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# Strategy-Proof Tie-Breaking\*

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## Abstract

We study a general class of priority-based allocation problems with weak priority orders and identify conditions under which there exists a *strategy-proof* mechanism which always chooses an *agent-optimal stable*, or *constrained efficient*, matching. A priority structure for which these two requirements are compatible is called *solvable*.

For the general class of priority-based allocation problems with weak priority orders, we introduce three simple necessary conditions on the priority structure. We show that these conditions completely characterize solvable environments within the class of *indifferences at the bottom* (IB) environments, where ties occur only at the bottom of the priority structure. This generalizes and unifies previously known results on solvable and unsolvable environments established in school choice, housing markets and house allocation with existing tenants. We show how the previously known solvable cases can be viewed as extreme cases of solvable environments. For sufficiency of our conditions we introduce a version of the agent-proposing deferred acceptance algorithm with exogenous and preference-based tie-breaking.

*JEL Classification:* C78, D61, D78, I20.

*Keywords:* Weak priorities, stability, constrained efficiency, strategy-proofness.

## 1 Introduction

In this paper we consider various classes of priority-based allocation problems where a set of indivisible objects is to be allocated among a finite set of agents and no monetary transfers are available. Agents have privately known strict preferences over available objects. For any object there is an exogenously given weak ordering specifying which agents have higher priority to be allocated that object. We restrict attention to *strategy-proof* (direct) mechanisms that provide

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agents with dominant strategy incentives to report preferences truthfully.<sup>1</sup> A *matching* (of agents to objects) is *stable*, if (i) no agent is worse off than receiving no object (*individual rationality*), (ii) no agent strictly prefers an unassigned object to her assignment (*non-wastefulness*), and (iii) there is no agent  $i$  who strictly prefers an object  $o$  (over her assignment) that was assigned to another agent  $j$  who has strictly lower priority for  $o$  than  $i$  (*fairness*).<sup>2</sup> A stable matching is *constrained efficient*, if it is not Pareto dominated by another *stable* matching. In this paper we search for tractable conditions which guarantee that a priority structure is *solvable* in the sense of admitting a constrained efficient and strategy-proof matching mechanism.

Important real-life examples of the class of problems we analyze are school choice, where a student's priority for a school is determined by objective criteria such as distance or the existence of siblings already attending the school, the allocation of dorm rooms, where an existing tenant is usually guaranteed priority for her room over others, and (live-donor) kidney exchange, where a potential donor who is immunologically incompatible with her intended recipient is only willing to give her kidney to someone else if her intended recipient receives a compatible kidney in exchange.<sup>3</sup> These three problems share the feature that priorities are exogenous and (strict) priority rankings are commonly known. Furthermore, stability is an important allocative desideratum: for the school choice problem, an unstable assignment is susceptible to appeals by unhappy parents and may be detrimental to public acceptance of an admissions procedure given the absence of a clear rationale for rejections at over-demanded schools. In the dorm allocation or the kidney exchange problem, a violation of stability means that some existing tenants/patients would have been strictly better off not participating in the assignment procedure (staying in their old room in the former, and sparing their incompatible donor the pain of kidney extraction in the latter case). While efficiency losses due to stability constraints<sup>4</sup> may thus be deemed acceptable, it is important to avoid any further efficiency losses and thus ensure constrained efficiency of the chosen matching. Given the private information that is inherent to the problems described above, whether a priority structure is solvable or not is an important and practically relevant question.

The solvability of strict priority structures, where no two distinct agents can ever have equal priority for a given object, is well known: the agent-proposing deferred acceptance (ADA) algorithm of Gale and Shapley (1962) produces *the* unique constrained efficient matching and the associated direct revelation mechanism is (group) strategy-proof for the agents (Dubins and Freedman, 1981; Roth, 1982a). Similarly, for the dorm/house allocation problem with existing tenants, the top trading cycles (TTC) algorithm produces a constrained efficient matching and

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<sup>1</sup>Strategy-proofness is the most widely used incentive compatibility requirement in the area of market design without monetary transfers (see Roth, 2008, as well as Sönmez and Ünver, 2011, for recent surveys). See Abdulkadiroglu et al. (2006) for a fairness rationale supporting strategy-proofness. Budish and Cantillon (2011) provide a critical perspective on the restriction to strategy-proof mechanisms.

<sup>2</sup>See Roth and Sotomayor (1990) for an excellent introduction into the theory and applications of stable matching mechanisms.

