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Abstract. Let $\mathcal{E}$ be a class of event. Conditionally Expected Utility decision makers are decision makers whose conditional preferences $\succsim_E$, $E \in \mathcal{E}$, satisfy the axioms of Subjective Expected Utility theory (SEU). We extend the notion of *unconditional preference that is conditionally EU* to unconditional preferences that are not necessarily SEU. We give a representation theorem for a class of such preferences, and show that they are Invariant Bi-separable in the sense of Ghirardato et al.[**7**]. Then, we consider the special case where the unconditional preference is itself SEU, and compare our results with those of Fishburn [**6**].

1. Introduction

In an interesting paper [**6**], Fishburn studied decision makers who are characterized by the property that their conditional preferences obey the axioms of Subjective Expected Utility (SEU) theory. That is, once they are informed that the true state lies in a certain subset of the state space, these decision makers evaluate acts by means of an expected utility criterion. In his study, Fishburn is motivated by the observation that Savage's derivation of SEU theory rests on the assumption that a sufficiently "rich" set of acts be available to the decision maker, but the assumption seems hardly met in many actual situations. Thus Fishburn, and Luce and Krantz [**10**] before

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him, proposed the model of conditional EU preferences as a way to remedy to this situation.

Once the assumption of conditional preferences is made, the problem becomes that of understanding how these decision makers would evaluate their options *ex-ante*, that is before they receive their information. Decision makers that display conditional EU preferences with respect to a certain class of events need not obey the SEU axioms *ex-ante*. An easy example that shows this point is as follows. Let S denote the state space, and assume that S is the interval $[0,1]$ endowed with the usual Borel σ-algebra. Consider a Maxmin Expected Utility decision maker who is described by a set of priors $\mathcal{C} = co\{\mu, \lambda\}$, the convex hull of two probabilities μ and λ. Assume further that μ has a density with respect to λ given by

$$f = \begin{cases} \frac{3}{2} & if\ x \in [0, \frac{1}{3}) \\ \frac{3}{4} & if\ x \in [\frac{1}{3}, 1] \end{cases}$$

and that λ is the Lebesgue measure on $[0,1]$ (i.e., λ has density $g \equiv 1$ on $[0,1]$). It is clear that this decision maker satisfies conditional expected utility with the respect to the family of events $\mathcal{E} = \{[0,1/3],[1/3,1]\}$ while, by assumption, he is Maxmin Expected Utility *ex-ante*.

In his paper, Fishburn provides axioms guaranteeing not only that the *ex-ante* preference be SEU, but also that the conditional preferences be obtained from the unconditional one by means of Bayes rule. In this paper, we are concerned with the study of conditional EU decision makers but from a different angle. Precisely, we are interested in characterizing all *ex-ante* (=unconditional) preferences that are compatible with conditional EU, and that satisfy certain mild assumptions listed below. Our interest is motivated not only by examples like the one above, but also by the examples in Fishburn [**6**, pp. 19-23], which seem to suggest that a variety of non-EU unconditional preferences might be compatible with conditional EU. We believe that this study might provide insights into the difficult problem of updating non-EU functionals, but we do not address this problem here. Here, we prove the following generalization of Fishburn's result. Fishburn's theorem can be thought of as saying that, under certain conditions, the unconditional preference is a "weighted average" of the conditional preferences. The operation of weighting is described by a probability measure. Here, we show that under our milder conditions, the unconditional preference is still a "weighted average" of the conditional preferences, but we allow for a more permissive notion of "weighting": Fishburn's probability is replaced by a capacity. Resulting unconditional preferences are a subclass of the Invariant Bi-separable Preferences of Ghirardato-Maccheroni-Marinacci [**7**], that is they satisfy the first five axioms in Gilboa-Schmeidler [**9**]. We then derive Fishburn's theorem as a special case of our result. By doing so, we clarify the role of Fishburn's axioms, and obtain a better understanding of the structure of conditional EU preferences. We employ three main tools. A theorem on the representation of monotone, translation invariant functionals, which we proved in [**1**], a theorem of Samet [**13**] on the characterization of common priors and a theorem of Phelps [**12**, Theorem 1] on the uniqueness of the Hahn-Banach extension.

The paper unfolds as follows. The formal setting is described in Section 2. In Section 3, we face the conceptual problem of extending the meaning of conditional expected utility outside the realm of SEU theory and Bayesian updating. Definition 1 of Section 3 gives our solution. We conclude that section by giving an example of a preference relation of Bewley's type (thus not SEU) which satisfies the criteria of our definition. In Section 4, we fully characterize those Conditional EU preferences which are, in addition, monotone and C-independent. In Section 5, we present a set of results which parallel those of Fishburn [**6**, pp. 19-23] and that, in fact, have a broader range of applicability. We discuss the relation between our assumptions and those of Fishburn [**6**, pp. 19-23] in Section 6.

2. Setting

In this section, we describe the primitives of the model as well as the notion of conditional EU preferences. The typical setting of decision making under uncertainty involves four primitives. The first three are:

(1) A measurable space (S, Σ) – Σ a σ-algebra of subsets of S – which is called the state space;

(2) A consequence space C, assumed to be a mixture space ([**3**], [**8**]);

(3) A set $\mathcal{A}$ of alternatives available to the decision maker, which are viewed as mappings $S \longrightarrow C$, and are called acts.

Usually, the fourth is a preference relation $\succsim$ on $\mathcal{A}$. The first three primitives are retained in this paper while the fourth is replaced by a family of preferences: the conditional preferences. Precisely, we have

(4) A family $\mathcal{E}$ of elements of Σ and, for each $E \in \mathcal{E}$, a preference relation $\succsim_E$ on $\mathcal{A}$, which is interpreted as a preference on $\mathcal{A}$ conditional on the true state being in E.

A consequence of this interpretation is that, when E is given, an act $f : S \longrightarrow C$ can be viewed as the restriction $f \mid_E$. Until Proposition 1 Section 5, we do not make any assumption on the family $\mathcal{E}$. In general, we will be mainly interested in the cases where either $\mathcal{E}$ is a (measurable) partition of S or $\mathcal{E}$ is a union of partitions of S. Clearly, this encompasses the cases of both σ-algebras and λ-systems, which are of special interest.

