the equilibrium path.

I have three variations from the original backward induction method. First, I have to approximate not only the aggregate law of motion for distributions but also the tax policy function that is endogenous. I find these mappings in a non-parametric way, as in Reiter (2010). Second, I arrange distributions for all types of off-the-equilibrium paths, taking the initial distribution of the simulations as the previous proxy distribution for each aggregate state. Finally, I modify the construction of the reference distributions in Reiter (2002, 2010), reflecting the features of economies in the MPE wherein how many times a policy off-equilibrium takes place is unknown before simulations.

Here, I show how to apply the algorithm to the case with proportional income taxes. Note that I solve all the value functions in the following steps with the Endogenous Grid Method of Carroll (2006).

A.1 Notation and Sketch of the Solution Method

The aggregate law of motion Γ and the tax policy function Ψ are evolved with the distribution μ that is an infinite dimensional equilibrium object, and thus it not not feasible in computations. To handle this issue, the Backward Induction method replaces μ with m, a set of moments from the distribution and discretize it. Here, I take the mean of the distribution and discretize it, $M = \{m_1, \cdots, m_{N_m}\}$. Furthermore, I discretze the tax policy, $T = \{\tau_1, \cdots, \tau_{N_\tau}\}$. This setting allows me to define the aggregate law motion and the tax policy function on each grid (m_{i_m}, τ_{i_τ}) such that $m' = G(m_{i_m}, \tau_{i_\tau}, \tau')$ where $\tau' = P(m_{i_m}, \tau_{i_\tau})$. Note that G and P do not rely on a parametric law.

Across a grid of aggregate states (m_{i_m}, τ_{i_τ}), each point selecting a proxy distribution, the Backward Induction method simultaneously solves for households' decision rules and an intratemporally consistent end-of-period distribution. This implies a future approximate aggregate state consistent with households' expectation ($m' = G(m_{i_m}, \tau_{i_\tau}, \tau')$). Likewise, the backward induction can find the tax policy function that is consistent with the choice of the government, by using household's value functions and the proxy distribution ($\tau^m = \tau' = P(m_{i_m}, \tau_{i_\tau})$). Theses mappings imply that G interacts with P. Given P, first, I find G during the iteration of value functions, and then update P with the value function and proxy distribution. I repeat this until P is convergent.

Given a distribution over individual states at each aggregate grid point (m_{i_m}, τ_{i_τ}), my goal is to obtain the law of motion for households distribution G and the tax policy function P that are intratemporally consistent with the end-of-period distribution and the choice of the government. Explicitly,

$$m' = G(m_{i_m}, \tau_{i_\tau}, \tau') \tag{37}$$

$$\tau' = P(m_{i_m}, \tau_{i_\tau}) \tag{38}$$

$$\tau' = \tau^m(m_{i_m}, \tau_{i_\tau}) \tag{39}$$

$$w = W(m_{i_m}, \tau_{i_\tau}) \tag{40}$$

$$T = TR(m_{i_m}, \tau_{i_\tau}) \tag{41}$$

(37) is to approximate Γ, (38) is to do Ψ, (39) is for the choice of the government, (40) is the mapping for the market wage, and (41) is the mapping for transfers.

The backward induction method explicitly computes G, P, τ^m, W, and T, given a set of proxy distributions before the simulation step. An issue is that computing $G(m_{i_m}, \tau_{i_\tau}, \tau')$ in solving the value is costly because it depends on τ' not only on the equilibrium path but also off-the-equilibrium path. To address this issue, I reduce $G(m_{i_m}, \tau_{i_\tau}, \tau')$ into $\tilde{G}(m_{i_m}, \tau_{i_\tau}) = G(m_{i_m}, \tau_{i_\tau}, P(m_{i_m}, \tau_{i_\tau}))$ while solving the value function; retrieve $G(m_{i_m}, \tau_{i_\tau}, \tau')$ with the converged value function and the proxy distribution. Note that $G(m_{i_m}, \tau_{i_\tau}, \tau')$ must also satisfy an intratemporal consistency.