<sup>3</sup>See Abdulkadiroglu and Sönmez (2003) for an analysis of school choice problems, Abdulkadiroglu and Sönmez (1999) for an analysis of the dorm room allocation problem, and Roth et al. (2004) for an analysis of the kidney exchange problem.

<sup>4</sup>Ergin (2002) shows that stability is often incompatible with full efficiency.

gives rise to a strategy-proof direct mechanism (Abdulkadiroglu and Sönmez, 1999).<sup>5</sup> These positive results for two very different classes of priority-based allocation problems, one without any ties in the priority structure and one in which almost no priority distinctions are made, encourage the search for other solvable priority structures. This is particularly important given that for many real-life applications, in particular school choice, priority structures do not belong to the two known solvable classes. Unfortunately, Erdil and Ergin (2008) establish the existence of unsolvable priority structures by presenting a simple unsolvable priority structure. The main difficulty in priority-based allocation problems lies in resolving *ties* in the priority structure: If tie-breaking does not condition on the submitted preferences of the agents, it creates additional stability constraints that can entail significant (cf. Abdulkadiroglu et al., 2009) welfare loss. On the other hand, relying on agents' preferences to break ties can, and sometimes will, destroy incentives for truthful preference revelation. This raises the question of whether the positive results for the two priority environments mentioned above are fortunate coincidences, or examples of a larger class of solvable environments. To the best of our knowledge, this paper is the first attempt to provide a systematic answer to this important question. Throughout our analysis we restrict attention to priority-based allocation problems where there is only one copy of each object. This formally excludes the school choice problem since public schools can typically admit multiple students. Our analysis, however, also presents an important leap towards understanding the conditions under which priority structures are solvable when more than one copy of some or all of the objects is available.

We begin our analysis with the derivation of three simple necessary conditions for the solvability of an arbitrary weak priority structure. The first condition is that a priority structure should be *acyclic*: there should not be a situation in which, no matter who out of a group of equal priority agents initially “wins” a tie at some object, at least one of the losing agents can “force” the subsequent rejection of the initial winner (because no stable allocation is compatible with keeping the initial winner in place). The other two conditions are concerned with *priority reversals*: consider a group of agents  $i_1, i_2, j_1, j_2$ , who all have equal priority for some object  $o$ . We say that a priority reversal occurs among this group of agents if there are two objects  $p$  and  $q$  such that (a)  $i_1$  has strictly higher priority for  $p$  than  $j_1$  and  $j_1$  has strictly higher priority for  $p$  than  $i_2$ , and (b)  $i_2$  has strictly higher priority for  $q$  than  $j_2$  and  $j_2$  has strictly higher priority for  $q$  than  $i_1$ . Our second condition is that a priority structure should not contain *strong priority reversals* where agent  $j_1$ , whose priority for  $p$  is in between that of  $i_1$  and  $i_2$ , is different from agent  $j_2$ , whose priority for  $q$  is in between that of  $i_2$  and  $i_1$ . Our third and final condition is that there should be *no inconsistent weak priority reversals*: if there is a (weak) priority reversal among a group of agents  $i_1, i_2, j$ , who have equal priority for some object  $o$ ,

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<sup>5</sup>Two important special cases that were studied prior to Abdulkadiroglu and Sönmez (1999), are housing markets (Shapley and Scarf, 1974) and house allocation (Hylland and Zeckhauser, 1979). In housing markets, each agent is endowed with one object and Gale's top trading cycles algorithm (TTC-algorithm) finds for each problem its unique (Roth and Postlewaite, 1977) core matching. Roth (1982b) was the first to show that the associated direct mechanism is strategy-proof. Ma (1994) showed that the TTC-algorithm is the unique mechanism satisfying individual rationality, efficiency and strategy-proofness. In house allocation all agents have equal priority for all objects. Here, the class of constrained efficient and strategy-proof mechanisms satisfying is very large (see Papai, 2000, and Pycia and Ünver, 2011, for characterization results).

then any other agent  $k$  with the same priority for  $o$  must either have weakly higher priority than agent  $j$  for all objects, or weakly lower priority than agent  $j$  for all objects. We establish that an arbitrary weak priority structure is solvable only if it is acyclic, contains no strong priority reversals, and contains no inconsistent weak priority reversals. As a corollary, we obtain that a priority structure which gives priority to *walk-zone students* is generically unsolvable.