In this paper, differently from Fishburn [**6**], we are not concerned with "economizing" on the derivation of the SEU theorem. Rather, we are interested in the model of conditional EU preferences *per se* and in its relation with non-necessarily additive unconditional preferences such as those studied in [**14**], [**9**], [**7**] and [**1**]. This motivates the following assumption which implies that the set of acts is "rich".

S0: $\mathcal{A}$ contains all measurable simple[1] acts $f : S \longrightarrow C$.

Assumption **S0** is customary in most models of decision making uncertainty and, in fact, it is usually not even stated as an assumption (see, for instance, [**3**], [**14**], [**9**], [**7**]). While in the usual setting **S0** is fairly uncontroversial, it is not so in models of conditional preferences: after all, one of the motivation of Fishburn [**6**] and Luce and Krantz [**10**] was exactly that of deriving the EU theorem when "few" acts are available to the decision maker. In Section 6, where we compare our results with Fishburn's, we will come back to this issue and we will provide an extensive discussion of the role played by assumptions like **S0** in models of conditional preferences.

Next, we are going to make two assumptions that we are going to maintain throughout the paper. The first identifies the scope of this paper as we are concerned only with conditional preferences satisfying the SEU axioms. We recall that a preference $\succsim_E$ satisfies the Anscombe and Aumann [**3**] SEU axioms if and only if (I) it is complete and transitive; (II) satisfies the Independence Axiom; (III) it is monotone; (IV) satisfies the Archimedean property; (V) it is non-trivial (see [**3**] for a formal statement of these axioms). Thus, we require that

S1: For each $E \in \mathcal{E}$, $\succsim_E$ satisfies the axioms of SEU.

[1]For a simple act $f : S \longrightarrow C$, let $\{c_1, ..., c_n\}$ be the set of its values in C. The act is measurable if $f^{-1}\{c_i\} \in \Sigma$ for every i.

As it is well-known, this implies that there exist a utility $u_E : C \longrightarrow \mathbb{R}$ and a unique[2] probability charge P_E such that

$$f \succsim_E g \qquad iff \qquad \int u_E(f) dP_E \geq \int u_E(g) dP_E$$

It is also clear that because of point (4) above, the charge P_E is supported by E.

The second assumption states that consequences are evaluated independently of the conditioning event. Precisely,

S2: The utilities $u_E : C \longrightarrow \mathbb{R}$ can be chosen so that $u_E \equiv u$ for all $E \in \mathcal{E}$.

It is easy to give a behavioral counterpart of assumption **S2**. For $x \in C$ and $E \in \mathcal{E}$, let x_E denote the act such that $x_E(s) = x$ for all $s \in E$. Assumptions **S0** guarantees that x_E is available to the decision maker for all $E \in \mathcal{E}$.

S2': For all $x \in C$ and for all $E_i, E_j \in \mathcal{E}$, $x_{E_i} \sim x_{Ej}$.

A similar assumption is made in Luce and Krantz [**10**]. Assumption **S2** is implied by Fishburn's axioms (A1) to (A6) (see [**6**, Lemma 6]).

Assumption **S2** permits to identify an act $f : S \longrightarrow C$ with the real-valued mapping $u \circ f$. Correspondingly, we identify the set of acts with the set of all bounded, Σ-measurable functions with values in $u(C) \subset \mathbb{R}$. In fact, since we are going to restrict (starting with Section 4) to functionals that are monotone and translation invariant, we will identify the set of acts with $B(\Sigma)$ – the Banach space of all bounded, Σ-measurable real-valued functions on S equipped with the sup-norm (see, for instance, [**9**] for the details of this procedure). To conclude this section, we only recall that the norm-dual of $B(\Sigma)$ is (isometrically isomorphic to) the space $ba(\Sigma)$ of bounded charges on Σ equipped with the variation norm. In what follows, the subset $ba_1^+(\Sigma) \subset ba(\Sigma)$ of finitely additive probability measures on Σ is always endowed with the weak*-topology produced by the duality $(ba(\Sigma), B(\Sigma))$.

3. Unconditional preferences

So far, we have focused on decision makers who are identified by a family of SEU preferences $\{\succsim_E\}_{E \in \mathcal{E}}$. In this section, we are going to introduce decision makers who, in addition, have an unconditional preference $\succsim$ on $\mathcal{A}$

[2] In our setting, uniqueness is a consequence of assumption **S0**.

with the property that the preferences $\{\succsim_E\}_{E\in\mathcal{E}}$ are the conditional preferences derived from $\succsim$. In order to do so, we must first define what we mean by *conditional preference.*

In his paper, Fishburn is concerned only with unconditional SEU preferences that are conditionally SEU. In that case, the meaning of "conditional preference" is clear, and one does not need to give a definition. Here, because our interest is not limited to unconditional SEU preferences, we need to give a general definition. In order to understand what kind of definition we should be giving, let us re-consider the SEU case as the notion is clear there. Let us suppose, for the purpose of illustration, that the family $\mathcal{E}$ is a partition of S. Then, one would say that a SEU unconditional preference is conditionally SEU with respect to $\mathcal{E}$ if the conditional measures can be computed from the unconditional one by means of Bayes rule. It is clear that such a definition cannot be generalized to the non-EU case. There is another way, however, of expressing the same requirement. Let f be an act (identified to an element in $B(\Sigma)$) and suppose that the unconditional SEU preference is described by the functional $I(f) = \int f dP$, $P \in ba_1^+(\Sigma)$. Let $S_{/\mathcal{E}}$ denote the quotient of S by the partition $\mathcal{E}$, that is the space whose elements are the cells of the partition. Then, one says that the unconditional preference is conditionally SEU if for all $f \in B(\Sigma)$ the following holds

$$\int_S f dP = \int_{S_{/\mathcal{E}}} \int f \mid_E dP_E dP' \tag{3.1}$$

where P' is the image (pushforward) measure of P under the canonical mapping $\pi : S \longrightarrow S_{/\mathcal{E}}$, that is $P'(A) = P(\pi^{-1}(A))$, $E \in \mathcal{E}$ and the P_E are the conditional measures. If we denote by I_E the conditional functionals, $I_E(\cdot) = \int \cdot dP_E$, this can be re-written as