A.2 Computing the Aggregate Mappings given a Set of Proxy Distributions

(1) Given $v^n(a, \epsilon; m, \tau)$ and $\tau' = P^q(m, \tau)$, where $n = 1, 2, \cdots$ and $q = 1, 2, \cdots$ denote the rounds of iteration, at grid (m_{i_m}, τ_{i_τ}), where $i_m = 1, \cdots, N_m$ and $i_\tau = 1, \cdots, N_\tau$ are grid indexes, solve for intratemporally consistent m'.

a) Guess m' . Using v^n and P^q, solve for $a' = g_a^{n+1}(a, \epsilon; m_{i_m}, \tau_{i_\tau})$ and $n = g_n^{n+1}(a, \epsilon; m_{i_m}, \tau_{i_\tau})$ using

$$v^{n+1}(a, \epsilon; m_{i_m}, \tau_{i_\tau}) = \max_{c, a', n} u(c, 1-n) + \beta \sum_{\epsilon'} \pi_{\epsilon, \epsilon'} v^n(a', \epsilon', m', \tau') \tag{42}$$

such that

$$c + a' = (1 - \tau_{i_\tau}) w(m_{i_m}, \tau_{i_\tau}) \epsilon n + (1 + (1 - \tau_{i_\tau}) r(m_{i_m}, \tau_{i_\tau})) a + T(m_{i_m}, \tau_{i_\tau})$$

$$\tau' = P^q(m_{i_m}, \tau_{i_\tau})$$

b) Using the proxy distribution, $\mu(a, \epsilon; m_{i_m}, \tau_{i_\tau})$, compute the distribution consistent with capital stock in the end of period $\tilde{m}'$, wage $\tilde{w}$, and transfers $\tilde{T}$.

$$\tilde{m}' = \int g_a^{n+1}(a, \epsilon; m_{i_m}, \tau_{i_\tau}) \mu(\mathbf{d}(a \times \epsilon); m_{i_m}, \tau_{i_\tau}) \tag{43}$$

$$\tilde{w} = (1 - \theta) \left(\frac{m_i}{N} \right)^\theta \tag{44}$$

$$\tilde{T} = \tau_{i_\tau} (r(m_{i_m}, \tau_{i_\tau}) m_i + w(m_{i_m}, \tau_{i_\tau}) N) \tag{45}$$

where

$$N = \int g_n^{n+1}(a, \epsilon; m_{i_m}, \tau_{i_\tau}) \epsilon \, \mu(\mathbf{d}(a \times \epsilon); m_{i_m}, \tau_{i_\tau})$$

c) If $\max \{ |\tilde{m}' - m'|, |\tilde{w} - w|, |\tilde{T} - T| \}$ >precision, update m', w, and T; set $r = \theta \left(\frac{w}{1-\theta} \right)^{\frac{\theta - 1}{\theta}} - \delta$; and return to a).

(2) Having found $m' = \tilde{G}^q(m_{i_m}, \tau_{i_\tau})$, $w = W^q(m_{i_m}, \tau_{i_\tau})$, and $T = TR^q(m_{i_m}, \tau_{i_\tau})$, use (42) to define $v^{n+1}(a, \epsilon; m, \tau)$ consistent with $v^n(a', \epsilon'; G^q(m_{i_m}, \tau_{i_\tau}), P^q(m_{i_m}, \tau_{i_\tau}))$. If $||v^{n+1} - v^n|| >$ precision, $n = n + 1$ and return to (1).

(3) For each aggregate grid $(m_{i_m}, \tau_{i_\tau}, \tau'_{i_\tau})$, retrieve $G^q(m_{i_m}, \tau_{i_\tau}, \tau'_{i_\tau})$ by solving for intratempollay consistent $\hat{m}'$.

a) For each $(m_{i_m}, \tau_{i_\tau}, \tau'_{i_\tau})$, guess $\hat{m}'$. With v^∞, solve for $a' = \hat{g}_a(a, \epsilon; m_{i_m}, \tau_{i_\tau}, \tau'_{i_\tau})$ and $n = \hat{g}_n(a, \epsilon; m_{i_m}, \tau_{i_\tau}, \tau'_{i_\tau})$ using

$$\hat{v}(a, \epsilon; m_{i_m}, \tau_{i_\tau}, \tau'_{i_\tau}) = \max_{c,a',n} u(c, 1-n) + \beta \sum_{\epsilon'} \pi_{\epsilon,\epsilon'} v^\infty(a', \epsilon', m', \tau'_{i_\tau})$$

such that

$$c + a' = (1 - \tau_{i_\tau})\hat{w}(m_{i_m}, \tau_{i_\tau}, \tau'_{i_\tau})\epsilon n + (1 + (1 - \tau_{i_\tau})\hat{r}(m_{i_m}, \tau_{i_\tau}, \tau'_{i_\tau}))a + \hat{T}$$

b) For each $(m_{i_m}, \tau_{i_\tau}, \tau'_{i_\tau})$, using the proxy distribution, $\mu(a, \epsilon; m_{i_m}, \tau_{i_\tau})$, compute the distribution consistent with the end of period aggregate capital stock.