Subsequently we restrict attention to environments in which equal priorities can only occur at the bottom of priority rankings. We show that the three necessary conditions above completely characterize the set of solvable priority structures within this class of *indifferences at the bottom* (IB) environments. Since all previously known solvable priority structures as well as the counterexample of Erdil and Ergin (2008) fall into this class of environments, this unifies and extends previous results on (un)solvable priority structures. This leaves us with the question of whether there exist solvable priority structures apart from those that were already known. To this end we show that without restrictions on how strict priority rankings vary across the objects, there are no additional solvable environments: if any pair of agents can be ranked either way in objects' priority rankings, a solvable priority structure with indifferences only at the bottom is either strict or corresponds to the house allocation with existing tenants problem. However, there do exist solvable priority structures which allow for some variability in strict priority rankings across objects and differ substantially from previously known solvable structures. Our proof of sufficiency is constructive: we show first that most ties can be broken exogenously, i.e. without relying on agents' reports, if a priority structure is solvable but does not correspond to the house allocation with existing tenants problem. We then introduce an algorithm which combines the ADA-algorithm for strict priority structures with a tie-breaking stage, where any remaining ties are broken according to agents' (reported) preferences. The algorithm reduces to the ADA when the priority structure is strict. In case of the house allocation with existing tenants problem, it reduces to the TTC algorithm. We show that whenever the priority structure satisfies the three requirements of acyclicity, no strong priority reversals and no inconsistent weak priority reversals, then our proposed algorithm yields a constrained efficient matching and is non-manipulable by (groups of) agents.

## Related Literature

In recent years several important contributions have analyzed priority-based allocation problems with weak priority orders. Here, the most closely related paper is Erdil and Ergin (2008). They study priority-based allocation problems with arbitrary weak priority structures. Their main result is that whenever a stable matching is not constrained efficient, it is possible to increase agents' welfare via a cyclical exchange of assignments that respects stability constraints. This leads them to propose the stable improvement cycles algorithm which finds a constrained efficient matching for any priority-based allocation problem. While our analysis relies on their main result, a major difference is that our proposed algorithm is strategy-proof (and constrained efficient) for all solvable priority structures within the class of IB environments. We provide a simple example of a solvable priority structure for which their algorithm must be manipulable.

Abdulkadiroglu et al. (2009) show that no strategy-proof mechanism can Pareto dominate the ADA resulting from some exogenous tie-breaking rule for all profiles of agents' preferences. We characterize the set of priority structures within the class of IB environments that are solvable by exogenous tie-breaking and show that it is a strict subset of the set of all solvable priority structures. In particular, there exist priority structures for which our proposed procedure is guaranteed to yield a constrained efficient matching, even though there is no exogenous tie-breaking rule which guarantees constrained efficiency of the ADA. Finally, Azevedo and Leshno (2010) show by example that equilibrium outcomes induced by the stable improvement cycles procedure may be Pareto dominated by the outcome of the ADA resulting from fixed tie-breaking. It follows from our results that the priority structure in their example is unsolvable.

From the literature on priority-based allocation problems with strict priority orders the most relevant paper Ergin (2002). He characterizes the set of strict priority structures for which stability is compatible with efficiency by means of an acyclicity condition.<sup>6</sup> The main difference to our analysis is that Ergin (2002) is concerned with the compatibility of two allocative criteria for problems that are known to admit a strategy-proof and constrained efficient mechanism.<sup>7</sup> Furthermore, we show that his condition is stronger than the combination of our three conditions and provide a slight extension of his main result to IB environments.

## Organization of the paper

The remainder of this paper is organized as follows: Section 2 introduces the model and reviews the most important existing results. Section 3 derives simple necessary conditions for the solvability of general weak priority environments. In Section 4, these conditions are shown to be sufficient for the solvability of environments where ties occur only at the bottom of the priority structure. Section 5 concludes and discusses our results. Most of the proofs are relegated to the Appendix.

## 2 Priority-Based Allocation Problems

A *priority-based allocation problem* is a 4-tuple  $(I, O, \succeq, R)$  consisting of

- a finite set of agents  $I \subset \mathbb{N}$ ,
- a finite set of objects  $O$ ,
- a priority structure  $\succeq = (\succeq_o)_{o \in O}$  where for each object  $o \in O$ ,  $\succeq_o$  is a weak ordering of  $I$ , and

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<sup>6</sup>He shows that the very same condition characterizes the sets of strict priority structures for which the ADA is strongly group strategy-proof and consistent, respectively. See Ehlers and Erdil (2010) for an extension of Ergin's result to priority-based allocation problems with weak priority structures.