$$I(f) = \int_{S_{/\mathcal{E}}} I_E(f) dP' \tag{3.2}$$

Clearly, this definition is equivalent to the one given above in terms of updating measures. But, in this form, the definition can be generalized to the non-EU case. To see this, let us begin by introducing the mapping κ which is defined by

$$\kappa : f \longmapsto \left(\int f dP_E\right)_{E\in\mathcal{E}} = (I_E(f))_{E\in\mathcal{E}} \tag{3.3}$$

that is, κ associates an act f with the set of all its conditional evaluations (we wrote f instead of $f \mid_E$ because, as noted above, the charge P_E is supported by E). Next, notice that there is an obvious identification between the set $S_{/\mathcal{E}}$ and the set $\{P_E\}_{E\in\mathcal{E}}$ given by $E \longmapsto P_E$. Hence, the image of f under κ can be thought of as the mapping $\kappa(f) : \{P_E\}_{E\in\mathcal{E}} \longrightarrow \mathbb{R}$ which associates the measure P_E to the number $\int f dP_E$, the SEU evaluation of f given $E \in \mathcal{E}$. Finally, by denoting the integral on the RHS of (3.2) by V, we can re-write (3.2) as

$$I(f) = V(\kappa(f)) \tag{3.4}$$

The factorization property, $I = V \circ \kappa$, in (3.4) expresses the crucial property of either equation (3.1) or equation (3.2): the unconditional preference depends only on the conditional ones. This is intuitively sound. If this requirement were violated, the link between conditional and unconditional preferences would be too tenuous, and probably nothing interesting could be said about their relation.

Motivated by this discussion, we give the following definition. Let $\mathcal{E}$ be a family of subsets of Σ (not necessarily a partition). For $f \in B(\Sigma)$, let κ be the mapping defined by

$$\kappa : f \longmapsto \psi_f$$

where $\psi_f : \{P_E\}_{E\in\mathcal{E}} \longrightarrow \mathbb{R}$ is defined by $\psi_f(P_E) = \int f dP_E$. It is useful to stress that the domain of the mappings ψ_f depends on the family $\mathcal{E}$, which is a primitive of the model. Also, because of the nature of conditional EU, the probability P_E is supported by the set E.

Definition 1. *Let $\mathcal{E}$ be a family of subsets of Σ. For a preference relation $\succsim$ on $B(\Sigma)$, denote by $I : B(\Sigma) \longrightarrow (\mathcal{Z}, \geq)$ the functional representing it, where $(\mathcal{Z}, \geq)$ is some ordered space.[3] We say that $\succsim$ satisfies the conditional EU property with respect to $\mathcal{E}$ if and only if there exists a functional $V : \kappa(B(\Sigma)) \longrightarrow (\mathcal{Z}, \geq)$ such that I factors as $I = V \circ k$.*

By noticing that any preference on $B(\Sigma)$ trivially admits a representation of the form $I : B(\Sigma) \longrightarrow (\mathcal{Z}, \geq)$ (take $\mathcal{Z} = B(\Sigma)$ and the ordering defined by the preference), the definition says in a precise way exactly what we said above: an unconditional preference is conditionally EU if only if it depends on the conditional preferences only. We conclude this section

[3]By an ordered space, we mean a set endowed with a quasi-order. While this is at odds with current usage (an ordered space is a set with a partial order), it simplifies the statement above. At any rate, this terminology is not used in the remainder of the paper.

with an example of a preference $\succsim$ which satisfies the requirement in the definition.

EXAMPLE 1. *Let $\mathcal{E} \subset \Sigma$, and let $\{\succsim_E\}_{E\in\mathcal{E}}$ be a family of SEU preferences. Define $\succsim$ on $\mathcal{A}$ by*

$$f \succsim g \qquad iff \qquad \psi_f \geq \psi_g$$

where $\psi_f \geq \psi_g$ means $\psi_f(P_E) \geq \psi_g(P_E)$ for all $E \in \mathcal{E}$ or, equivalently, $\int f dP_E \geq \int g dP_E$ for all $E \in \mathcal{E}$. This preference relation $\succsim$ is incomplete if $\mathcal{E}$ contains more than one element. It is reminiscent of Bewley's preferences [**4**] *and of the unambiguous preference relation of Ghirardato et al.* [**7**].

4. $\mathcal{D}$-preferences

With this section, we are going to narrow down the focus of our study. In doing so, we will exclude from consideration incomplete preferences like the one seen in the example at the end of the previous section. In order to avoid duplications, we state our assumptions directly in terms of the properties of the functionals involved in the representation. Corresponding axioms on preferences are discussed below, following our representation theorem. For lack of a better name, the preferences object of our interest will be called $\mathcal{D}$-preferences. Let $\mathcal{E}$ be a family of subsets of Σ, and let $\{\succsim_E\}_{E\in\mathcal{E}}$ be the corresponding family of conditional expected utility preferences. Let the mapping κ be as in the previous section.

DEFINITION 2. *A preference relation $\succsim$ on $\mathcal{A}$ $(= B(\Sigma))$ is a $\mathcal{D}$-preference with respect to family $\mathcal{E}$ if it satisfies the following properties:*

(1) The functional I that represents it factors as $I = V \circ k$;

(2) The functional V is real-valued;

(3) The functional V is translation invariant: for all $\lambda \geq 0$, $\beta \in \mathbb{R}$ and for all $\psi \in \kappa(B(\Sigma))$

$$V(\lambda\psi + \beta\mathbf{1}) = \lambda V(\psi) + \beta V(\mathbf{1})$$

where $\mathbf{1}$ denotes the function identically equal to 1 on its domain.

(4) The functional V is monotone: for $\psi, \varphi \in \kappa(B(\Sigma))$

$$\psi \geq \varphi \qquad \Longrightarrow \qquad V(\psi) \geq V(\varphi)$$

Property (1) simply says that we are still in the realm of preferences satisfying the conditional EU property. Properties (2) to (4) imply that $\mathcal{D}$-preferences are complete, transitive and (as we shall see) Archimedean. With regard to Property (3), observe that, having identified $\mathcal{A}$ with $B(\Sigma)$, $\psi \in \kappa(B(\Sigma))$ implies $\lambda\psi + \beta\mathbf{1} \in \kappa(B(\Sigma))$. Thus, Property (3) is legitimate. Its introduction is motivated by the fact that the utility function $u : C \longrightarrow \mathbb{R}$ of assumption **S2** is unique only up to positive affine transformations. As such, we consider Property (3) a mild as well as a very reasonable assumption. Property (4) in combination with Property (1) postulates that if an act f is better than an act g for each of the possible conditioning events, then f should be preferred to g unconditionally. Again, we consider this a very mild assumption.

