Centre interuniversitaire de
recherche en économie quantitative

Cahier 01-2021

Non-Manipulable House Exchange under (Minimum) Equilibrium Prices

Lars-Gunnar Svensson, Tommy Andersson and Lars Ehlers

www.cireqmontreal.com

@CIREQMTL

facebook.com/CIREQ

Le **Centre interuniversitaire de recherche en économie quantitative (CIREQ)** regroupe des chercheurs dans les domaines de l'économétrie, la théorie de la décision, la macroéconomie et les marchés financiers, la microéconomie appliquée ainsi que l'économie de l'environnement et des ressources naturelles. Ils proviennent principalement des universités de Montréal, McGill et Concordia. Le CIREQ offre un milieu dynamique de recherche en économie quantitative grâce au grand nombre d'activités qu'il organise (séminaires, ateliers, colloques) et de collaborateurs qu'il reçoit chaque année.

*The **Center for Interuniversity Research in Quantitative Economics (CIREQ)** regroups researchers in the fields of econometrics, decision theory, macroeconomics and financial markets, applied microeconomics as well as environmental and natural resources economics. They come mainly from the Université de Montréal, McGill University and Concordia University. CIREQ offers a dynamic environment of research in quantitative economics thanks to the large number of activities that it organizes (seminars, workshops, conferences) and to the visitors it receives every year.*

Cahier 01-2021

Non-Manipulable House Exchange under (Minimum) Equilibrium Prices

Lars-Gunnar Svensson, Tommy Andersson and Lars Ehlers

CIREQ, Université de Montréal
C.P. 6128, succursale Centre-ville
Montréal (Québec) H3C 3J7
Canada
Téléphone : (514) 343-6557
Télécopieur : (514) 343-7221
cireq@umontreal.ca
http://www.cireqmontreal.com

Ce cahier a également été publié par le Département de sciences économiques de l'Université de Montréal sous le numéro (2021-01).

This working paper was also published by the Department of Economics of the University of Montreal under number (2021-01).

Dépôt légal - Bibliothèque nationale du Canada, 2021, ISSN 0821-4441
Dépôt légal - Bibliothèque et Archives nationales du Québec, 2021
ISBN-13 : 978-2-89382-776-6

Non-Manipulable House Exchange under (Minimum) Equilibrium Prices

Lars-Gunnar Svensson[*], Tommy Andersson[†] and Lars Ehlers[‡]

December 15, 2020

Abstract

We consider a market with indivisible objects, called houses, and money. On this market, each house is initially owned (or rented) by some agent and each agent demands precisely one house. The problem is to identify the complete set of direct allocation mechanisms that can be used to reallocate the houses among the agents. The focus is on price mechanisms, i.e., mappings of preference profiles to price equilibria, that are strategy-proof and satisfy an individual rationality condition. We prove that the only mechanism that satisfies these conditions is a price mechanism with a minimal equilibrium price vector. The result is not true in full preference domain. Instead, we identify a smaller domain, that contains almost all profiles, where the result holds.

JEL Classification: C71; C78; D71; D78.
Keywords: Public housing; existing tenants; equilibrium; minimum equilibrium prices; domain.

1 Introduction

This paper analyzes a market with indivisible objects, called houses, and money. There is the same finite number of agents and houses, and each agent demands precisely one house. Each house is primarily assigned to an agent, and we think of the primary assignment as a distribution

[*]Department of Economics, Lund University, Box 7082, 220 07 Lund, Sweden; e-mail: lars-gunnar.svensson@nek.lu.se.

[†]Department of Economics, Lund University, Box 7082, 220 07 Lund, Sweden; e-mail: tommy.andersson@nek.lu.se.

[‡]Département de Sciences Économiques and CIREQ, Université de Montréal, C.P. 6128, Succursale Centre-Ville, Montréal, Québec H3C 3J7, Canada; e-mail: lars.ehlers@umontreal.ca.

of endowments, i.e., that each agent initially owns a house. Other interpretations are also possible, for instance that each agent rents a house, owned by a local government authority, and a market is opened where the houses can be sold to those who are renting.[1]

We examine the problem to identify a specific set of direct allocation mechanisms that can be used to reallocate the houses among the agents. Our focus is on price mechanisms, i.e., mappings of preference profiles to price equilibria, that are also strategy-proof and satisfy an individual rationality condition. Here, individual rationality means that an agent buys a house only if this is a weakly better alternative than the option to keep the house that she owns (or continue renting the house she currently lives in). Unfortunately, the characterization problem has no solution in the full preference domain. As a consequence, one of the main tasks in the paper is to reduce the preference domain, partly to obtain a solution, partly to make the restriction of the domain in such a way that it makes the reduction "negligible." The latter means that the characterization result is true for "almost all" profiles in the full preference domain.

Andersson, Ehlers and Svensson (2016) analyzed price mechanisms in the same setting as the one considered in this paper. They demonstrated that a minimum price mechanisms is strategy-proof on a restricted preference domain that contains almost all preference profiles. They also showed that the mechanism is manipulable in the full preference domain if the number of agents is strictly greater than three. The present paper considers the more fundamental question of characterizing the entire class of strategy-proof price mechanisms. The main finding demonstrates that there is a restricted domain, containing almost all preference profiles, such that this class contains only one mechanism, namely the minimum price mechanism.

The seminal contribution by Hurwicz (1972) showed the manipulability of Walrasian price mechanisms in classical exchange economies with divisible commodities. As explained in the above, this conclusion also holds on the full preference domain in our model with indivisible objects and money, but the minimum price mechanism is the only non-manipulable (equilibrium) price mechanism on a restricted domain containing almost all profiles. This is also a key difference to the literature discussed below which always considers the full preference domain.

The strategy-proofness property of the minimum price mechanism is well-known from a number of papers analyzing equilibrium in two-sided housing markets, i.e., markets where buyers and sellers are distinct groups and, hence, buyers do not initially own houses. Vickrey's (1961) second-price auction model with one house and a number of bidders is one example where strategy-proofness is achieved. In generalizations to multi-object models, e.g., Demage and Gale (1985), Leonard (1983), Sun and Yang (2003), it is shown that the minimum price mechanism satisfies strategy-proofness.[2] The characterization problem for two-sided markets is

[1] This interpretation was thoroughly discussed by Andersson, Ehlers and Svensson (2016) in connection to the "Right to Buy Act" in the U.K. Housing Act 1980.

[2] For recent one-sided strategy-proofness results in trading networks with money, see e.g. Hatfield, Kojima and Kominers (2017), Jagadeesan, Kominers and Rheingans-Yoo (2018), and Schlegel (2018). See also Fleiner et al. (2019) for trading networks with frictions.

analyzed in, e.g., Miyake (1998), Morimoto and Serizawa (2014), Svensson (2009). The findings are that the minimum price mechanism is the only possible one.

The characterization problem is also analyzed in Ma (1994) in a housing market without money where each agent initially owns one indivisible good. It is shown that the only mechanism that satisfies strategy-proofness and is onto is the core mechanism. Here, the assignment is given by Gale's top trading cycle principle. A similar result is found in Miyagawa (2001) who characterizes the class of mechanisms that are strategy-proof, individual rational, non-bossy[3] and onto in a model with money.

Like the present paper, Miyagawa (2001) studies a housing market where each agent owns one house and can buy another at a given price. However, the non-bossy condition implies that such mechanisms are characterized by a matrix (p_{ah}) of fixed prices, i.e., prices not related to agents' preferences. An agent a pays p_{ah} when he receives house h and leaves his own house. Budget-balance is achieved when $p_{ah} = p_h - p_a$ for some price vector p. Also here, the assignment is obtained from Gale's top trading cycle principle.

A main difference between the problem considered by Miyagawa (2001) and ours is the non-bossiness condition. It is an appealing condition, but it excludes a number of useful mechanisms, e.g., the Vickrey second-price mechanism and its generalizations to multi-objects models. Non-bossiness does not exclude strategy-proofness and budget-balance but price equilibrium. In our model, price equilibrium and strategy-proofness are fundamental properties, and, as a consequence, we have to give up non-bossiness and budget-balance. Hence, in general, with fixed prices allocation inefficiency prevails, while with non budget-balance inefficiency in the form of "waste" of money occurs.

Note, finally, that one interpretation of our model is a two-sided market where the houses are owned, e.g., by a local government authority, and the agents are primarily renting the houses. If the sum of the selling prices exceeds the sum the agents pay, there is simply a positive profit for the seller. But the minimum price mechanism minimizes this profit. Alternatively, the gap between selling and buying prices can be seen as a transaction tax, or in a labor market interpretation, taxes on wages.[4]

The remaining part of this paper is organized as follows. Section 2 introduces the formal model together with assumptions and definitions that are used throughout the paper. The main theorem is given in Section 3, while the concept of a "negligible" subset of preference profiles is defined in Section 4. The entire Section 5 is devoted to the proof of the main theorem. Section 6 provides a foundation for the use of price mechanisms. The proofs of all lemmas are provided in the Appendix.