<sup>7</sup>Several other papers have investigated consequences of the structural properties of strict priority structures, see e.g. Kesten (2006) and Ehlers and Klaus (2006).

- a preference profile  $R = (R_i)_{i \in I}$  where for each agent  $i \in I$ ,  $R_i$  is a strict ordering of  $O \cup \{i\}$ .

Given a weak ordering  $\succeq_o$ , we denote by  $i \succ_o j$  that agent  $i$  has strictly higher priority for object  $o$  than agent  $j$  and by  $i \sim_o j$  that  $i$  has the same priority for  $o$  as  $j$ . For two subsets  $J, J' \subseteq I$ ,  $J \succ_o J'$  means that  $i \succ_o i'$  for all  $i \in J$  and  $i' \in J'$ , and  $J \sim_o J'$  means that  $i \sim_o j$  for all  $i, j \in J \cup J'$ . Given  $J \subseteq I$ , let  $\succeq_o|_J$  denote the restriction of  $\succeq_o$  to  $J$ , and  $\succeq|_J = (\succeq_o|_J)_{o \in O}$  denote the restriction of  $\succeq$  to  $J$ . Similarly, given a strict ordering  $R_i$ , we denote by  $oP_i p$  that agent  $i \in I$  strictly prefers object  $o$  to object  $p$ , by  $oP_i i$  that  $i$  strictly prefers object  $o$  to not being assigned any object at all, and by  $iP_i o$  that  $i$  strictly prefers not being assigned any object at all than being assigned object  $o$ . If  $oP_i i$ , then object  $o$  is called *acceptable*. We often write preferences as  $R_i = o_1, \dots, o_m$ , which means that  $o_l P_i o_{l+1}$  for all  $l \leq m-1$  and  $iP_i p$  for all  $p \in O \setminus \{o_1, \dots, o_m\}$ . For the following,  $\mathcal{R}_i$  denotes all strict orderings of  $O \cup \{i\}$  for  $i \in I$ , and  $\mathcal{R} = \times_{i \in I} \mathcal{R}_i$  denotes the set of all preference profiles.

The sets of agents and objects as well as the priority structure  $\succeq = (\succeq_o)_{o \in O}$  are taken as fixed throughout, and a priority-based allocation problem  $(I, O, \succeq, R)$  is for short given by the profile  $R = (R_i)_{i \in I}$ . We will refer to the triple  $(I, O, \succeq)$  as an *environment*. Apart from the shape of the priority structure, our results depend on the environment only in so far that we sometimes have to distinguish cases according to the *numbers* of agents and objects. Hence, we will often think of an environment  $(I, O, \succeq)$  as being given simply by its weak priority structure  $\succeq$  and will use the terms *environment* and *priority structure* interchangeably.

An *assignment* is a mapping  $\mu : I \rightarrow I \cup O$  such that  $\mu(i) \in O \cup \{i\}$  for all  $i \in I$ . Given an assignment  $\mu$ , let  $\mu(o) := \{i \in I : \mu(i) = o\}$ . An assignment  $\mu$  is a *matching*, if  $\mu(i) \neq \mu(j)$  for all distinct  $i, j \in I$ . A matching  $\mu$  is *stable* for problem  $R$ , if it is

- (i) *individually rational*, that is,  $\mu(i)R_i i$  for all agents  $i \in I$ ,
- (ii) *fair*, that is, there is no agent-object pair  $(i, o)$  such that  $oP_i \mu(i)$  and  $i \succ_o i'$  for some  $i' \in \mu(o)$ , and
- (iii) *non-wasteful*, that is, there is no agent-object pair  $(i, o)$  such that  $oP_i \mu(i)$  and  $\mu(o) = \emptyset$ .