Next, we are going to provide a representation for $\mathcal{D}$-preferences. We first need an easy lemma. Set $\mathcal{C} = \{P_E\}_{E\in\mathcal{E}} \subset ba_1^+(\Sigma)$ and let $\tilde{\mathcal{C}} = \overline{co}(\mathcal{C})$ be the weak*-closed convex hull of $\mathcal{C}$. For $f \in B(\Sigma)$, denote by ψ_f and $\tilde{\psi}_f$ the mappings defined by

$$\psi_f(P) = \int_S f dP \quad , \qquad P \in \mathcal{C} \qquad \textit{and} \qquad \tilde{\psi}_f(\tilde{P}) = \int_S f d\tilde{P} \quad , \qquad \tilde{P} \in \tilde{\mathcal{C}}$$

and let $\kappa : f \longmapsto \psi_f$ and $\tilde{\kappa} : f \longmapsto \tilde{\psi}_f$.

Lemma 1. *Let $V : \kappa(B(\Sigma)) \longrightarrow \mathbb{R}$ be a functional satisfying properties (3) and (4) above. Then, there exists a unique functional $\tilde{V} : \tilde{\kappa}(B(\Sigma)) \longrightarrow \mathbb{R}$ such that $V(\psi_f) = \tilde{V}(\tilde{\psi}_f)$, for all $f \in B(\Sigma)$. Moreover, $\tilde{V}$ satisfies properties (3) and (4).*

Proof. By properties (3) and (4), $\left(\sup_{\mathcal{C}} \psi_f\right) V(\mathbf{1}) \geq V(\psi_f) \geq \left(\inf_{\mathcal{C}} \psi_f\right) V(\mathbf{1})$. Hence, for each ψ_f there exists $\alpha(\psi_f) \in [0,1]$ such that $V(\psi_f) = [\alpha(\psi_f) \inf_{\mathcal{C}} \psi_f + (1 - \alpha(\psi_f)) \sup_{\mathcal{C}} \psi_f] V(\mathbf{1})$. Now, observe that

$$\inf_{\mathcal{C}} \psi_f = \inf_{\tilde{\mathcal{C}}} \tilde{\psi}_f \qquad \textit{and} \qquad \sup_{\mathcal{C}} \psi_f = \sup_{\tilde{\mathcal{C}}} \tilde{\psi}_f$$

and that the mapping $\psi_f \longmapsto \tilde{\psi}_f$ from $\kappa(B(\Sigma) \longrightarrow \tilde{\kappa}(B(\Sigma))$ is clearly one-to-one and onto. Hence, we can define $\tilde{\alpha} : \tilde{\kappa}(B(\Sigma)) \longrightarrow [0,1]$ by $\tilde{\alpha}(\tilde{\psi}_f) = \alpha(\psi_f)$. Then, $\tilde{V} : \tilde{\kappa}(B(\Sigma)) \longrightarrow \mathbb{R}$ defined by $\tilde{V}(\tilde{\psi}_f) = [\tilde{\alpha}(\tilde{\psi}_f) \inf_{\mathcal{C}} \tilde{\psi}_f + (1 - \tilde{\alpha}(\tilde{\psi}_f)) \sup_{\mathcal{C}} \tilde{\psi}_f] V(\mathbf{1})$ is the unique functional satisfying $V(\psi_f) = \tilde{V}(\tilde{\psi}_f)$, for all $f \in B(\Sigma)$. The second part is immediate. □

By means of the lemma, the problem of representing V is transformed into that of representing $\tilde{V}$. The advantage of doing so is that the domain of $\tilde{V}$ is the space $A(\tilde{\mathcal{C}})$ of all weak*-continuous affine functions on the convex, weak*-compact set $\tilde{\mathcal{C}}$. Moreover, $\tilde{\kappa}$ is the canonical linear mapping

$B(\Sigma) \longrightarrow A(\tilde{\mathcal{C}})$. Thus, the characterization of $\mathcal{D}$-preferences becomes an easy consequence of a theorem that we proved in [**1**].

THEOREM 1. *A preference relation $\succsim$ on $\mathcal{A}$ $(= B(\Sigma))$ is a $\mathcal{D}$-preference with respect to the family $\mathcal{E}$ if and only if there exists a capacity*[4] Γ *on* $\tilde{\mathcal{C}} = \overline{co}\{P_E\}_{E\in\mathcal{E}}$ *such that,* $\forall f \in B(\Sigma)$

$$I(f) = V(\kappa(f)) = \tilde{V}(\tilde{\kappa}(f)) = \int_{\tilde{\mathcal{C}}} \tilde{\kappa}(f) d\Gamma$$

where the integral is taken in the sense of Choquet.

PROOF. Let $\psi, \varphi \in \tilde{\kappa}(B(\Sigma)) = A(\tilde{\mathcal{C}})$. If ψ and φ are non-constant, then ψ and φ are comonotonic if and only if they are isotonic [**1**, Prop. 2]. In such a case, there exist $\lambda > 0$ and $\beta \in \mathbb{R}$ such that $\varphi = \lambda\psi + \beta\mathbf{1}$. Then, by property (3)

$$\tilde{V}(\psi + \varphi) = \tilde{V}((1 + \lambda)\psi + \beta) = \tilde{V}(\psi) + \tilde{V}(\varphi)$$

Combined with the previous observation, this implies that $\tilde{V}$ is comonotonic additive on its domain. By property (4), $\tilde{V}$ is monotone as well. By [**1**, Cor. 1], these functionals can be represented by Choquet integrals. The converse statement – that any preference defined by $\int_{\tilde{\mathcal{C}}} \tilde{\kappa}(f) d\Gamma$ is a $\mathcal{D}$-preference – follows immediately from the properties of the Choquet integral. □

It is now easy to provide a behavioral foundation for $\mathcal{D}$-preferences. We recall that a preference relation $\succsim$ on $\mathcal{A}$ is an Invariant Bi-separable (IB) preference if it satisfies the axioms of (I') completeness and transitivity; (II') Constant-independence; (III') Archimedean property; (IV') Monotonicity; and (V') Non-triviality (see Ghirardato et al. [**7**] for a formal statement). The next corollary states that $\mathcal{D}$-preferences are a sub-class of IB preferences.