[3]A mechanism is non-bossy if an agent cannot change the outcome of the mechanism for others without changing the outcome for himself (Satterthwaite and Sonnenschein, 1981).

[4]Taxation in matching markets is also studied in Dupuy, Galichon, Jaffe and Kominers (2017). The focus of their analysis is, however, efficiency, and not incentive properties as in our study.

2 The Formal Model

Let $A = \{1, 2, \ldots, n\}$ be a finite set of *agents* and $H = \{1, 2, \ldots, n\}$ a finite set of *houses*. The endowment of agent $a \in A$ is house $h \in H$ if $h = a$. We consider a market where endowments can be reallocated through a system of equilibrium prices, i.e., a market where each agent can buy a most preferred house. In this market, there are two types of prices, a vector of *fixed* selling prices $\underline{p} \in \mathbb{R}^n$ and a vector $p \in \mathbb{R}^n$ of buying (equilibrium) prices.

We have two different interpretations of the endowments in mind. The first one, call it E_o, entails that each agent a owns house $h = a$. Agent a receives $\underline{p}_a$ when selling house $h = a$, and pays p_h when buying house $h \neq a$. The second one, call it E_r, entails that there is an owner of the houses different from the agents $a \in A$, while each agent a rents house $h = a$. The owner wants to sell the houses to the group of renting agents, but not necessarily house a to agent a. However, agent a has a particular right to the house she is renting; she can continue to rent "her" house, but she has also the option to buy her house to the fixed price $\underline{p}_a$ (see also footnote 1). The fixed price vector $\underline{p}$ defines the owners' reservation prices in E_o as well in E_r. The common feature in the two interpretations of endowments is that continuing to own or rent the "own" house is a fixed alternative in contrast to buying another house, where the price that the agent has to pay depends on the market valuation. In an reallocation process, *individual rationality* means that an agent never is assigned an alternative that is worse than her fixed alternative.

A reallocation of the endowments is given by an *assignment* μ that is a bijection $\mu : A \rightarrow H$. A *state* is a pair $x = (\mu, p)$ of an assignment and a feasible price vector. Here, $x_a = (\mu_a, p)$ means that agent a is assigned house μ_a at the price vector p.

We assume that the sellers' reservation prices $\underline{p}$ constitute a lower bound on *feasible prices*, $p \geq \underline{p}$. Since the lower bound is fixed in the analysis, without loss of generality, let $\underline{p}_h = 0$ for all $h \in H$.

Each agent $a \in A$ has rational *preferences* R_a on houses and prices, i.e., on bundles of type $(h, p_h) \in H \times \mathbb{R}$. To simplify notation, let $(h, p) \equiv (h, p_h)$, i.e., by (h, p) we mean house h at price p_h in price vector p. Preferences are further assumed to be strictly *monotonic*, i.e., $(h, p'_h)\, P_a\, (h, p_h)$ if $p'_h < p_h$, for all houses $h \in H - \{a\}$, while constant for the own house, i.e., $(a, p'_a)\, I_a\, (a, p_a)$ for all $p_a, p'_a \in \mathbb{R}$. The reason for assuming price independence of the own house is simply that an agent a always pays the reservation price $\underline{p}_a = 0$ for the own house, and this reservation price is independent of the price vector p that specifies the buying prices. Finally, preferences are assumed to be *continuous* and *boundedly desirable.* Continuity means that for all $h \in H$, the sets $\{p_h \in \mathbb{R} : (h, p_h)\, R_a\, (h, p'_h)\}$ and $\{p_h \in \mathbb{R} : (h, p'_h)\, R_a\, (h, p)\}$ are closed for all $p'_h \in \mathbb{R}$. Bounded desirability means that if the price of a house is sufficiently high, the agents will strictly prefer to keep the house they are currently living in rather than buying some other house, i.e., $(a, p) P_a (h, p_h)$ for each agent $a \in A$ and for each house $h \in H$ for p_h sufficiently high. However, we do not exclude the case that an agent $a \in A$ does not demand a particular house $h \in H - \{a\}$ to any price p_h, i.e., $(a, 0)\, P_a\, (h, p_h)$ for all p_h.

For $a \in A$, the set of rational, monotonic, continuous and boundedly desirable preferences on $H \times \mathbb{R}$ defined in this way is denoted $\mathcal{R}_a$. A (preference) *profile* is a list $R = (R_a)_{a \in A}$ of agents' preferences. The set of profiles is denoted $\mathcal{R}$, where $\mathcal{R} = \times_{a \in A} \mathcal{R}_a$, and where agent a's preferences are in the set $\mathcal{R}_a$. The notation $\mathcal{R}_{-a}$ is used for the set $\mathcal{R}_{-a} = \times_{a' \in A - \{a\}} \mathcal{R}_{a'}$.

Definition 1. *For $R \in \mathcal{R}$, a state $x = (\mu, p)$ is a weak equilibrium state if (i) for all $a \in A$, $x_a R_a (h, p)$ for all $h \in H$ and (ii) $\mu_a = a$ and $p_a > 0$ only if $x_{a'} I_{a'} (a, p)$ for some $a' \neq a$. It is an equilibrium state if it also satisfies (iii) the number of houses $h \in H$ such that $h = \mu_a = a$ is minimal among all states satisfying (i) and (ii) with price vector p.*

Condition (i) is the usual equilibrium condition, i.e., at prices p each agent has been assigned a most preferred alternative. Note also that the assignment is individually rational since if $\mu_a = a$, agent a pays 0. Condition (ii) is introduced to avoid trivial price vectors. Since an agent's utility of the own house does not depend on the price of his house, any sufficiently high price would be an equilibrium price without condition (ii) in cases where an agent prefers his own house to all other houses. That trade is better than no trade is reflected by condition (iii). This condition does not directly influence the utility of the agents. However, if there is an external owner of the houses and the agents primarily are renting, then the profit of the owner is weakly larger when a house is sold to an agent not renting it than to the agent that rents it.[5]

For a given profile $R \in \mathcal{R}$, the set of equilibria is denoted $\mathcal{E}_R$ and the set of corresponding equilibrium price vectors Π_R. Hence, $p \in \Pi_R$ precisely when there is a state $(\mu, p) \in \mathcal{E}_R$. Moreover, the set of all equilibrium states is denoted $\mathcal{E}$, where $\mathcal{E} = \cup_{R \in \mathcal{R}} \mathcal{E}_R$. Note also that the sets $\mathcal{E}_R$ are nonempty.[6]

3 The Main Result

This section demonstrates that a mechanism which is strategy-proof must be a minimum price mechanism. The result is not true on the entire domain $\mathcal{R}$. Instead, we identify a domain that contains "almost all" profiles, denoted by $\mathring{\mathcal{R}}$, where the result holds. This domain is formally defined in the next section together with another important domain, denoted by $\tilde{\mathcal{R}}$, that was considered by Andersson, Ehlers and Svensson (2016).

Before we can state the main result, we need to define a (minimum) price mechanism and a few concepts related to manipulability.

Definition 2. *A price mechanism, or for short a mechanism, is a mapping $f : \mathcal{R} \to \mathcal{E}$ of profiles to equilibrium states such that $f(R) \in \mathcal{E}_R$ for all $R \in \mathcal{R}$.*

[5]A similar condition is part of the "efficiency condition" in Morimoto and Serizawa (2015).

[6]See Andersson, Ehlers and Svensson (2016, Proposition 1).

A mechanism f is *manipulable* at a profile $R \in \mathcal{R}$ by an agent $a' \in A$ if there is a profile $(R'_{a'}, R_{-a'}) \in \mathcal{R}$ such that for $f(R) = x$ and $f(R'_{a'}, R_{-a'}) = x'$, $x'_{a'} P_{a'} x_{a'}$. Let $\breve{\mathcal{R}} \subset \mathcal{R}$ be a subset of profiles. The mechanism f is *strategy-proof* on the domain $\breve{\mathcal{R}}$ if no agent can manipulate at any profile $R \in \breve{\mathcal{R}}$. Note that if f is strategy-proof on a domain $\breve{\mathcal{R}}$ and $(R_a, R_{-a}) \in \breve{\mathcal{R}}$, then a cannot manipulate by using any preferences R'_a with $(R'_a, R_{-a}) \in \mathcal{R}$.

The use of a minimal price vector will be central in the main characterization result. Let $R \in \breve{\mathcal{R}}$ and denote by $p^m \in \Pi_R$ a price vector that is minimal in Π_R, i.e., if $p \in \Pi_R$ and $p \leq p^m$ then $p = p^m$.