It is important to note that only strict rankings in the priority structure matter for stability. Intuitively, randomly breaking ties between distinct agents introduces additional stability constraints, which may be detrimental to agents' welfare. To formalize this intuition, we first define the concept of Pareto dominance: given  $R \in \mathcal{R}$  and two matchings  $\mu$  and  $\mu'$ ,  $\mu$  *Pareto dominates*  $\mu'$  if for all  $i \in I$ ,  $\mu(i)R_i \mu'(i)$ , and  $\mu(j)P_j \mu'(j)$  for at least one  $j \in I$ . A matching  $\mu$  is *efficient*, if there is no other matching which Pareto dominates it. As shown by Ergin (2002), stability is often incompatible with efficiency. However, given the finiteness of the problem there always exists (at least one) stable matching which is not Pareto dominated by any other *stable* matching with respect to the welfare of the agents. We call a matching with this property *constrained efficient* and given a problem  $R$ , we denote by  $CE^\succeq(R)$  the set of constrained efficient matchings. If priorities are strict,  $CE^\succeq(R)$  contains exactly one matching which can

be found using the agent-proposing deferred acceptance (ADA) algorithm that we describe in Section 2.2. In the presence of ties in the priority structure there may, however, be multiple constrained efficient matchings. Furthermore, as discussed in Ehlers (2007) and Erdil and Ergin (2008), one cannot in general find a constrained efficient outcome by first *exogenously*, i.e. irrespective of submitted preferences, breaking all ties and then conducting the resulting ADA (for strict priority structures). Erdil and Ergin (2008) develop an algorithm which always finds a constrained efficient matching. Their construction is based on the observation that if a stable matching  $\mu$  is *not* constrained efficient, then it is possible to increase welfare of the agents via a cyclical exchange that respects stability constraints. More formally, let  $\mu$  be a stable matching for the problem  $R$ . Then agent  $i$  *desires object  $o$  at  $\mu$*  if  $oP_i\mu(i)$ . For each object  $o$ , let  $D_o(\mu)$  denote the set of highest  $\succeq_o$ -priority agents among those who desire  $o$  at  $\mu$ . A *stable improvement cycle (SIC)* of  $\mu$  at  $R$  consists of  $m$  distinct agents  $i_1, \dots, i_m$  such that for all  $l = 1, \dots, m$ ,  $i_l \in D_{\mu(i_{l+1})}(\mu)$  (where  $m + 1 := 1$ ). Erdil and Ergin (2008) show that a stable matching  $\mu$  is constrained efficient if and only if  $\mu$  admits no stable improvement cycle (SIC) of  $\mu$  at  $R$ . As they point out, this implies that a constrained efficient matching can be found by first breaking all ties in the priority structure arbitrarily, running the associated ADA algorithm, and then successively implementing stable improvement cycles.

A *matching mechanism* is a function  $f$  that assigns a matching  $f(R)$  to each problem  $R \in \mathcal{R}$ . Given a rule  $f$  and a profile  $R$ , we denote agent  $i$ 's allotment by  $f_i(R)$ . Given a set of agents  $J \subseteq I$  and a profile  $R \in \mathcal{R}$ ,  $R_{-J}$  denotes restriction of  $R$  to  $I \setminus J$ . A mechanism  $f$  is *group strategy-proof*, if for all  $J \subseteq I$  and all preference profiles  $R$ , there does not exist a joint manipulation  $\tilde{R}_J \in \times_{i \in J} \mathcal{R}_i$  such that  $f_i(\tilde{R}_J, R_{-J})P_i f_i(R)$  for all  $i \in J$ . It is *strategy-proof*, if it is group strategy-proof for all groups that contain only one agent. A mechanism  $f$  is *strongly group strategy-proof*, if for all  $J \subseteq I$  and all preference profiles  $R$ , there does not exist a joint manipulation  $\tilde{R}_J \in \times_{i \in J} \mathcal{R}_i$  such that  $f_i(\tilde{R}_J, R_{-J})R_i f_i(R)$  for all  $i \in J$  and  $f_j(\tilde{R}_J, R_{-J})P_j f_j(R)$  for at least one  $j \in J$ . Finally, a mechanism  $f$  is *constrained efficient*, if  $f(R) \in CE^\succeq(R)$  for all preference profiles  $R$ .

An environment  $(I, O, \succeq)$  is *solvable*, if there exists a strategy-proof and constrained efficient matching mechanism  $f$ . The next example, due to Erdil and Ergin (2008), shows that unsolvable environments exist.

**Example 1** (Example 2 of Erdil and Ergin (2008)). Let  $I = \{1, 2, 3\}$  and  $O = \{o, p, q\}$ , and  $\succeq$  be given by

$\succeq_o$	$\succeq_p$	$\succeq_q$
3	1	3
1, 2	2	2
	3	1

Erdil and Ergin (2008) show that  $\succeq$  is unsolvable.