COROLLARY 1. *$\mathcal{D}$-preferences are IB preferences.*

PROOF. If $\succsim$ is a $\mathcal{D}$-preference with respect to some family $\mathcal{E}$, then it is represented by a real-valued functional that factors as $I = V \circ k$. Since κ is linear, the translation invariance of V implies the translation invariance of I. If $f, g \in B(\Sigma)$ are such that $f \geq g$, then $\kappa(f) \geq \kappa(g)$ and monotonicity

[4]By a capacity Γ on $\tilde{\mathcal{C}} = \overline{co}\{P_E\}_{E\in\mathcal{E}}$ we mean a capacity on the Borel σ-algebra generated by the weak*-topology on $\tilde{\mathcal{C}}$.

of V implies $I(f) \geq I(g)$, that is I is monotone. The following well-known argument shows that I is sup-norm continuous. For $f, g \in B(\Sigma)$, one has

$$\begin{aligned} f &= g + f - g \leq g + \|f - g\| \\ g &= f + g - f \leq f + \|f - g\| \end{aligned}$$

Hence, by monotonicity and translation invariance

$$|I(f) - I(g)| \leq \|f - g\| \, I(\mathbf{1})$$

which is the sup-norm continuity of I. From this, the Archimedean property follows immediately. Thus, $\mathcal{D}$-preferences satisfy axioms (I') to (V') above. □

By virtue of the corollary, we then have

Corollary 2. *A preference relation $\succsim$ on $\mathcal{A}$ is a $\mathcal{D}$-preference if and only if it satisfies axioms (I') to (V') above plus assumptions* ***S1*** *and* ***S2'****.*

Note that general IB-preferences are not necessarily $\mathcal{D}$-preferences as the preferences conditional on $E \in \mathcal{E}$ (however defined) need not be SEU.

5. SEU unconditional preferences

In this section, we are going to restrict further the class of preferences under consideration, and focus on $\mathcal{D}$-preferences that are unconditionally SEU preferences. In the next section, we are going to compare our results with Fishburn's. To begin, let us consider the case in which the family $\mathcal{E}$ is a finite partition of S, $\mathcal{E} = \{E_1, ..., E_n\}$. As a corollary to Theorem 1, we have

Corollary 3. *Let $\mathcal{E} = \{E_1, ..., E_n\}$ be a finite partition of S. A $\mathcal{D}$-preference with respect to $\mathcal{E}$ is a SEU preference iff V is linear. In such a case, I has the representation $I(f) = \int_s f dP_\Gamma$, $P_\Gamma \in ba_1^+(\Sigma)$ and for all $f \in B(\Sigma)$ it holds that*

$$\int_S f dP_\Gamma = \sum_{i=1}^{n} P_\Gamma(E_i) \int f dP_{E_i}$$

Proof. Clearly, a $\mathcal{D}$-preference is SEU if and only if the functional I representing it is linear. Since $\tilde{\kappa}$ is a linear mapping, $\tilde{V}$ must be a linear functional on $A(\tilde{\mathcal{C}})$. By Hahn-Banach, $\tilde{V}$ can be extended to a linear functional on $C(\tilde{\mathcal{C}})$, the Banach space of all weak*-continuous functions on the compact, convex set $\tilde{\mathcal{C}}$ endowed with the sup-norm. It then follows from the

Riesz representation theorem that the capacity Γ in the theorem is a Borel probability measure on $\tilde{\mathcal{C}}$. By [**11**, Proposition 1.1], Γ has a unique barycenter $P_\Gamma \in \tilde{\mathcal{C}}$. That is, P_Γ is such that $\psi(P_\Gamma) = \int \psi d\Gamma$ for every continuous affine function ψ on $\tilde{\mathcal{C}}$. Thus, by Theorem 1

$$I(f) = \int\limits_{\tilde{\mathcal{C}}} \tilde{\kappa}(f) d\Gamma = \tilde{\kappa}(f)(P_\Gamma) = \int_S f dP_\Gamma$$

Since $P_\Gamma \in \tilde{\mathcal{C}} = \overline{co}\{P_{E_i}\}_{i=1}^n$, P_Γ can be written as a convex combination of the P_{E_i}' s: $P_\Gamma = \sum_{i=1}^n P"(P_{E_i})P_{E_i}$, where $P"(\cdot)$ is a probability measure on the set of extreme points of $\tilde{\mathcal{C}}$, that is on the set $\{P_{E_i}\}_{i=1}^n$ (recall that E_i's are pairwise disjoint). Define a mapping $K : \{P_{E_1}, ..., P_{E_n}\} \longrightarrow \{E_1, ..., E_n\} = S_{/\mathcal{E}}$ by $K(P_{E_i}) = E_i$. Clearly, K is a bijection. Let P' be the image measure of $P"$ under K. Define $Q_\Gamma = \sum_{i=1}^n P'(E_i)P_{E_i}$; then Q_Γ defines a continuous linear functional on $B(\Sigma)$. It is easy to check that this coincides with the continuous linear functional defined by P_Γ for all simple functions. Since the set of all simple functions is norm dense in $B(\Sigma)$, the two functionals coincide on $B(\Sigma)$ and we have $Q_\Gamma = P_\Gamma = \sum_{i=1}^n P'(E_i)P_{E_i}$. Finally, since P_{E_i} is supported by E_i, we conclude that $P_\Gamma(E_i) = P'(E_i)$ for each i. □

A useful, yet trivial, observation that emerges from the proof of Corollary 3 is recorded below. Notice, however, that Corollary 4 is Fishburn's theorem [**6**, Theorem 1] specialized to the case of a partition.