$$\tilde{m}' = \int \hat{g}_a(a, \epsilon; m_{i_m}, \tau_{i_\tau}, \tau'_{i_\tau})\mu(\mathbf{d}(a \times \epsilon); m_{i_m}, \tau_{i_\tau})$$

$$\tilde{w} = (1-\theta)\left(\frac{m_i}{N}\right)^\theta$$

$$\tilde{T} = \tau_{i_\tau}(\hat{r} m_i + \hat{w} N)$$

where

$$N = \int \hat{g}_n(a, \epsilon; m_{i_m}, \tau_{i_\tau}, \tau'_{i_\tau})\epsilon\, \mu(\mathbf{d}(a \times \epsilon); m_{i_m}, \tau_{i_\tau})$$

c) If $\max\left\{|\tilde{m}' - \hat{m}'|, |\tilde{w} - \hat{w}|, |\tilde{T} - \hat{T}|\right\} >$ precision, update $\hat{m}', \hat{w}$, and $\hat{T}$; set $\hat{r} = \theta\left(\frac{\hat{w}}{1-\theta}\right)^{\frac{\theta-1}{\theta}} - \delta$; and return to a).

(4) Having found $m' = G^q(m_{i_m}, \tau_{i_\tau}, \tau'_{i_\tau})$, keep $G^q(m_{i_m}, \tau_{i_\tau}, \tau'_{i_\tau})$. Note that here is no update of the value.

(5) For each aggregate grid (m_{i_m}, τ_{i_τ}), find $\tau^{m,q}(m_{i_m}, \tau_{i_\tau})$.

a) Given $(a, \epsilon; m_{i_m}, \tau_{i_\tau})$, using $\hat{v}(a, \epsilon; m_{i_m}, \tau_{i_\tau}, \tau'_{i_\tau})$ in (3) - a), solve $\Psi^q(m, \tau)$ as follows:

$$\Psi^q(m_{i_m}, \tau_{i_\tau}) = \underset{\tilde{\tau}'}{\text{argmax}}\, \hat{V}(m_{i_m}, \tau_{i_\tau}, \tilde{\tau}') = \int \hat{v}(a, \epsilon; m_{i_m}, \tau_{i_\tau}, \tilde{\tau}')\mu(\mathbf{d}(a \times \epsilon); m_{i_m}, \tau_{i_\tau}) \tag{46}$$

The golden section search is used to find $\Psi^q(m_{i_m}, \tau_{i_\tau})$ with a cubic spline for $\hat{V}$ over τ'.

b) For each aggregate grid (m_{i_m}, τ_{i_τ}), if $P^q(m_{i_m}, \tau_{i_\tau}) = \Psi^q(m_{i_m}, \tau_{i_\tau})$, G^q and P^q are the solutions, given the proxy distribution. Then, go to the next step. Otherwise, they are not the solutions. Take $P^{q+1} = \omega \cdot P^q + (1 - \omega) \cdot \Psi^q$, and go back to (1).

A.3 Constructing the Reference Distributions

Until now, I have solved G and P for a given set of proxy distributions. In the following step, I will simulate the economy and update the distribution selection function, as in Reiter (2002, 2010); but, the simulation step in this paper is substantially different from that in his method. He addresses Krusell and Smith (1998) model where aggregate uncertainty exists. Thus, what matters in his papers is to obtain the Ergodic set that is not affected by the initial distribution.

However, in economies without government commitment, it is important to obtain not only distributions on the equilibrium path but also those off-the-equilibrium path. For example, let us think of an economy without commitment in the stationary equilibrium. Then, there will be a unique value of $\tau^* = P(m^*, \tau^*)$ and $m^* = G(m^*, \tau^*, \tau^*)$. In this case, I may not know the value of other alternatives because this economy has nothing but the unique equilibrium path. This difficulty might lead the previous studies to employ local solution methods in solving this type of the MPE. By constrast, my approach is a global solution method, which means I need proxy distributions over all types of off-the-equilibrium paths.