Definition 3. *A mechanism f is a minimum price mechanism on the domain $\breve{\mathcal{R}} \subset \mathcal{R}$ if for all $R \in \breve{\mathcal{R}}$, $f(R) = x \in \mathcal{E}_R$ and $x = (\mu, p^m)$ with p^m minimal in Π_R.*

Andersson, Ehlers and Svensson (2016) demonstrated that if the number of agents is strictly greater than three, a minimal price vector is not necessarily unique on the domain $\mathcal{R}$ and the minimum price mechanism is manipulable on the domain $\mathcal{R}$. They also showed that a minimal price vector is unique on a reduced domain $\tilde{\mathcal{R}}$ and, furthermore, that the minimum price mechanism is strategy-proof on the domain $\tilde{\mathcal{R}}$.

Our objective is to characterize the set of all strategy-proof mechanisms on a domain, denoted $\mathring{\mathcal{R}}$, containing almost all profiles in $\mathcal{R}$. As already stated in the above, the sets $\tilde{\mathcal{R}}$ and $\mathring{\mathcal{R}}$ will be defined in next section, but they are related as follows: $\mathring{\mathcal{R}} \subset \tilde{\mathcal{R}} \subset \mathcal{R}$. We are now ready to present our main result whose proof can be found in Section 5.

Theorem 1. *Let f be a price mechanism. There is a domain $\mathring{\mathcal{R}} \subset \mathcal{R}$ such that $R \in \mathring{\mathcal{R}}$ for almost all $R \in \mathcal{R}$, and the following holds: f is strategy-proof on $\mathring{\mathcal{R}}$ if and only if f is a minimal price mechanism on $\mathring{\mathcal{R}}$.*

Theorem 1 shows that a mechanism which is strategy-proof on the domain $\mathring{\mathcal{R}}$ must be a minimum price mechanism on this domain. For the other direction, we use Theorem 2 in Andersson, Ehlers and Svensson (2016) that shows that minimum price mechanisms are strategy-proof on the domain $\tilde{\mathcal{R}}$ and note that $\mathring{\mathcal{R}} \subset \tilde{\mathcal{R}}$ (see Section 4). Hence, on the domain $\mathring{\mathcal{R}}$, minimum price mechanisms completely characterize the class of strategy-proof price mechanisms.

Note that Theorem 1 is the first characterization result which is obtained for a domain containing almost all profiles (see the discussion in the introduction section). Furthermore, the result does not hold on the full domain. The latter conclusion follows since any minimal price mechanism is manipulable on the domain $\mathcal{R}$ (see Andersson, Ehlers and Svensson, 2016, Proposition 2).

Remark 1. *The result in Theorem 1 may be used to support a (normative) definition of fairness. In a market model with private ownership net trades are considered fair primarily because no agent envies any other agent's net trade. However, in general no envy is not sufficient as a fairness criterion. First, no envy is not sufficient for obtaining a unique allocation and second, fairness should reasonably be based on agent's true preferences. According to Theorem 1,* *the outcome of the minimal price mechanism is envy free and non-manipulable. It is also the only price mechanism that satisfies those two conditions. Hence, the outcome of the minimal price mechanism seems to be a strong candidate for a definition of (procedural) fairness. The foundation for the use of price mechanisms in general is further discussed in Section 6.*

4 Construction of the Subset of Profiles $\mathring{\mathcal{R}}$

The purpose of this section is to construct a reduced preference domain $\mathring{\mathcal{R}} \subset \mathcal{R}$, and show that the subset $\mathcal{R} - \mathring{\mathcal{R}} \subset \mathcal{R}$ can be considered "negligible."

The idea with the concept of a *negligible* subset S' of a set S is that the number of elements in S' is "small" compared to the number of elements in S. For example, that the set S contains an uncountable number of elements while the number of elements in S' is countable. With a measurable set, S may have a positive measure while the measure of S' is zero. The presumption for the analysis is that agents' true preferences are exogenously given by nature. If a set of preference profiles can reasonably be considered improbable to be the outcome of the natural lottery, that set is here considered negligible.

We will define the domain $\mathring{\mathcal{R}} \subset \mathcal{R}$ in two steps. In Subsections 4.1 and 4.2, we define the subsets $\mathcal{R}' \subset \mathcal{R}$ and $\mathcal{R}'' \subset \mathcal{R}$, respectively, and show that both these subsets are considered negligible. Given the constructions of $\mathcal{R}'$ and $\mathcal{R}''$, we define $\mathring{\mathcal{R}} = \mathcal{R} - (\mathcal{R}' \cup \mathcal{R}'')$. Clearly, $\mathcal{R}' \cup \mathcal{R}''$ is negligible, meaning that almost all profiles $R \in \mathcal{R}$ belong to $\mathring{\mathcal{R}}$. By the constructions of $\mathcal{R}'$ and $\mathcal{R}''$, it will also follow that $\mathring{\mathcal{R}} \subset \tilde{\mathcal{R}} \subset \mathcal{R}$. The latter conclusion holds since $\tilde{\mathcal{R}} = \mathcal{R} - \mathcal{R}'$ (see Subsection 4.1).

4.1 Construction of the Subset of Profiles $\mathcal{R}'$

To construct the subset $\mathcal{R}' \subset \mathcal{R}$, the set of profiles $\tilde{\mathcal{R}} \subset \mathcal{R}$ where no two houses are "connected by indifference" will be important.[7] To define this concept formally, let $\mathcal{S}$ be the set of sequences $s = (h_j, a_j)_{j=1}^{q}$ of distinct houses $h_j \in H$ and distinct agents $a_j \in A$ such that $h_1 = a_1$, $h_j \neq a_j$ for all j such that $1 < j \leq q$ and $h_{j+1} \neq a_j$for $j < q$.

Definition 4. *For a given profile $R \in \mathcal{R}$, two distinct houses, $h', h'' \in H$, are connected by indifference if there is a sequence $s \in \mathcal{S}$, and a corresponding price vector $p \in \mathbb{R}^n_+$, such that $h' = a_1$ and $h'' = a_q$, and $(h_j, p) I_{a_j} (h_{j+1}, p)$ for $1 \leq j < q$ and $(h_q, p) I_{a_q} (a_q, p)$. The subset of $\mathcal{R}$ where no two houses are connected by indifference, at any profile, is denoted by $\tilde{\mathcal{R}}$.*

Note that for all such price vectors in Definition 4, prices p_{h_j}, $1 < j \leq q$, are uniquely determined by continuity and monotonicity of the preferences.

[7]The concept of "connected by indifference" was first used in Andersson and Svensson (2014). They used a slightly different version compared to the one presented in this paper.

Let $R \in \mathcal{R}$ be a profile and $a \in A$ an agent, and denote by $\mathcal{R}_{aR}^{con}$ the set of preference profiles $R'_a \in \mathcal{R}_a$ such that there are two houses $h', h'' \in H$, with $h'' = a$, that are connected by indifference at the profile (R'_a, R_{-a}). From Definition 4, it now follows that $\tilde{\mathcal{R}} = \mathcal{R} - \mathcal{R}'$, where:

$$\mathcal{R}' = \{R \in \mathcal{R} : R_a \in \mathcal{R}_{aR}^{con} \text{ for some } a \in A\}.$$

We can think of a profile in $\mathcal{R}'$ as the outcome in two steps of the natural lottery, where, for some agent a, the first outcome is R_{-a} and the second is $R_a \in \mathcal{R}_{aR}^{con}$. Now, if $\mathcal{R}_{aR}^{con}$ can be considered negligible, we can also consider $\mathcal{R}'$ negligible since there is only a finite number of agents and houses.

We next demonstrate that $\mathcal{R}_{aR}^{con}$ can be considered negligible. Note first that if $R'_a \in \mathcal{R}_{aR}^{con}$, there is a sequence $s = (h_j, a_j)_{j=1}^{q}$ and a corresponding price vector $p \in \mathbb{R}_+^n$ such that $h' = a_1$, $h'' = a_q = a$, $(h_j, p) I_{a_j} (h_{j+1}, p)$ for $1 \leq j < q$ and $(h_q, p) I_a (a, p)$. The price p_{h_q} is uniquely determined by the profile R but independent of preferences R_a. Let now preferences R'_a be represented by utility functions u'_{ah}, where $u'_{ah}(p)$ is the agent's willingness-to-pay for house $h \in H$. The indifference $(h_q, p) I'_a (a, p)$ prevails if and only if $u'_{aa}(p) = p_{h_q}$. When preferences are chosen by nature, it is reasonable to assume that $u'_{aa}(p) \neq p_{h_q}$ is the case for most preferences $R_a \in \mathcal{R}_a$. Since there is only a finite number of sequences $s \in \mathcal{S}$, we thus consider the set $\mathcal{R}_{aR}^{con}$ negligible. It then follows, by the above arguments, that also $\mathcal{R}'$ can be considered negligible.