COROLLARY 4. *Let $\mathcal{E}$ be a finite partition $\mathcal{E} = \{E_1, ..., E_n\}$ of S, and let $\{\succsim_E\}_{E\in\mathcal{E}}$ be a family of conditional preferences satisfying the axioms of SEU. Then, there always exists an unconditional SEU preference $\succsim$ such that all the $\succsim_E$ are the conditional preferences derived from $\succsim$ by means of Bayes' rule.*

PROOF. Take any $P \in co\{P_{E_i}\}_{i=1}^n$. From the proof of Corollary 3, it follows that the unconditional SEU preference defined by

$$f \succsim g \qquad iff \qquad \int f dP \geq \int g dP$$

satisfies the desired property. □

When we consider, as Fishburn does, families $\mathcal{E}$ more general then those in Corollary 4, we should expect that the requirements for the existence of an unconditional SEU preference with the assigned conditional properties become more stringent. Let P be the measure representing an unconditional SEU preference, and suppose that the family $\mathcal{E}$ contains a partition $\{E_1, ..., E_n\}$ of S. We have seen that P satisfies the assigned conditional properties if and only if $P \in co\{P_{E_i}\}_{i=1}^n$. With this in mind, the next proposition comes easily. It parallels an observation made by Samet [**13**] in the context of games with incomplete information. For $\mathcal{E}$ a class of events, let Π be the collection of all finite partitions of S which consist of events belonging to $\mathcal{E}$.

PROPOSITION 1. *Let $\mathcal{E}$ be a class of events which is closed under complementation, and let $\{\succsim_E\}_{E\in\mathcal{E}}$ be a family of conditional preferences satisfying the axioms of SEU. For $\pi \in \Pi$, denote by $\{P_{\pi_i}\}$ the corresponding collection of conditional probabilities. Then, there exists an unconditional SEU preference $\succsim$ such that all the $\succsim_E$ are the conditional preferences derived from $\succsim$ if and only if*

$$\bigcap_{\pi\in\Pi} co\{P_{\pi_i}\} \neq \varnothing$$

In such a case, if $P \in \cap_{\pi\in\Pi} co\{P_{\pi_i}\}$, then for every $\pi \in \Pi$, $\pi = \{E_1, ..., E_n\}$, and every $f \in B(\Sigma)$ we have

$$\int_S f dP = \sum_{i=1}^{n} P(E_i) \int f dP_{E_i} \tag{5.1}$$

Proposition 1 delivers the same statement as Fishburn's Theorem 1, although it is established under a different set of assumptions. Notice, however, that the proposition not only covers the case where $\mathcal{E}$ is an arbitrary algebra, as in Fishburn, but also more general cases. For instance, $\mathcal{E}$ could be a λ-system, a case of special interest in the study of unambiguous events (see [**2**])

PROOF. Since $\mathcal{E}$ is closed under complementation, any $E \in \mathcal{E}$ belongs to at least one π. If $P \in \cap_{\pi\in\Pi} co\{P_{\pi_i}\}$, then P satisfies (5.1) for every $\pi \in \Pi$ and every $f \in B(\Sigma)$ by Corollary 4. Conversely, we are going to show that if $\cap_{\pi\in\Pi} co\{P_{\pi_i}\} = \varnothing$, then no P in $ba_1^+(\Sigma)$ can satisfy the assigned conditional properties. Let $P \in ba_1^+(\Sigma)$. Since $\cap_{\pi\in\Pi} co\{P_{\pi_i}\} = \varnothing$, there exists a $\pi \in \Pi$ such that $P \notin co\{P_{\pi_i}\}$. By the separation theorem, there exists a weak*-continuous linear functional L on $ba(\Sigma)$ such that $L(P) > L(\tilde{P})$ for all $\tilde{P} \in co\{P_{\pi_i}\}$. By definition, this means that $\exists f \in B(\Sigma)$ (which can be taken to be valued in $u(C)$) such that $\int f dP > \int f d\tilde{P}$ for all $\tilde{P} \in co\{P_{\pi_i}\}$.

Thus, for such an f (5.1) fails as

$$\int_S f dP > \sum_{i=1}^{n} P(E_i) \int f dP_{E_i}$$

That is, if $\cap_{\pi \in \Pi} co \{P_{\pi_i}\} = \varnothing$, then no $P \in ba_1^+(\Sigma)$ can satisfy (5.1) $\square$

It is possible to extend the above results to the case of infinite (non-necessarily countable) partitions by adding some measurability conditions. These do not appear in Fishburn because his statements apply to finite partitions only. To see why additional conditions are necessary, notice that in the infinite case the condition $\int_S f dP = \sum_{i=1}^{n} P(E_i) \int f dP_{E_i}$ reads as

$$\int_S f dP = \int_{S_{/\mathcal{E}}} \int f dP_{E_i} dP' \tag{5.2}$$

which makes sense only if the integral on the RHS exists. Clearly, this requires that the mapping $S_{/\mathcal{E}} \longrightarrow \mathbb{R}$ defined by $E_i \longmapsto \int f dP_{E_i}$ be a measurable mapping when the quotient $S_{/\mathcal{E}}$ is endowed with the measurable structure carried by the canonical projection $\rho : S \longrightarrow S_{/\mathcal{E}}$ (that is, the finest σ-algebra on $S_{/\mathcal{E}}$ which makes ρ is measurable). This requirement is trivially satisfied when the partition $\{E_i\}$ is finite and all the E_i's are elements of Σ (as we have assumed throughout), but might fail when $\{E_i\}$ is infinite. Thus, when dealing with infinite partitions, one must restrict to those partitions such that the mapping $E_i \longmapsto \int f dP_{E_i}$ is measurable. In general, however, this condition is necessary but not sufficient. A set of sufficient conditions guarateeing that the decomposition in (5.2) holds is given in [**5**, pp. 125-129]. Once those are assumed, the statements above extend without further modifications.

6. Comparison with Fishburn's axioms

Let $\mathcal{E} \subset \Sigma$ be a family of subsets of S which is closed under complementation, and let $\succsim_E$, $E \in \mathcal{E}$, be a family of preferences satisfying the Anscombe-Aumann axioms for each $E \in \mathcal{E}$. We have seen that under assumption **S0** and **S2** as well as the condition $\cap_{\pi \in \Pi} co \{P_{\pi_i}\} \neq \varnothing$, there exists a $P \in ba_1^+(\Sigma)$ such that for every finite partition $\pi = \{E_1, ..., E_n\} \subset \mathcal{E}$, and every $f \in B(\Sigma)$

$$\int_S f dP = \sum_{i=1}^{n} P(E_i) \int f dP_{E_i} \tag{6.1}$$

In addition, if we assume, as in Fishburn [**6**], that $\mathcal{E} = \Sigma$ – the algebra defining the measurable structure on the state space – then the measure P is unique as a consequence of the Riesz representation theorem (given assumption **S0**). The decomposition (6.1) along with the uniqueness of the measure P is essentially Fishburn's Theorem 1. We are now going to compare our conditions with those of Fishburn [**6**].