To reserve distributions off-the-equilibrium path, I use the proxy distributions in the previous step as the initial distribution for the simulation. For each (m_{i_m}, τ_{i_τ}), I run a simulations for T periods from the proxy distribution $\mu_0 = \mu(a, \epsilon; m_{i_m}, \tau_{i_\tau})$, implying the number of simulations is $N_m \times N_\tau$ and that of simulation outcomes is $T \times N_m \times N_\tau$. Note that any type of (m_{i_m}, τ_{i_τ}) will be observed at least once in the simulations. For each (m_{i_m}, τ_{i_τ}), using $\mu_0 = \mu(a, \epsilon; m_{i_m}, \tau_{i_\tau})$ and v^∞ from the previous step, I simulate the economy in a forward manner. I compute the market cleared w_t and r_t and transfers T_t satisfying the government budget condition for each simulation period $t = 1, \cdots, T$. In addition, I solve the government's problem τ_t^m for each simulation period $t = 1, \cdots, T$ with the $m' = G(m_{i_m}, \tau_{i_\tau}, \tau'_{i_\tau})$ obtained in the previous step.

I gather all the simulated distributions and rearrange the index as $\tilde{t} = 1, \cdots, T \times N_m \times N_\tau$.

In creating the reference distributions from the simulation, I need a measure of distance for the moments of a distribution. For (m, τ), define an inverse norm

$$d((m_0, \tau_0), (m_1, \tau_1)) = (m_0 - m_1)^{-4} + (\tau_0 - \tau_1)^{-4} \tag{47}$$

In contrast to an economy with uncertainty, the initial simulation results should be preserved, having to be used to construct the reference distributions off-the-equilibrium path (non-Ergodic set). For each t, when (m_t, τ_t) with $m_t \in [m_k, m_{k+1})$ and $\tau_t \in [\tau_s, \tau_{s+1})$,

$$\begin{aligned}
d_1(m_k, \tau_s) &= d_1(m_k, \tau_s) + (m_t - m_k)^{-4} + (\tau_t - \tau_s)^{-4} \\
d_1(m_{k+1}, \tau_s) &= d_1(m_{k+1}, \tau_s) + (m_t - m_{k+1})^{-4} + (\tau_t - \tau_s)^{-4} \\
d_1(m_k, \tau_{s+1}) &= d_1(m_k, \tau_{s+1}) + (m_t - m_k)^{-4} + (\tau_t - \tau_{s+1})^{-4} \\
d_1(m_{k+1}, \tau_{s+1}) &= d_1(m_{k+1}, \tau_{s+1}) + (m_t - m_{k+1})^{-4} + (\tau_t - \tau_{s+1})^{-4}
\end{aligned}$$

Above m_k (τ_s) is the k-th (s-th) grid point for m (τ). Note that distances between a given node and non-adjacent moments are not taken into account, which is different from the corresponding step in Reiter (2002, 2010).

I construct the reference distributions for each (m_{i_m}, τ_{i_τ}) using the above, when $(m_{\tilde{t}}, \tau_{\tilde{t}}) \in \Big([m_{i_m}, m_{i_m+1}), [\tau_{i_\tau}, \tau_{i_\tau+1})\Big)$,

$$\mu^r(a, \epsilon; m_{i_m}, \tau_{i_\tau}) = \sum_{\tilde{t}=1}^{T \times N_m \times N_\tau} \frac{d((m_{i_m}, \tau_{i_\tau}), (m_{\tilde{t}}, \tau_{\tilde{t}}))}{d_1(m_{i_m}, \tau_{i_\tau})} \mu_{\tilde{t}}(a, \epsilon). \tag{48}$$

Each reference distribution is a weighted sum of distributions over the simulation only when simulated moments are adjacent to a given pair of grid points (m_{i_m}, τ_{i_τ}). Since the simulation moments are not on an Ergodic set, this should be considered.

I arrange the finite grid, which is the distribution support, as explicit. The distribution over (a, ϵ) used below size $(N_a \times N_\epsilon)$ with $\epsilon \in E = \{\epsilon_1, \cdots, \epsilon_{N_\epsilon}\}$ and $a \in A = \{a_1, \cdots, a_{N_a}\}$. I represent $\mu^r(a, \epsilon; m_{i_m}, \tau_{i_\tau})$ using $\mu^r_{i_a, i_\epsilon}(i_m, i_\tau)$, indexing $(a_{i_a}, \epsilon_{i_\epsilon})$ over $A \times E$ for (m_{i_m}, τ_{i_τ}). The moment of a reference distribution, $\sum_{i_\epsilon}^{N_\epsilon} \mu^r_{i_a, i_\epsilon}(i_m, i_\tau) a_{i_a}$, will not be consistent with m_{i_m}. However, the proxy distribution at (i_m, i_τ) will have this property.