Our main research question is whether such examples are an exception or the rule in priority-based allocation problems with weak priority structures. We will attempt to answer this question via a characterization of solvable priority structures by means of simple and tractable

conditions.

## 2.1 A taxonomy of environments

The next definition summarizes several classes of priority structures, or *environments*, that will play an important role for our analysis.

**Definition 1.** *An environment  $(I, O, \succeq)$  is*

- (i) *a strict (S) environment, if there is no object  $o \in O$  such that  $i \sim_o j$  for two distinct  $i, j \in I$ .*
- (ii) *a house allocation with existing tenants (HET) environment, if there is no object  $o \in O$ , such that  $\{i, j\} \succ_o k$  for three distinct agents  $i, j, k \in I$ .*
- (iii) *an indifference at the bottom (IB) environment environment, if there is no object  $o \in O$  such that  $i \sim_o j \succ_o k$  for three distinct agents  $i, j, k \in I$ .*

The interpretation and motivation for the first two classes is straightforward: Strict environments were the dominant assumption in the school-choice literature prior to Erdil and Ergin (2008) and Ehlers (2007). The class of HET environments has been introduced by Abdulkadiroglu and Sönmez (1999). The interpretation here is that an object  $o$  such that  $i \succ_o j$  for some  $i \in I$  and all  $j \in I \setminus \{i\}$  is *owned*, or *occupied*, by agent  $i$ , while an object  $o$  such that  $i \sim_o j$  for all  $i, j \in I$  is *unoccupied* and owned by all of the agents jointly. Two special cases of HET environments that have been widely studied are *housing markets* (Shapley and Scarf, 1974) and *house allocation problems* (Hylland and Zeckhauser, 1979). A housing market is a special case of a HET environment where each object/house is occupied and each agent owns exactly one object. In a house allocation problem, no object is occupied (i.e.  $i \sim_o j$  for all  $i, j \in I$  and all  $o \in O$ ).

The third class of environments is new to this paper and is the most general of the three classes described above. One interpretation for this class is that objects are specialized tasks and an agent’s priority for a task represents her task-specific (commonly known) skill. Each task has a lower bound on qualifications, above which agents are ranked strictly according to their qualification, while all agents who fall below the threshold are considered equally worse.<sup>8</sup> An interesting special case of IB environments is the one where the set of objects  $O$  can be partitioned into two sets  $O_1$  and  $O_2$ , such that objects in  $O_1$  have a strict priority ranking of agents, while objects in  $O_2$  assign equal priority to all agents. In the context of *school choice*, one may think of “schools” in  $O_1$  as selective schools for which priorities are determined by scores in some pre-school examination. Schools in  $O_2$  on the other hand are “open-access” in the sense that they do not offer prioritized access to anyone.<sup>9</sup> In Section 3 we provide a full

<sup>8</sup>Interestingly, Ehlers (2002) characterized IB environments as the unique maximal domain of preferences (containing all strict preferences) such that a strongly group strategy-proof and efficient mechanism exists.

<sup>9</sup>Subsequently to this paper, Abdulkadiroglu (2011) has studied this setting. He refers to schools/objects in  $O_1$  as *stability constrained* and to objects in  $O_2$  as *unconstrained*. His main interest lies in the construction of a constrained efficient mechanism that combines the ADA and the TTC procedures.

characterization of solvable environments within the class of IB environments that generalizes and unifies all previously known results on (un)solvable priority structures.

**Remark 1.** (1) Strictly speaking, our definition of HET environments are a slight generalization of those studied in Abdulkadiroglu and Sönmez (1999), since they assume that each agent may own at most one object, which is not required in (ii) of Definition 1. However, their proofs apply almost verbatim to the HET environments we consider here and we will henceforth neglect this subtle difference.

(2) For HET environments, constrained efficiency is equivalent to the combination of the requirements of *individual rationality for owners*, i.e. no agent should be worse off than consuming one of the objects she owns, and *efficiency*. It is obvious that individual rationality together with efficiency implies constrained efficiency. For the other direction, note that whenever an individually rational matching is not efficient, we can find a matching that Pareto improves on it. Given that the old matching was individually rational, the Pareto improving matching must clearly also have this property.