4.2 Construction of the Subset of Profiles $\mathcal{R}''$

The idea underlying the construction of the subset $\mathcal{R}''$ is similar as the one used to construct $\mathcal{R}'$ in the previous subsection, but the construction of the set $\hat{\mathcal{R}}_{aR}^{con}$ is somewhat more involved. The whole point is again to define a set:

$$\mathcal{R}'' = \{R \in \mathcal{R} : R_a \in \hat{\mathcal{R}}_{aR}^{con} \text{ for some } a \in A\},$$

and demonstrate that $\hat{\mathcal{R}}_{aR}^{con}$ can be considered negligible. It then follows, using the same arguments as in the previous subsection, that also $\mathcal{R}''$ can be considered negligible. The set $\hat{\mathcal{R}}_{aR}^{con}$ will be defined in three steps. First, we define utility functions u_a and $\hat{u}_a$ that represent preference relations denoted by R_a and $\hat{R}_a$, respectively. Second, we observe that the profiles $\hat{R}_a$ and R_a can be related through a function g_{aR}. Third, the set $\hat{\mathcal{R}}_{aR}^{con}$ is constructed using the function g_{aR}.

Let $R \in \tilde{\mathcal{R}}$ be a profile and $x^m = (\mu^m, p^m) \in \mathcal{E}_R$ an equilibrium where p^m is minimal in Π_R.[8] Let preferences R_a for agent $a \in A$ be represented by a utility function u_{ah} where $u_{ah}(p)$ for $h \in H$ is the agent's willingness-to-pay for house h, and define $\bar{u}_a = \max_{h \in H} u_{ah}(p^m)$. Let now preferences $\hat{R}_a$ for agent $a \in A$ be represented by a utility function $\hat{u}_{ah}$ according to (i)

[8]Note that by Andersson, Ehlers and Svensson (2016, Theorem 1), p^m is unique when $R \in \tilde{\mathcal{R}}$.

$\hat{u}_{aa}(p) = \bar{u}_a$ and (ii) $\hat{u}_{ah}(p) = u_{ah}(p)$ for all $h \neq a$. Note also that the definition of preferences $\hat{R}_a$ is independent of the particular utility representation u_a of preferences R_a.

Now, preferences $\hat{R}_a$ and R_a can be related according to $\hat{u}_a = g_{aR}(u_a)$, i.e., the function g_{aR} changes the value of agent a's own house, $h = a$, to be the maximal value at the (unique) minimal price vector, while other utilities are unchanged.

In the following, let the preference relations R_a and $\hat{R}_a$ be represented by u_a and $\hat{u}_a$, respectively. For each agent $a \in A$ and profile $R \in \tilde{\mathcal{R}}$, let $\hat{\mathcal{R}}_{aR}$ be the set of preferences:

$$\hat{\mathcal{R}}_{aR} = \{\hat{R}_a \in \mathcal{R}_a : \hat{u}_a = g_{aR}(u_a) \text{ and } R_a \in \mathcal{R}_a\}.$$

Define $\hat{\mathcal{R}}^{con}_{aR}$ according to:

$$\hat{\mathcal{R}}^{con}_{aR} = \{R_a \in \mathcal{R}_a : \hat{R}_a \in \mathcal{R}^{con}_{aR} \text{ where } \hat{u}_a = g_{aR}(u_a)\}.$$

Given that the set $\hat{\mathcal{R}}^{con}_{aR}$ has been defined, it only remains to demonstrate that the set $\hat{\mathcal{R}}^{con}_{aR}$ can be considered negligible. If $R'_a \in \hat{\mathcal{R}}^{con}_{aR}$, there is a sequence $s = (h_j, a_j)_{j=1}^{q}$ and a corresponding price vector $p \in \mathbb{R}^n_+$, such that $h' = a_1$ and $h'' = a_q = a$, and $(h_j, p) I_{a_j} (h_{j+1}, p)$ for $1 \leq j < q$ and $(h_q, p) \hat{I}_a (a, p)$, where preferences R'_a and $\hat{R}_a$ are represented by u'_a and $\hat{u}_a = g_{aR}(u'_a)$, respectively. The price p_{h_q} is uniquely determined by the profile R but independent of preferences R_a. The indifference $(h_q, p)\hat{I}(a, p)$ prevails if and only if $\hat{u}_{aa}(p) = p_{h_q}$. Here, $\hat{u}_{aa}(p) = \max_{h \in H} u_{ah}(p^m) = \bar{u}_a$ since the minimal price vector p^m is the same at profiles R and $(\hat{R}_a, R_{-a})$ according to Lemma 3 in Section 5. When preferences are chosen by nature, it is reasonable to assume that $u_{aa}(p) = \bar{u}_a \neq p_{h_q}$ is the case for most preferences $R_a \in \mathcal{R}_a$. Since there is only a finite number of sequences $s \in \mathcal{S}$, we consider the set $\hat{\mathcal{R}}^{con}_{aR}$ negligible.

4.3 Illustration with Quasi-Linear Preferences

This subsection illustrates the negligible subset of profiles for quasi-linear preferences. For each agent $a \in A$, preferences over bundles (h, p) are quasi-linear if there exists real numbers $(v_{ah})_{h \in H}$ such that R_a is represented by the utility function $u_{ah}(p) = v_{ah} - p_h$. Let $\mathcal{Q}_a \subset \mathcal{R}_a$ denote the set of all quasi-linear preferences, and $\mathcal{Q} = \times_{a \in A} \mathcal{Q}_a$ denote the set of profiles of quasi-linear preferences. Let now $\tilde{\mathcal{Q}} = \mathcal{Q} \cap \tilde{\mathcal{R}}$, and similarly for $\mathcal{Q}'$, $\mathcal{Q}''$ and $\mathring{\mathcal{Q}}$.

Note that any $R_a \in \mathcal{Q}_a$ has a representation of values $(v_{ah})_{h \in H}$ and by adding the same constant to all these values induces the same quasi-linear preferences. Below we use the canonical representation of R_a where $v_{aa} = 0$. Using this convention, $\mathcal{Q}_a$ corresponds to $\mathbb{R}^{n-1}$ and $\mathcal{Q}$ to $\mathbb{R}^{n(n-1)}$.

Let $R \in \mathcal{Q}$ be a profile of quasi-linear preferences. Suppose that houses h_1 and h_{q+1} are connected by indifference, i.e., that there exist sequences of distinct agents $(a_1, \ldots, a_q)$ and distinct houses $(h_1, \ldots, h_{q+1})$, and a price vector p such that:

(i) $h_1 = a_1$ and $h_{q+1} = a_q$,

(ii) $v_{a_1h_1} = v_{a_1h_2} - p_{h_2}$ and $v_{a_qh_q} - p_{h_q} = v_{a_qh_{q+1}}$,

(iii) $v_{a_jh_j} - p_{h_j} = v_{a_jh_{j+1}} - p_{h_{j+1}}$ for $2 \leq j \leq q-1$.

Summing all left-hand sides and all right-hand sides yields:

$$\sum_{j=1}^{q}(v_{a_jh_j} - v_{a_jh_{j+1}}) = 0. \tag{1}$$

Note that (1) is independent of the price vector p and this is a hyperplane in $\mathbb{R}^{n(n-1)}$ with measure zero in $\mathcal{Q}$. As the set of houses and the set of sequences is finite, it follows that $\mathcal{Q}'$ has measure zero in $\mathcal{Q}$ and $\mathcal{Q}'$ is negligible. Now, $\tilde{\mathcal{Q}} = \mathcal{Q} - \mathcal{Q}'$.