A.4 Updating the Proxy Distributions

Following Reiter (2002, 2010), for each aggregate grid (i_m, i_τ), I solve for μ_{i_a, i_ϵ}, the proxy distribution, as the solution to a problem that minimizes the distance to the reference distribution while imposing that each type of sums to its reference value and moment consistency.

$$\min_{\{\mu_{i_a,i_\epsilon}\}_{i_a=1,i_\epsilon=1}^{N_a,N_\epsilon}} \sum_{i_a=1}^{N_a}\sum_{i_\epsilon=1}^{N_\epsilon}\left(\mu_{i_a,i_\epsilon} - \mu^r_{i_a,i_\epsilon}(i_m, i_\tau)\right)^2 \tag{49}$$

$$\sum_{i_a=1}^{N_a} \mu_{i_a,1_\epsilon} = \sum_{i_a=1}^{N_a} \mu^r_{i_a,i_\epsilon}(i_m, i_\tau) \text{ for } i = 1, \cdots, N_\epsilon \tag{50}$$

$$\sum_{i_\epsilon=1}^{N_\epsilon}\sum_{i_a=1}^{N_a} \mu_{i_a,i_\epsilon} \cdot a_{i_a} = m_{i_m} \tag{51}$$

$$\mu_{i_a,i_\epsilon} \geq 0 \tag{52}$$

The first-order condition for μ_{i_a,i_ϵ} with λ_i as the LaGrange multiplier for (50) and ω the multiplier (51) is

$$2(\mu_{i_a,i_\epsilon} - \mu^r_{i_a,i_\epsilon}(i_m, i_\tau)) - \lambda_i - \omega \cdot a_{i_a} = 0 \tag{53}$$

If I ignore the non-negative constraints for probabilities in (52), I have N_ϵ constraint in (50). 1 constraint in (51) and $N_a \times N_\epsilon$ first-order conditions in (52). These are a system of $N_a \times N_\epsilon + N_\epsilon + 1$ linear equations in $\left(\{\mu_{i_a,i_\epsilon}\}_{i_a=1,i_\epsilon=1}^{N_a,N_\epsilon}, \{\lambda_{i_\epsilon}\}_{i_\epsilon}^{N_\epsilon}, \omega\right)$.

I construct a column vector **x**. The first block of **x** are the stack of the elements from the proxy distribution, such that $\mathbf{x}(j) = \mu_{i_a,i_\epsilon}$ where $j = (i_\epsilon - 1) \times N_a + i_a$. Next are the N_ϵ multipliers λ_i, followed by one multiplier ω. I solve for **x** using a system of linear equations, $\mathbf{Ax} = \mathbf{b}$ in Figure 12. The non-zero element of **A** and **b** are described here. The coefficients for μ_{i_a,i_ϵ} are entered into **A** as

$$\mathbf{A}((i_\epsilon - 1) \times N_a + i_a, (i_\epsilon - 1) \times N_a + i_a) = 2 \tag{54}$$

$$\mathbf{A}(N_\epsilon \times N_a + i_\epsilon, (i_\epsilon - 1) \times N_a + i_a) = 1 \text{ for } i_\epsilon = 1, \cdots, N_\epsilon \tag{55}$$

$$\mathbf{A}(N_\epsilon \times N_a + N_\epsilon + 1, (i_\epsilon - 1) \times N_a + i_a)) = a_{i_a}. \tag{56}$$

The coefficient for λ_i are entered in **A**, for $i_\epsilon = 1, \cdots N_\epsilon$ and $i_a = 1, \cdots, N_a$, as

$$\mathbf{A}((i_\epsilon - 1) \times N_a + i_a, N_\epsilon \times N_a + i_\epsilon) = -1 \tag{57}$$

The coefficients for ω sets the following elements of **A**, for $i_\epsilon = 1, \cdots N_\epsilon$ and $i_a = 1, \cdots, N_a$,

$$\mathbf{A}((i_\epsilon - 1) \times N_a + i_a, N_\epsilon \times N_a + N_\epsilon + 1) = -a_{i_a}. \tag{58}$$

The elements of **b** are, for $i_\epsilon = 1, \cdots N_\epsilon$ and $i_a = 1, \cdots, N_a$,

$$\mathbf{b}((i_\epsilon - 1) \times N_a + i_a) = 2\mu^r_{i_a, i_\epsilon}(i_m, i_\tau) \tag{59}$$

$$\mathbf{b}(N_\epsilon \times N_a + i_\epsilon) = \sum_{i_a=1}^{N_a} \mu^r_{i_a, i_\epsilon}(i_m, i_\tau) \tag{60}$$

$$\mathbf{b}(N_\epsilon \times N_a + N_\epsilon + 1) = m_{i_m}. \tag{61}$$

I solve $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$ iteratively using an active set method corresponding to probabilities that are not set to 0.