Without the non-triviality condition embedded therein, our assumption **S1** is equivalent to the axioms (H1) to (H3) mentioned in [**6**, p. 9] and, as such, it is weaker than the axioms (A1) to (A3) used by Fishburn [**6**, p. 8]. Moreover, it is easy to see that Fishburn's axioms (A1) to (A6) imply our **S2** as well as our non-triviality conditions. Thus, Fishburn's (A1) to (A6) imply our **S1** and **S2**. It remains to analyze the role of **S0** and of the condition $\underset{\pi\in\Pi}{\cap} co\{P_{\pi_i}\} \neq \varnothing$.

The next Proposition shows that Fishburn's (A1) to (A6) imply the condition $\underset{\pi\in\Pi}{\cap} co\{P_{\pi_i}\} \neq \varnothing$. For the ease of the reader, we report Fishburn's (A4) below, as it is the crucial axiom leading to this result.

> **Fishburn A4:** Let $A, B \in \Sigma$ be such that $A \cap B = \varnothing$, and for $E \in \Sigma$, let f_E denote the evaluation of the act f given $E \in \Sigma$. Then,
>
> $$f_A \succsim f_B \qquad \Longrightarrow \qquad f_A \succsim f_{A\cup B} \succsim f_B$$

Proposition 2. *Fishburn's (A1) to (A6) imply* $\underset{\pi\in\Pi}{\cap} co\{P_{\pi_i}\} \neq \varnothing$.

Before we give the proof of the Proposition, a remark is in order. Under assumptions (A1) to (A6), one can still identify an act with a real-valued function on S. But, without **S0** one can no longer identify the set of acts to the whole $B(\Sigma)$. Rather, the set of acts corresponds to a convex subset of $A \subset B(\Sigma)$ (since Fishburn assumes that the set of acts is a mixture set).

Proof. By [**6**, Theorem 3], (A1) to (A6) imply that there exists a linear functional $U : \mathcal{A} \longrightarrow \mathbb{R}$ representing the unconditional preference. U can be first extended to the subspace generated by $\mathcal{A}$ by homogeneity, and then extended to the whole $B(\Sigma)$ by Hahn-Banach. Let $P \in ba(\Sigma)$ be the charge representing it. Fishburn's result [**6**, Theorem 3] guarantees that P can be taken to be a probability. Now, suppose by the way of contradiction that $\underset{\pi\in\Pi}{\cap} co\{P_{\pi_i}\} = \varnothing$. Thus, there exists a partition $\pi \in \Pi$, $\pi = \{E_1, ..., E_n\}$, such that $P \notin co\{P_{\pi_i}\}$. By the separation theorem, $\exists f$ such that for any choice of weights α_i, $\alpha_i \geq 0$ and $\sum_1^n \alpha_i = 1$, we have

$$\int f dP > \sum_1^n \alpha_i \int f dP_{E_i}$$

Define $A = E_1$ and $B = \cup_2^n E_i$. By A4, $P_B(\cdot)$ is an average of the E_i's, $i = 2, ..., n$. Hence, we choose the weights α_i's so that $\sum_2^n \alpha_i P_{E_i}(\cdot) = P_B(\cdot)$. But then, the preceding implies

$$\int_{S=A\cup B} f dP > P(A) \int f dP_A + P(B) \int f dP_B$$

which contradicts A4. □

Summing up, Fishburn's (A1) to (A6) imply our **S1** and **S2** as well as the condition $\underset{\pi\in\Pi}{\cap} co\{P_{\pi_i}\} \neq \varnothing$. We now move to studying the effect of assumption **S0**.

Assumption **S0** buys us the following uniqueness properties:

I. The unconditional measure is unique; and

II. The "weights" $P(E_i)$ in the decomposition (6.1) are unique.

Once the existence of an unconditional SEU preference with the desired conditional properties is established, both these properties follow from the Riesz representation theorem: **S0** allows us to identify the set of acts with $B(\Sigma)$, and the Riesz representation theorem tells us that there exists a unique measure representing the SEU preference. Fishburn [**6**, Theorem 3] obtains the same uniqueness properties without imposing **S0**: in addition to his axioms A1 to A6 mentioned above, he uses a much richer preference relation (Fishburn's preference relation is defined on $\mathcal{A} \times \Sigma$) and two additional assumptions (Fishburn's A7 and A8).[5] It is possible, but rather tedious, to give a direct proof that these assumptions essentially imply our **S0**.[6] An indirect approach, however, would be much more telling: in fact, the theorem of Phelps [**12**, Theorem 1] below tells us that *any* set of assumptions leading to the uniqueness of the unconditional measure is essentially equivalent to **S0**. Let us denote by **SW** an arbitrary assumption. We will say that Assumption **SW** is *essentially weaker* than **S0** if it implies that (by following the same procedure as in [**9**]; see end of Section 2 of this paper) the set of

[5]It is readily seen that Fishburn's A7 and A8 are necessary conditions for the uniqueness properties above.
[6]The meaning of the qualification "essentially" will be clarified below.

acts is identified to a proper closed subspace $F \subset B(\Sigma)$. With the regard to the uniqueness of the unconditional measure, we have

COROLLARY 5. *Assume* ***S1***, ***S2***, $\bigcap_{\pi \in \Pi} co\{P_{\pi_i}\} \neq \varnothing$ *and* ***SW***. *If* ***SW*** *is essentially weaker than* ***S0***, *then the unconditional measure is not unique.*

The result is a consequence of the following observations. Under the conditions in the corollary, Proposition 1 guarantees the existence of an unconditional measure, hence of a continuous linear functional defined on a subspace of $(B(\Sigma), \|\cdot\|_\infty)$. Since **SW** is essentially weaker than **S0**, this subspace is a proper closed subspace of $(B(\Sigma), \|\cdot\|_\infty)$. From Phelps' theorem (see below) and the fact that the dual of the Banach space $(B(\Sigma), \|\cdot\|_\infty)$ is not strictly convex, we then conclude that this continuous linear functional admits, generally speaking, more than one norm-preserving extension. All these extensions represent the same unconditional preference, and are associated to different probability measures on Σ.