For the construction of $\mathcal{Q}''$, let $R \in \tilde{\mathcal{Q}}$. Now if $R \in \mathcal{Q}''$, then for some $a \in A$ we have $R_a \in \hat{\mathcal{Q}}_a$ and $\hat{R}_a \in \mathcal{Q}^{con}_{aR}$ where for $\bar{u}_a = \max\{0, \max_{h \in H} v_{ah} - p^m_h\}$ the preference relation $\hat{R}_a$ is quasi-linear with $\hat{v}_{ah} = v_{ah} - \bar{u}_a$ for all $h \neq a$ (and $\hat{v}_{aa} = 0$). Using the same construction as in the above, if houses h_1 and $h_{q+1} = a = a_q$ are connected by indifference, conditions (i)–(iii) hold, and by summing all left-hand sides and all right-hand sides we get:

$$\sum_{j=1}^{q}(v_{a_jh_j} - v_{a_jh_{j+1}}) = \bar{u}_a. \tag{2}$$

Note that the equality holds since $\hat{v}_{a_qh_q} = v_{a_qh_q} - \bar{u}_a$, and condition (2) is independent of the price vector p. Furthermore, any profile $R \in \tilde{\mathcal{Q}}$ has a unique minimum price vector p^m and we may denote by $\tilde{\mathcal{Q}}|_{\bar{u}_a}$ the set of profiles of quasi-linear preferences where agent a's maximal utility from p^m is equal to $\bar{u}_a \in \mathbb{R}_+$. Now, again the profiles satisfying (2) is a hyperplane in $\mathbb{R}^{n(n-1)}$ and this set is negligible in $\tilde{\mathcal{Q}}|_{\bar{u}_a}$. This remains true for the profiles in $\tilde{\mathcal{Q}}|_{\bar{u}_a}$ satisfying (2) for some sequence ending at house a (as the set of houses and the set of sequences is finite), and they are a negligible set in $\tilde{\mathcal{Q}}|_{\bar{u}_a}$.

Let $\bar{u} = (\bar{u}_a)_{a \in A} \in \mathbb{R}^n_+$ and $\tilde{\mathcal{Q}}|_{\bar{u}} = \cap_{a \in A} \tilde{\mathcal{Q}}|_{\bar{u}_a}$ denote the set of profiles in $\tilde{\mathcal{Q}}$ where any agent a's utility from p^m is equal to $\bar{u}_a$. Then from the above it follows that the profiles in $\tilde{\mathcal{Q}}|_{\bar{u}}$ satisfying (2) for some agent a is a negligible set in $\tilde{\mathcal{Q}}|_{\bar{u}}$. As any profile in $\tilde{\mathcal{Q}}$ belongs to exactly one $\tilde{\mathcal{Q}}|_{\bar{u}}$, the set $\mathcal{Q}'' - \mathcal{Q}'$ is negligible in $\tilde{\mathcal{Q}}$.[9]

Hence, it follows that $\mathcal{Q}' \cup \mathcal{Q}''$ is negligible in $\mathcal{Q}$ and $\mathring{\mathcal{Q}} = \mathcal{Q} - (\mathcal{Q}' \cup \mathcal{Q}'')$ has measure one in $\mathcal{Q}$. Finally, we remark that Theorem 1 remains true for $\mathring{\mathcal{Q}}$, i.e., if we restrict mechanisms to the quasi-linear domain, then $\mathring{\mathcal{Q}}$ any strategy-proof mechanism must be a minimum price mechanism.

[9]Note that in the construction of $\mathring{\mathcal{R}}$, after eliminating $\mathcal{R}'$ it is sufficient to eliminate $\mathcal{R}'' - \mathcal{R}'$ where $\mathcal{R}'' - \mathcal{R}' = \{R \in \tilde{\mathcal{R}} : R_a \in \hat{\mathcal{R}}^{con}_{aR} \text{ for some } a \in A\}$.

5 Proof of Theorem 1

To prove Theorem 1 some lemmas are useful. Lemma 1 is an important consequence of a condition called the "minimal price condition," Lemma 2 is a characterization of minimal price vectors, while Lemma 3 shows upon an invariance property of minimal prices when some preferences are changed. The proofs of these lemmas can be found in the Appendix.

Definition 5. *Let $R \in \tilde{\mathcal{R}}$ be a profile and $x = (\mu, p)$ a weak equilibrium at the profile R. Then the state x satisfies the minimal price* (MP) *condition if for each nonempty set $S \subset \{h \in H : p_h > 0\}$, there is a house $h \in S$ and an agent $a \in A$, $a \neq h$, such that $\mu_a \notin S$ and $x_a I_a (h, p)$.*

Note that the MP condition is not satisfied at x if $p_h > 0$ for all $h \in H$.

Definition 6. *Let $R \in \mathcal{R}$ be a profile and $x = (\mu, p)$ and $x' = (\mu', p)$ two weak equilibria where $p \in \Pi_R$. A sequence $(a_j)_{j=1}^{t+1}$ of agents $a_j \in A$, different for $j \leq t$ but $a_{t+1} = a_1$, is called a* trading cycle *at x if $h_j = \mu_{a_j}$ and $\mu'_{a_j} = h_{j+1}$ for all j. If, in addition, also $h_{j+1} \neq a_j$ for all j and $h_j = a_j$ for some j, the trading cycle is* strong.

Clearly, if $(a_j)_{j=1}^{t+1}$ is a strong trading cycle at x, then x cannot be an equilibrium since, in that case, trade cannot be maximal at all weak equilibrium states at prices p.

Lemma 1. *Let $R \in \tilde{\mathcal{R}}$ be a profile and x and x' two weak equilibria, where the corresponding price vectors satisfy: $p, p' \in \Pi_R$ and $p' \leq p$, $p' \neq p$. Assume that x satisfies the* MP *condition. Then there exists a strong trading cycle $(a_j)_{j=1}^{t+1}$ at state x.*

Lemma 2. *Let $R \in \tilde{\mathcal{R}}$ be a profile. A price vector p is minimal in Π_R, if and only if, for each equilibrium $(\mu, p) \in \mathcal{E}_R$, the* MP *condition holds.*

Lemma 3. *Let $R \in \tilde{\mathcal{R}}$ and let p^m be a minimal vector in Π_R. Then, for all $\hat{R}_a \in \hat{\mathcal{R}}_{aR} - \hat{\mathcal{R}}_{aR}^{con}$, p^m is a minimal vector also in $\Pi_{\hat{R}}$, where $\hat{R} = (\hat{R}_a, R_{-a})$.*

Proof of Theorem 1. Suppose that the mechanism f is strategy-proof, but not a minimal price mechanism. Then there is a profile $R \in \mathring{\mathcal{R}}$ such that $f(R) = x \equiv (\mu, p)$ and $p \geq p^m$, $p \neq p^m$, where p^m is minimal in Π_R. Hence, there is a house $h \in H$ such that $p_h > p_h^m \geq 0$. Without loss of generality, let $p_1 > p_1^m \geq 0$. Then, by Definition 1(ii), there is an agent $a' \in A$, $a' \neq h = 1$, such that $x_{a'} I_{a'} (1, p)$. Note that the case $\mu_{a'} = 1$ is not excluded. Further, by monotonicity, it follows that $x_{a'}^m P_{a'} x_{a'}$, since $x_{a'}^m R_{a'} (1, p^m) P_{a'} (1, p) I_{a'} x_{a'}$. Finally, let $S_{a'} = \{h \in H : (h, p^m) I_{a'} x^m\}$. If $h = a'$, then $h \notin S_{a'}$ from $x_{a'}^m P_{a'} x_{a'}$.

Suppose now that agent a' manipulates by using preferences $R'_{a'} \in \hat{\mathcal{R}}_{a'R} - \hat{\mathcal{R}}_{a'R}^{con}$. Then $R' = (R'_{a'}, R_{-a'}) \in \mathring{\mathcal{R}}$ and, by Lemma 3, p^m is minimal in $\Pi_{R'}$.

Let $f(R') = x' \equiv (\mu', p')$ and $\mu'_{a'} \equiv h'$. Then $p' \geq p^m$ and $h' \in S_{a'} \cup \{a'\}$ by the definition of $R'_{a'}$. In addition, it follows directly that $x'_{a'} P_{a'} (h, p')$ for all $h \notin S_{a'} \cup \{a'\}$.

We will prove that $x'_{a'} P_{a'} x_{a'}$. In such a case, agent a' can manipulate which contradicts our presumption. Let the utility function $u_{a'}$ represent preferences $R_{a'}$, and consider the utility difference $(u_{a'h'}(p') - u_{a'1}(p)) \equiv \alpha + \beta$, where $\alpha = (u_{a'h'}(p^m) - u_{a'1}(p))$ and $\beta = (u_{a'h'}(p') - u_{a'h'}(p^m))$. Clearly, $x'_{a'} P_{a'} x_{a'}$ if and only if $\alpha + \beta > 0$.

Consider first the case when $h' \neq a'$. Then $h' \in S_{a'}$. We know that $x^m_{a'} P_{a'} x_{a'}$, so $u_{a'h'}(p^m) > u_{a'1}(p)$ and, hence, $\alpha > 0$. We also have $x' \in \mathcal{E}_{R'}$, and at equilibrium $u'_{a'h'}(p') \geq u'_{a'a'}(p') = \bar{u}$. Further, when $h' \in S_{a'}$ and $h' \neq a'$ then $u_{a'h'}(p^m) = u'_{a'h'}(p^m) = \bar{u}$, so $\beta \geq \bar{u} - \bar{u} = 0$. Hence, $\alpha + \beta \geq \alpha > 0$. Then a' can manipulate if $h' \in S_{a'}$, so $h' = a'$ must be the case.