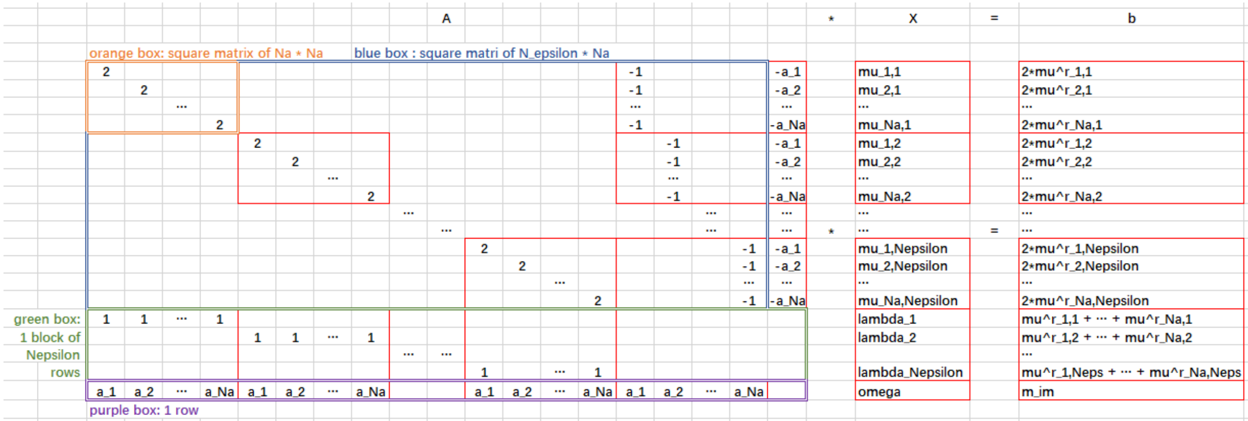

Figure 12: $\mathbf{A} \times \mathbf{x} = \mathbf{b}$

To solve the linear system, I use the active set approach to non-negative constraints in Reiter (2002, 2010). If any of the first $N_\epsilon \times N_a$ elements of **x** are negative, the constraint $\mu_{i_a, 1_\epsilon} \geq 0$ has been violated for some $(i_\epsilon - 1)N_a + i_a = j \in J_0$ where

$$J_0 = \{j | 1 \leq j \leq N_a \times N_\epsilon \text{ and } \mathbf{x}(j) < 0\}. \tag{62}$$

For some $O > 0$, set the most negative O elements indexed in J_0 to 0, $\mu_{i_a, i_\epsilon} = 0$. Remove the $j - th$ row and column of A along with the $j - th$ element of **b**. Solve the reduced system with O less rows. If any of the $N_\epsilon \times (N_a - O)$ elements are negative, again discard the most negative O. I repeat this procedure until the most negative elements of **x** is larger than a precision level. This iteratively implements the non-negativity of probabilities (62).

Table 6 shows the setting of the grids in this paper. With this setting, I continue to repeat the whole steps above until no improvement in accuracy statistic proposed by Den Haan (2010). I find that the mean errors on the equilibrium path are sufficiently small (considerably less than 0.02% for all cases) and the maximum of the errors are also reasonably small (not exceeding 0.03% for all cases). Furthermore, the method is substantially efficient in a usual personal computer.

Table 6: Setting for Computation

	num. of nodes	**Description**
N_a	400(400)	asset (distribution)
N_ϵ	10	persistence wage process
N_m	10	aggregate capital (aggregate)
N_τ	9	income tax (aggregate)

Table 7: Accuracy and Efficiency of the Solution Method

	Proportional Income Tax
Run time	51.43 min
AVG(DH) of m	0.014%
AVG(DH) of w	0.004%
AVG(DH) of τ	0.006%
MAX(DH) of m	0.018%
MAX(DH) of w	0.013%
MAX(DH) of τ	0.022%

$AVG(\cdot)$ and $MAX(\cdot)$ are computed on the equilibrium path.
Processor: AMD Ryzen Threadripper 3960X @ 3.8GHz, RAM: 256GB

Appendix B Ramsey Problem with Capital Income Tax

The results are consistent with those in Dyrda and Pedroni (2023). This front-loaded capital income tax is to quickly reduce overall inequality. Afterward, the Ramsey planner balances between inequality and redistribution over the transition path.