When assumption **S0** is replaced by a weaker assumption, thus the set of acts is identified to a proper closed subset $F \subset B(\Sigma)$, the problem of the uniqueness of the "weights" $P(E_i)$ in the decomposition (6.1) is still meaningful despite the non-uniqueness of the unconditional measure (the reader should think, for example, of the case of a partition $\mathcal{E}$ of S). A necessary and sufficient condition for the uniqueness of these weights is given in the next corollary, which is once again an immediate implication of Phelps' theorem. Rather than deducing it from general principles, we chose to provide a direct proof so to make the argument immediately accessible. However, as the proof uses Phelps' theorem, for the reader's convenience we report it below.

THEOREM 2 (Phelps [**12**, Theorem 1]). *Let X be a normed linear space. A closed subspace $V \subset X$ has the unique extension property iff its annihilator $V^\perp \subset X^*$ is Chebyshev.*

We recall that a closed subspace V of a normed linear space X has the unique extension property if every linear functional $V \longrightarrow \mathbb{R}$ has a unique norm-preserving extension to the whole X. We also recall that a V is said to be Chebyshev if and only if for each $x \in X$ there exists exactly one element of best approximation, that is, an element $y \in V$ such that $\|x - y\| = dist(x, V)$.

For simplicity, we are going to focus on the case where the family $\mathcal{E}$ is a finite partition. More complicated cases where $\mathcal{E}$ is either infinite or contains more than one partition are easily dealt with by introducing the obvious modifications. So assume that $\mathcal{E} = \{E_1, E_2, ..., E_n\}$ is a partition of S. An unconditional measure P with the desired conditional properties is completely determined by the weights $P(E_i)$ that it assigns to the events E_i. Thus, P is completely defined by $n-1$ numbers. Any act $f \in F \subset B(\Sigma)$ defines a function $S_{/\mathcal{E}} \longrightarrow \mathbb{R}$ by $E_i \longmapsto \int f dP_{E_i}$, where P_{E_i} is the conditional measure for the event E_i. That is, the partition $\mathcal{E}$ along with the family of conditional measures $\{P_{E_i}\}_{E_i \in \mathcal{E}}$ induces a mapping $\Pi : F \longrightarrow B(S_{/\mathcal{E}})$, where $B(S_{/\mathcal{E}})$ denotes the set of mappings on the quotient $S_{/\mathcal{E}}$. Clearly, $B(S_{/\mathcal{E}})$ can be identified to a subset of $\mathbb{R}^n$. Let $lin\{1\}$ denote the one-dimensional subspace consisting of the constant functions on $S_{/\mathcal{E}}$. We have

COROLLARY 6. *The weights $P(E_i)$ are unique iff $\Pi(F) \cup lin\{1\} = \mathbb{R}^n$.*

PROOF. For any measure P on Σ with the assigned conditional properties, let $\tilde{P} = (P(E_1), P(E_2), ..., P(E_n))$. $\tilde{P}$ defines a linear functional on $V = \Pi(F) \cup lin\{1\}$. Thus, $\tilde{P} \in (\mathbb{R}^n, \|\cdot\|_1)$, where, for $x = \{x_1, x_2, ..., x_n\}$, $\|x\|_1 = \sum_{i=1}^{n} |x_i|$, and $V \subset (\mathbb{R}^n, \|\cdot\|_\infty)$.

Sufficiency: $V = \mathbb{R}^n$ obviously implies that $V^\perp$ is Chebyshev. Thus, by Phelps' theorem $\tilde{P}$ is unique.

Necessity: By assumption, $1 \in V$. Then $\lambda \in V^\perp$ implies $\sum_{i=1}^{n} \lambda_i = 0$. Moreover, $\lambda \in V^\perp$ implies $k\lambda \in V^\perp$ for all $k \in \mathbb{R}$. Let $x = (x_1, ..., x_n)$ be such that $x_i > 0$ for all i. For $\lambda \in V^\perp$, we have $\|x - k\lambda\|_1 = \sum_{i=1}^{n} |x_i - k\lambda_i|$. Partition $N = \{1, 2, ..., n\}$ into two subsets $N = I^+ \cup I^-$ where $I^+ = \{j \in N \mid |x_j - k\lambda_j| \geq 0\}$ and I^- is defined accordingly. Then,

$$\begin{aligned} \|x - k\lambda\|_1 &= \sum_{i \in I^+} (x_i - k\lambda_i) + \sum_{i \in I^-} (k\lambda_i - x_i) \\ &\geq \sum_{i \in I^+} (x_i - k\lambda_i) + \sum_{i \in I^-} (x_i - k\lambda_i) = \|x\|_1 \end{aligned}$$

Now, if $V \neq \mathbb{R}^n$, then, $V^\perp$ is non-trivial. That is, $\exists y \in V^\perp$, $y \neq 0$. By the preceding, if x is such that $x_i > 0$ for all i, then $\forall \lambda \in V^\perp$ we have $\|x - k\lambda\|_1 \geq \|x\|_1$. Clearly, the value $\|x\|_1$ is attained by setting $k = 0$, that is $dist(x, V^\perp) = \|x\|_1$ and $0 \in \arg\min_{\lambda \in V^\perp} \|x - \lambda\|_1$. However, if $y \in V^\perp$ and $y \neq 0$, then $\sup_i y_i > 0$ (because $y \in V^\perp$ implies $\sum_{i=1}^{n} y_i = 0$) and any z in the set

$$\left\{ z = ky \in V^{\perp} \mid \frac{\inf_i x_i}{\sup_i y_i} \geq k > 0 \right\}$$

also belong to $0 \in \arg\min_{\lambda \in V^{\perp}} \|x - \lambda\|_1$ as (the second inequality below follows from $x_i \geq \inf_i x_i \geq k \sup_i y_i \geq k y_i$)

$$\|x - ky\|_1 = \sum_{i=1}^{n} |x_i - ky_i| = \sum_{i=1}^{n} (x_i - ky_i) = \sum_{i=1}^{n} x_i = \|x\|_1$$

That is, $V^{\perp}$ is not Chebyshev and $\tilde{P}$ is not unique. □

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