Suppose now that $h' = a'$, and consider the following two cases: (i) $p' = p^m$ and (ii) $p' \geq p^m$, $p' \neq p^m$.

(i) Suppose that $p' = p^m$. Let $(a_j)_{j=1}^{t+1}$ be a sequence of agents that constitutes a trading cycle (from μ' to μ^m) at the equilibrium x' such that $a_1 = a'$, $\mu^m_{a_j} = \mu'_{a_{j+1}}$ for $j \leq t$, and $\mu^m_{a_{t+1}} = \mu'_{a_1}$. Clearly, there is such a cycle when $p' = p^m$ since $p^m \in \Pi_{R'}$, i.e., x^m is a weak equilibrium at the profile R'. Denote by h_j houses μ'_{a_j}. Then $h_1 = a_1$. If $\mu^m_{a_j} = \mu'_{a_{j+1}} = a_j$ for some $j > 1$, i.e., $h_{j+1} = a_j$, then houses h_1 and h_{j+1} are connected by indifference and the profile does not satisfy the "not connected by indifference condition." Hence, for each trading cycle $\mu^m_{a_j} \neq a_j$ for $j > 1$. Since $\mu'_{a'} = a'$, this means that the trade at μ^m is larger than the trade at μ'. Hence, $x' = (\mu', p^m)$ cannot be be an equilibrium and, hence, cannot be the outcome of the mechanism f. Thus, it cannot be the case that $p' = p^m$.

(ii) Suppose that $p' \geq p^m$, $p' \neq p^m$. Let also $a' = a^1$, $R'_{a'} = R^1_{a^1}$, and $R^1 = (R^1_{a^1}, R_{-a^1})$. Suppose now that agent $a^2 \neq a^1$ manipulate with preferences $R^2_{a^2} \in \hat{\mathcal{R}}_{a^2R^1} - \hat{\mathcal{R}}^{con}_{a^2R^1}$. Then, $R^2 = (R^1_{a^1}, R^2_{a^2}, R_{-\{a^1,a^2\}}) \in \mathring{\mathcal{R}}$ and, by Lemma 3, p^m is minimal in Π_{R^2}. The outcome of this profile is $f(R^2) = x^2 \equiv (\mu^2, p^2)$ and $\mu^2_{a^2} \equiv h^2$. Then $p^2 \geq p^m$ and $h^2 \in S_{a^2} \cup \{a^2\}$ by the definition of $R^2_{a'}$. Now, just as in the above, there will be two cases: (i) $p^2 = p^m$ and (ii) $p^2 \geq p^m$, $p^2 \neq p^m$. But case (i) cannot not prevail, by the above conclusion, so it must be case (ii).

Now, we can repeat the above analysis a number of times with a sequence $A' = (a_j)_{j=1}^k$ of different agents. Let k be maximal, meaning that there is an agent $a^{k+1} \neq h$ and $a^{k+1} \neq a^j$, $j \leq k$, such that $x^k_{a^{k+1}} I_{a^{k+1}} (h, p^k)$. But then a^{k+1} can manipulate by using preferences $R^{k+1}_{a^{k+1}} \in \hat{\mathcal{R}}_{a^{k+1}R^k} - \hat{\mathcal{R}}^{con}_{a^{k+1}R^k}$, which is a contradiction to k being maximal. Hence, $p^k = p^m$ must be the case. But then we have case (i) above, which cannot be an equilibrium since trade is not maximal. In conclusion, the mechanism f must be a minimal price mechanism. $\square$

6 Foundation of Price Mechanisms

This section provides a foundation for the use of price mechanisms. Recall that a state x is a pair (μ, p) where $\mu : A \to H$ is a bijection and $p \geq \underline{p} = 0$. Let $\mathcal{X}$ denote the set of all states.

Given profile R, a state $x = (\mu, p)$ is *fair* (under profile R) if for all $a \in A$, $x_a R_a(h, p)$ for all $h \in H$. Note that fairness incorporates individual rationality since $x_a R_a(a, p)$, and envy-freeness because $x_a R_a(h, p)$ for all $h \neq a$.

A *social choice correspondence* is a mapping $F : \mathcal{R} \to \mathcal{X}$ associating with each profile R a non-empty set of states $F(R)$. We require the following properties on a social choice correspondence F:

Essentially Single-Valuedness. For all $R \in \mathcal{R}$ and all $x, y \in F(R)$ we have $x_a I_a y_a$ for all $a \in A$.

Pareto Indifference. For all $R \in \mathcal{R}$, all $x \in F(R)$, and all $y \in \mathcal{X}$, if $x_a I_a y_a$ for all $a \in A$, then $y \in F(R)$.

Fairness. For all $R \in \mathcal{R}$ and all $x \in F(R)$, x is fair (under profile R).

Essentially single-valuedness means that all chosen states are welfare equivalent. Pareto indifference means that any state, which is welfare equivalent to a chosen state, should also be chosen. In addition, we say that F is *manipulable at* R if there exists $a \in A$ and R'_a such that for some $x' \in F(R'_a, R_{-a})$ we have $x'_a P_a x_a$ for all $x \in F(R)$. Furthermore, F is *strongly manipulable at* R if there exists $a \in A$ and R'_a such that $x'_a P_a x_a$ for all $x' \in F(R'_a, R_{-a})$ and all $x \in F(R)$.

Note that a price mechanism f always associates a unique state with each profile, which we denote below by $f(R)$. We say that f and F are *welfare equivalent* if and only if for all $R \in \mathcal{R}$ and all $x \in F(R)$, we have $x_a I_a f_a(R)$ for all $a \in A$.

Theorem 2. *If f is a price mechanism, then there exists a unique welfare equivalent social choice correspondence F satisfying essentially single-valuedness, Pareto indifference and fairness. Furthermore, for any profile R, if f is manipulable at R, then F is manipulable at R.*

Proof. The first part follows directly, for any $R \in \mathcal{R}$, by setting $F(R) = \{x \in \mathcal{X} : x_a I_a f_a(R)$ for all $a \in A\}$.

For the second part, let $R \in \mathcal{R}$, $a \in A$ and $(R'_a, R_{-a}) \in \mathcal{R}$. Suppose that $f_a(R'_a, R_{-a}) P_a f_a(R)$, i.e., that f is manipulable at R. Then by welfare equivalence of f and F and Pareto indifference of F, we have $f(R'_a, R_{-a}) \in F(R'_a, R_{-a})$ and $x_a I_a f_a(R)$ for all $x \in F(R)$. Hence, $f_a(R'_a, R_{-a}) P_a x_a$ for all $x \in F(R)$, and F is manipulable at R. □

Theorem 3. *If F is essentially single-valued, Pareto indifferent and fair, then there exists a unique (in terms of welfare) welfare equivalent price mechanism f. Furthermore, for any profile R, if F is strongly manipulable at R, then f is manipulable at R.*

Proof. For the first part, let F be essentially single-valued, Pareto indifferent and fair. Let $R \in \mathcal{R}$ and $x = (\mu, p) \in F(R)$. If for some $h \in H$, $\mu_h = h$, $p_h > 0$ and $x_a P_a(h, p)$ for all $a \in A$ with $\mu_a \neq h$, then decrease p_h to p'_h such that for some $a \in A$, $x_a I_a(h, p'_h)$ and $\mu_a \neq h$ or $p'_h = 0$. Let p' be the obtained price vector and $x' = (\mu, p')$. Then for all $a \in A$, $x'_a I_a x_a$ and by Pareto indifference of F, $x' \in F(R)$. Now, x' satisfies (i) and (ii) of Definition 1, i.e., x' is a weak equilibrium state. If x' does not satisfy (iii), then there exists $x'' = (\mu', p')$ satisfying (i) and (ii) in Definition 1, where trade is maximal. But then, by fairness applied to x' and x'', we obtain $x'_a I_a x''_a$ for all $a \in A$. Again, by $x' \in F(R)$ and Pareto indifference, we have $x'' \in F(R)$. Now we set $f(R) = x''$. But then f is a price mechanism.

For the second part, let $R \in \mathcal{R}$, $a \in A$ and $(R'_a, R_{-a}) \in \mathcal{R}$. Suppose that $x'_a P_a x_a$ for all $x' \in F(R'_a, R_{-a})$ and all $x \in F(R)$, i.e., that F is strongly manipulable at R. By welfare equivalence of f and F and Pareto indifference, we have $f(R) \in F(R)$ and $f(R'_a, R_{-a}) \in F(R'_a, R_{-a})$. Hence, $f_a(R'_a, R_{-a}) P_a f_a(R)$ and f is manipulable at R. □

Let $\breve{\mathcal{R}} \subset \mathcal{R}$ be a subset of profiles. The social choice correspondence F is *weakly strategy-proof* on the domain $\breve{\mathcal{R}}$ if no agent can strongly manipulate at any profile $R \in \breve{\mathcal{R}}$. We now obtain the following corollary to Theorems 1–3.