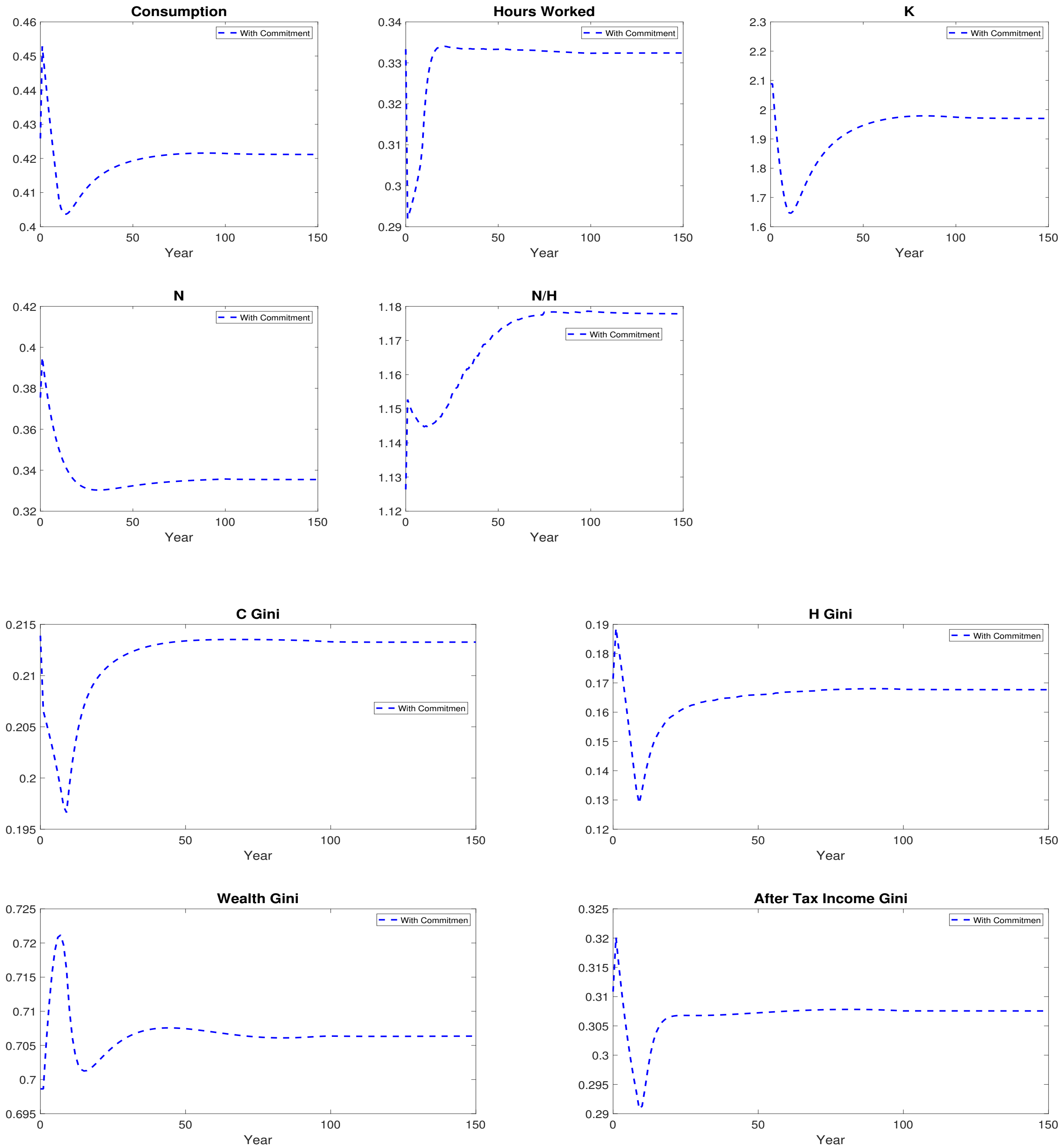

Figure 13: Results with Capital Income Tax in the Ramsey Problem

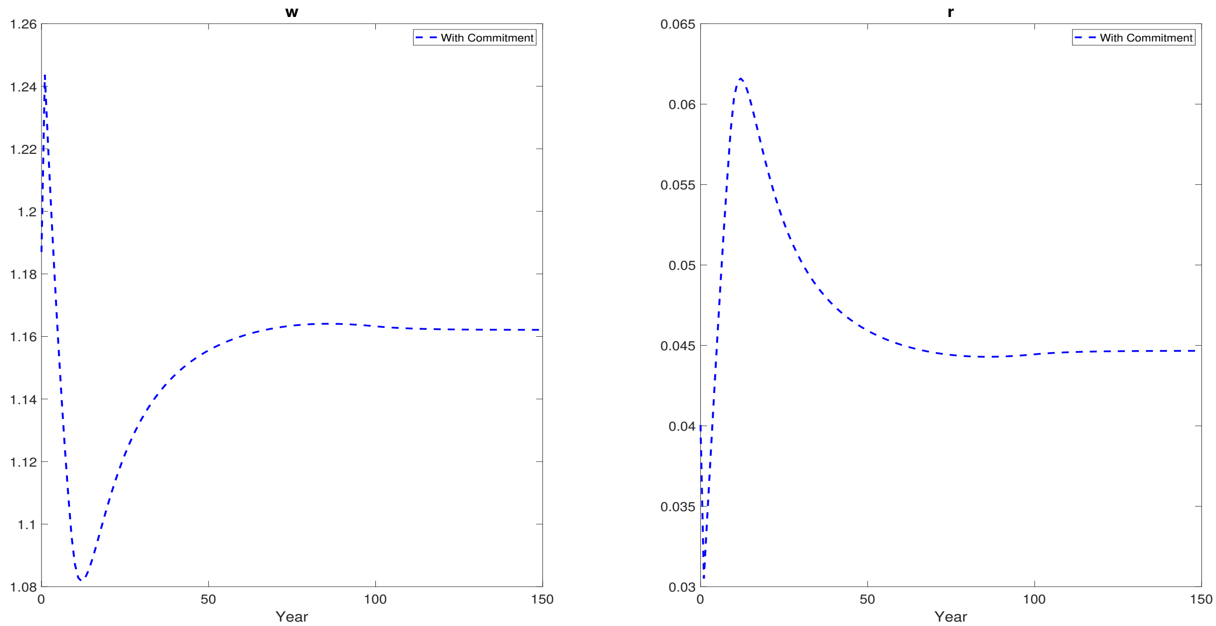

Figure 14: Time-Inconsistent Capital Income Taxes: Dynamics of w and r

Appendix C Definition of $\frac{\Delta K_{t+s}}{\Delta\tau_{t+1}}$ and $\frac{\Delta\tau_{t+s}}{\Delta\tau_{t+1}}$

$$\frac{\Delta K_{t+s}}{\Delta\tau_{t+1}} = \frac{\partial K_{t+1}}{\partial\tau_{t+1}} \cdot \frac{\Xi\Gamma_{t+s-1}}{\Xi K_{t+1}} + \frac{\Xi\Gamma_{t+s-1}}{\Xi\tau_{t+1}}$$

$$\frac{\Delta\tau_{t+s}}{\Delta\tau_{t+1}} = \frac{\partial K_{t+1}}{\partial\tau_{t+1}} \cdot \frac{\Xi\Psi_{t+s-1}}{\Xi K_{t+1}} + \frac{\Xi\Psi_{t+s-1}}{\Xi\tau_{t+1}}$$

where

$$\frac{\Xi\Gamma_{t+s-1}}{\Xi K_t} = F^K_{t+s}(F^K_{t+s-1}, G^K_{t+s-1}) = \begin{cases} \Gamma_{K_t} & \text{if s=1} \\ \Gamma_{K_{t+s-1}} F^K_{t+s-1} + \Gamma_{\tau_{t+s-1}} G^K_{t+s-1} & \text{if } s \geq 2 \end{cases}$$