Corollary 1. *Let F be a social choice correspondence satisfying essentially single-valuedness, Pareto indifference and fairness. There is a domain $\mathring{\mathcal{R}} \subset \mathcal{R}$ such that $R \in \mathring{\mathcal{R}}$ for almost all $R \in \mathcal{R}$, and the following holds: if F is weakly strategy-proof on $\mathring{\mathcal{R}}$, then F is welfare equivalent to the minimum price mechanism on the domain $\mathring{\mathcal{R}}$.*

Finally, we note that the "no trade mechanism," obtained by setting prices high enough such that each agent prefers keeping her endowment, satisfies all properties in Corollary 1 except for Pareto indifference. The no trade mechanism also satisfies conditions (i) and (ii) in Definition 1, by setting prices high enough such that for any house at least one agent is indifferent between keeping his endowment and buying the house, i.e., it chooses for any profile a weak equilibrium state.

Appendix: Proofs of Lemmas

Lemma 1. *Let $R \in \tilde{\mathcal{R}}$ be a profile and x and x' two weak equilibria, where the corresponding price vectors satisfy: $p, p' \in \Pi_R$ and $p' \leq p$, $p' \neq p$. Assume that x satisfies the* MP *condition. Then there exists a strong trading cycle $(a_j)_{j=1}^{t+1}$ at state x.*

Proof. Let $H' = \{h \in H : p'_h < p_h\}$ and $H'' = H - H'$. Then, $H' \neq \emptyset$ since $p' \neq p$. Furthermore, $H'' \neq \emptyset$ since $H'' = \emptyset$ means that $p_h > p'_h$ for all h, and hence, $p_h > 0$ for all h, which is not consistent with the MP condition (for $S = N$).

Let now $h_j = \mu_{a_j}$ and define a first part $(a_j)_{j=1}^{k}$ of the sequence $(a_j)_{j=1}^{t+1}$, where $h_j \in H'$ for all $1 \leq j < k \leq t$ while $h_k \in H''$, in the following way:

- Let h be an arbitrary house in H' and consider a sequence $(h'_i)_{i=1}^{k'}$, where $h'_1 = h$ and h'_i, $i < k'$, are different houses in H', while $h'_k \in H''$. Also let $(a'_i)_{i=1}^{k'}$ be the corresponding sequence of agents, where $\mu_{a'_i} = h'_i$. Further, the sequence has to satisfy: for each $q < k'$

and set $\{h'_i\}_{i=1}^q$, $x_{a'_{q+1}} I_{a'_{q+1}} x_{a'_j}$ for some $j \leq q$ and $h'_j \neq a'_{q+1}$. The sequence $(h'_i)_{i=1}^{k'}$ is obtained recursively in the following way:

- Let $h'_1 = h$. If we have obtained the sequence for q houses, i.e., the sequence $\{h'_i\}_{i=1}^q$, then, according to the MP condition, there is a house $h'_r \in \{h'_i\}_{i=1}^q$ and an agent, say a'_{q+1}, with $a'_{q+1} \neq h'_r$, such that $\mu_{a'_{q+1}} \notin \{h'_i\}_{i=1}^q$ and $x_{a'_{q+1}} I_{a'_{q+1}} (h'_r, p)$. Then let $h'_{q+1} = \mu_{a'_{q+1}}$. If $h'_{q+1} \in H''$ stop, and let $k' = q+1$, otherwise continue. The sequence stops at some time k' since H is finite. Note that h'_{q+1} cannot stop in H' since the MP condition implies that $H'' \neq \emptyset$.

- The sequence $(h'_i)_{i=1}^{k'}$ clearly contains a subsequence $(h'_{i_j})_{j=1}^k$ such that $(a'_{i_j})_{j=1}^{k-1}$ are different agents and $x_{a'_{j+1}} I_{a'_{j+1}} x_{a'_j}$ for $1 \leq j < k$. Then, define the sequence $(h_j)_{j=1}^k$ as $h_j = h'_{i_j}$ and $a_j = a'_{i_j}$.

We next define the second part $(a_j)_{j=k}^l$ of the sequence $(a_j)_{j=1}^{t+1}$. To do this, denote the houses associated with $(a_j)_{j=k}^l$ by $(h_j)_{j=k}^l$, and note that $h_k \in H''$ by the above construction. Let now $a_{k+1} \in A$ be given by $\mu'_{a_{k+1}} = \mu_{a_k} = h_k \in H''$. Continue to define a_j, $j \geq k+1$ in a similar way, i.e., $\mu'_{a_{j+1}} = \mu_{a_j} = h_j \in H''$. The sequence ends at h_l if $h_l \in H'$.

Before continuing to define the sequence $(a_j)_{j=1}^{t+1}$, we note that $a_l \neq a_i$ for all i, $k \leq i < l$. To see this, assume that $a_l = a_i$ for some i, $k \leq i < l$. Then $i = k$ because of the rule $\mu'_{a_{j+1}} = \mu_{a_j}$. Furthermore, $x_{a_l} I_{a_l} x_{a_{l-1}}$ since $p'_{h_{l-1}} = p_{h_{l-1}}$. But then $x'_{a_l} I_{a_l} x_{a_l}$. Further, $x_{a_k} I_{a_k} x_{a_{k-1}}$, $p'_{h_{k-1}} < p_{h_{k-1}}$ and $a_k \neq h_{k-1}$ so, by monotonicity, $(h_{k-1}, p') P_{a_k} x_{a_k}$. But then $x'_{a_k} P_{a_k} x_{a_k}$, contradicting $x'_{a_l} I_{a_l} x_{a_l}$ when $a_l = a_k$. Hence, this case cannot prevail.

Note next that since the sequence $(h_j)_{j=k}^l$ ends at $h_l \in H'$, by construction, we can expand the sequence $(h_j)_{j=1}^l$ to a sequence $(h_j)_{j=1}^{k'}$, $k' > l$, in the same way as $(h_j)_{j=1}^k$ was constructed, where $h_j \in H'$ for $l \leq j < k'$ and $h_{k'} \in H''$. Moreover, all h_j are different for $1 \leq j \leq k$ and $l \leq j \leq k'$ by the construction.

Further expansion to $(h_j)_{j=1}^{l'}$, is obtained by the rule $\mu'_{a_{j+1}} = \mu_{a_j} = h_j \in H''$, for $k' \leq j \leq l'$. In this way, we further continue the expansion to a sequence $(h_j)_{j=1}^r$. The expansion of the sequence is stopped at the first agent a_r such that $a_r = a_i$ for some $i < r$. Then we have a cycle $(a_j)_{j=i}^r$ where all agents are different for $i \leq j < r$ and $a_r = a_i$. In addition, $x_{a_j} I_{a_j} x_{a_{j-1}}$, for $i < j \leq r$, and $x_{a_1} I_{a_1} x_{a_r}$. Moreover, the sequence satisfies:

1. If $h_j \in H'$ and $h_{j-1} \in H''$ then, by monotonicity, $h_j = \mu_{a_j} = a_j$.

2. $h_j \neq a_{j-1}$ for all j, since if $h_{j-1} = a_{j'}$ for some j' then $h_{j'}$ and $h_{j''}$ are connected by indifference where j'' satisfies $h_{j''} \in H'$ and $h_{j-1} \in H''$. This is not consistent with the "not connected by indifference condition."

Finally, given points 1 and 2 above, and after a renumbering, the cycle $(a_j)_{j=i}^r$ constitutes a strong trading cycle. $\square$

Lemma 2. *Let $R \in \tilde{\mathcal{R}}$ be a profile. A price vector p is minimal in Π_R, if and only if, for each equilibrium $(\mu, p) \in \mathcal{E}_R$, the* MP *condition holds.*

Proof. We first prove that the MP condition is a necessary condition. For this purpose, let $x = (\mu, p) \in \mathcal{E}_R$ be an equilibrium and suppose that the MP condition is not satisfied at x. Then there is a nonempty set $S \subset \{h \in H : p_h > 0\}$ such that there is no $h \in S$ and $a \in A$, with $h \neq a$ and $\mu_a \notin S$, such that $x_a I_a (h, p)$. This means that all agents $a \in A$, with $\mu_a \notin S$ and $a \neq h$, strictly prefer x_a to (h, p) for all $h \in S$. On the other hand, for $a = h$ the utility of h is independent of p_h. Then there is a price vector $p' \in \Pi_R$ such that $p' \leq p$, $p' \neq p$ (Alkan, Demange and Gale, 1991). Hence, p cannot be a minimal price vector in Π_R.