$$\frac{\Xi\Psi_{t+s-1}}{\Xi K_t} = G^K_{t+s}(F^K_{t+s-1}, G^K_{t+s-1}) = \begin{cases} \Psi_{K_t} & \text{if s=1} \\ \Psi_{K_{t+s-1}} F^K_{t+s-1} + \Psi_{\tau_{t+s-1}} G^K_{t+s-1} & \text{if } s \geq 2 \end{cases}$$

$$\frac{\Xi\Gamma_{t+s-1}}{\Xi\tau_t} = F^\tau_{t+s}(F^\tau_{t+s-1}, G^\tau_{t+s-1}) = \begin{cases} \Gamma_{\tau_t} & \text{if s=1} \\ \Gamma_{K_{t+s-1}} F^\tau_{t+s-1} + \Gamma_{\tau_{t+s-1}} G^\tau_{t+s-1} & \text{if } s \geq 2 \end{cases}$$

$$\frac{\Xi\Psi_{t+s-1}}{\Xi\tau_t} = G^\tau_{t+s}(F^\tau_{t+s-1}, G^\tau_{t+s-1}) = \begin{cases} \Psi_{\tau_t} & \text{if s=1} \\ \Psi_{K_{t+s-1}} F^\tau_{t+s-1} + \Psi_{\tau_{t+s-1}} G^\tau_{t+s-1} & \text{if } s \geq 2 \end{cases}$$