We next prove that the MP condition is a sufficient condition. Suppose that the MP condition is satisfied at an equilibrium $x = (\mu, p) \in \mathcal{E}_R$ but that $p \in \Pi_R$ is not minimal in Π_R. Then there is an equilibrium $x' = (\mu', p') \in \mathcal{E}_R$ such that $p' \leq p$, $p' \neq p$. Then, according to Lemma 1, there is a strong trading cycle $(a_j)_{j=1}^{t+1}$. But then trade cannot be maximal at x. To see this, let a state x'' be defined as: for all $a \notin \{a_j\}_{j=1}^{t}$ let $x''_a = x_a$, and for $a \in \{a_j\}_{j=1}^{t}$ let $x''_{a_{j+1}} = x'_{a_j}$ for $1 \leq j \leq t$. It then follows directly from Definition 6 that x'' is a weak equilibrium and that trade is larger at x'' than at x. This is a contradiction to x being an equilibrium. Hence, the MP condition is sufficient for p being a minimal price vector. $\square$

Lemma 3. *Let $R \in \tilde{\mathcal{R}}$ and let p^m be a minimal vector in Π_R. Then, for all $\hat{R}_a \in \hat{\mathcal{R}}_{aR} - \hat{\mathcal{R}}_{aR}^{con}$, p^m is a minimal vector also in $\Pi_{\hat{R}}$, where $\hat{R} = (\hat{R}_a, R_{-a})$.*

Proof. Let $x^m = (\mu^m, p^m) \in \mathcal{E}_R$ and let $\hat{R}_a$ be represented by a utility function $\hat{u}_a$. Since $x^m \in \mathcal{E}_R$, it follows directly from the definition of $\hat{u}_a$ that x^m is a weak equilibrium given $\hat{R}$ and, hence, $\hat{x} = (\hat{\mu}, p^m) \in \mathcal{E}_{\hat{R}}$ for some assignment $\hat{\mu}$. If $p_h^m = 0$ for all $h \in H$, we are done. If $p_h^m > 0$ for some $h \in H$, let $S \subset \{h \in H : p_h^m > 0\}$ and $S \neq \emptyset$. Such a set S exists since $p_h^m > 0$ for some $h \in H$. Then, by the necessary part of Lemma 2, the MP condition holds at x, i.e., there is a house $h \in S$ and an agent $a' \in A$, $a' \neq h$, such that $\mu_{a'}^m \notin S$ and $x_{a'} I_{a'} (h, p^m)$. If $a' = a$, then also $\hat{x}_a \hat{I}_a (h, p^m)$ by the construction of $\hat{u}_a$. Hence, the MP condition is satisfied at $\hat{x} \in \mathcal{E}_{\hat{R}}$. Then, by the sufficiency part of Lemma 2, p^m is a minimal vector in $\Pi_{\hat{R}}$. $\square$

References

Alkan, A., Demange, G., and Gale, D., 1991. Fair allocation of indivisible objects and criteria of justice, *Econometrica* 59, 1023–1039.

Andersson, T., Ehlers, L., and Svensson, L.-G., 2016. Transferring ownership of public housing to existing tenants: A market design approach, *Journal of Economic Theory* 165, 643–671.

Andersson, T., and Svensson, L.-G., 2014. Non-manipulable house allocation with rent control, *Econometrica* 82, 507–539.

Demange, G., and Gale, D., 1985. The strategy structure of two-sided matching markets, *Econometrica* 53, 873–888.

Dupuy, A., Galichon, A., Jaffe, S., and Kominers, S.D., 2017. Taxation in matching markets, Working Paper.

Fleiner, T., Jagadeesan, R., Jankó, Z., and Teytelboym, A., 2019. Trading networks with frictions, *Econometrica* 87, 1633–1661.

Hatfield, J.W., Kojima, F., and Kominers, S.D., 2017. Strategy-proofness, investment efficiency, and marginal returns: An equivalence, Working Paper.

Hurwicz, L., 1972. On informationally decentralized systems, in *Decision and Organisation: A Volume in Honor of Jacob Marschak* ed. by C.B. McGuire and Roy Radner, Amsterdam: North-Holland.

Jagadeesan, R., Kominers, S.D., and Rheingans-Yoo, R., 2018. Strategy-proofness of worker-optimal matching with continuously transferable utility, *Games and Economic Behavior* 108, 287–294.

Leonard, H.B., 1983. Elicitation of honest preferences for the assignment of individuals to positions, *Journal of Political Economy* 91, 461–479.

Ma J., 1994. Strategy-proofness and the strict core in a market with indivisibilities, *International Journal of Game Theory* 23, 75–83.

Miyagawa, E. 2001, House allocations with transfers, *Journal of Economic Theory* 100, 329–355

Miyake, M., 1998. On the incentive properties of multi-item auctions, *International Journal of Game Theory* 27, 1–19.

Morimoto, S., and Serizawa, S., 2015. Strategy-proofness and efficiency with non-quasi-linear preferences: A characterization of minimum price Walrasian rule, *Theoretical Economics* 10, 445–487.

Satterthwaite, M.A., and Sonnenschein, H., 1981. Strategy proof allocation mechanisms at differentiable points, *Review of Economic Studies* 48, 587–597.

Schlegel, J.C., 2018. Trading networks with general preferences, Working Paper.

Sun, N., and Yang, Z., 2003. A general strategy proof fair allocation mechanism, *Economics Letters* 81, 73–79.

Svensson, L.-G., 2009. Coalitional strategy-proofness and fairness, *Economic Theory* 40, 227–245.

Vickrey, W., 1961. Counterspeculation, auctions, and competitive sealed tenders, *Journal of Finance* 16, 8–37.

Récents cahiers de recherche du CIREQ
Recent Working Papers of CIREQ

La liste complète des cahiers de recherche du CIREQ est disponible à l'adresse suivante : http://www.cireqmontreal.com/cahiers-de-recherche .

The complete list of CIREQ's working papers is available at the following address : http://www.cireqmontreal.com/cahiers-de-recherche .

06-2020	L. Kahn, F. Lange, D. Wiczer, "Labor Supply in the Time of COVID19", avril 2020, 7 pages
07-2020	Gottlieb, C., J. Grobovsek, M. Poschke, "Working from Home across Countries", avril 2020, 21 pages
08-2020	Benatia, D., R. Godefroy, J. Lewis, Estimating COVID-19 Prevalence in the United States : A Sample Selection Model Approach, avril 2020, 24 pages
09-2020	Kahn, L., F. Lange, D. Wiczer, Labor Demand in the Time of COVID-19 : Evidence from Vacancy Postings and UI Claims, avril 2020, 18 pages
10-2020	Lange, F., T. Papageorgiou, Beyond Cobb-Douglas : Flexibly Estimating Matching Functions with Unobserved Matching Efficiency, avril 2020, 37 pages
11-2020	Baccini, L., A. Brodeur, Explaining Governors' Response to the COVID-19 Pandemic in the United States, avril 2020, 12 pages
12-2020	Gomme, P., Labor Market and Fiscal Policy During and After the Coronavirus, avril 2020, 28 pages
13-2020	Horan, S., Y. Sprumont, Two-Stage Majoritarian Choice, mai 2020, 12 pages
14-2020	Bounie, D., Y. Camara, J. Galbraith, Consumers' Mobility, Expenditure and Online-Offline Substitution Response to COVID-19 : Evidence from French Transaction Data, mai 2020, 54 pages
15-2020	Beaulieu, M.-C., J.-M. Dufour, L. Khalaf, Arbitrage Pricing, Weak Beta, Strong Beta : Identification-Robust and Simultaneous Inference, avril 2020, 57 pages
16-2020	Bahel, E., Y. Sprumont, Strategy-proof Choice under Monotonic Additive Preferences, mai 2020, 7 pages
17-2020	Jones, S.R.G., F. Lange, W.C. Riddell, C. Warman, Waiting for Recovery: The Canadian Labour Market in June 2020, juin 2020, 29 pages
18-2020	Gottlieb, C., J. Grobovsek, M. Poschke, F. Saltiel, Lockdown Accounting, octobre 2020, 13 pages
19-2020	Matveev, D., F. Ruge-Murcia, Tariffs and the Exchange Rate : Evidence from Twitter, septembre 2020, 18 pages
20-2020	Riboni, A., F. Ruge-Murcia, The Power of the Federal Reserve Chair, juin 2020, 56 pages
21-2020	Bouakez, H., O. Rachedi, E. Santoro, The Sectoral Origins of the Spending Multiplier, décembre 2020, 66 pages