## 2.2 Two Mechanisms for Solvable Environments

For strict environments, the *agent-proposing deferred acceptance algorithm* (ADA-algorithm) of Gale and Shapley (1962) plays a central role.<sup>10</sup> Given a strict environment  $\succeq$  and a problem  $R$ , the ADA-algorithm proceeds as follows:

### Algorithm 1: ADA

**Round 1:** Each agent applies to her most preferred acceptable object.

Each object tentatively accepts the proposal from the highest priority agent and rejects all others.

If all rejected agents have applied to all acceptable objects, stop. Otherwise, proceed to Round 2.

⋮

**Round  $t$ :** Each agent who was rejected by some object in Round  $t - 1$ , applies to her next most preferred acceptable object (if any).

Each object tentatively accepts the proposal from the highest priority agent and rejects all others.

If all rejected agents have applied to all acceptable objects, stop. Otherwise, proceed to Round  $t + 1$ .

⋮

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<sup>10</sup>See Roth (2008) for a recent survey of deferred acceptance algorithms in theory and practice.

Let  $ADA^{\succeq}(R)$  denote the matching chosen by the above algorithm when the profile of submitted preferences is  $R$  and the strict priority structure is given by  $\succeq$ . It is well known that if  $\succeq$  is strict, then for any profile of strict preferences  $R$ ,  $ADA^{\succeq}(R)$  is the unique constrained efficient matching (Gale and Shapley, 1962) and  $ADA^{\succeq}$  is group strategy-proof (Dubins and Freedman, 1981; Roth, 1982a).

For HET environments, the *top-trading cycles algorithm* (TTC-algorithm) of Abdulkadiroglu and Sönmez (1999) plays a central role. Given a HET environment  $\succeq$  and a problem  $R$ , the TTC-algorithm proceeds as follows:

### Algorithm 2: TTC

**Round 1:** Construct a directed graph on  $I \cup O$  by letting each agent point to her most preferred acceptable object (or herself, if she does not find any object in  $O$  acceptable), each occupied object point to its owner, and all other objects point to the highest indexed agent.

Assign each agent who belongs to a directed cycle of this graph to the object she points to (or let this agent be unassigned, if she points to herself), remove all agents and objects who belong to a directed cycle, and let  $I_2$  and  $O_2$  be the sets of remaining agents and objects.

If  $I_2 = \emptyset$ , stop. Otherwise, proceed to Round 2.

⋮

**Round  $t$ :** Construct a directed graph on  $I_t \cup O_t$  by letting each agent point to her most preferred acceptable object in  $O_t$  (or herself, if she does not find any object in  $O_t$  acceptable), each object in  $O_t$  occupied by an agent in  $I_t$  point to its owner, and all other objects point to the highest indexed agent in  $I_t$ .

Assign each agent who belongs to a directed cycle of this graph to the object she points to (or let this agent be unassigned, if she points to herself), remove all agents and objects who belong to a directed cycle, and let  $I_{t+1}$  and  $O_{t+1}$  be the sets of remaining agents and objects.

If  $I_{t+1} = \emptyset$ , stop. Otherwise, proceed to Round  $t + 1$ .

⋮

Let  $TTC^{\succeq}(R)$  denote the matching chosen by this algorithm when the profile of submitted preferences is  $R$  and the HET environment is given by  $\succeq$ . It is well known that  $TTC^{\succeq}$  is individually rational, efficient and (strongly group) strategy-proof (Abdulkadiroglu and Sönmez, 1999).<sup>1112</sup>

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<sup>11</sup>Papai (2000) shows more generally that all rules belonging to the class of *hierarchical exchange rules*, such as the TTC procedure, are strongly group strategy-proof and efficient.

<sup>12</sup>For the intermediate case of HET environments where no agent owns more than one object, Sönmez and Ünver (2010) show that the TTC-algorithm can be characterized by individual rationality, efficiency, strategy-proofness, weak neutrality, and consistency.

For the case of housing markets, the above algorithm reduces to Gale's TTC-algorithm described in Shapley and Scarf (1974). It finds the unique strong core allocation (Roth and Postlewaite, 1977) and the TTC mechanism is the unique individually rational, efficient and strategy-proof mechanism (Ma, 1994).<sup>13</sup> By Remark 1, the TTC mechanism is the unique constrained efficient and strategy-proof rule for housing markets.

For house allocation problems there is a large class of efficient and (strongly group) strategy-proof mechanisms studied in e.g. Papai (2000) and Pycia and Ünver (2009). In this case the TTC-algorithm reduces to a *serial dictatorship* where agents take turns in choosing their most preferred object that is still available, with the highest indexed agent taking the first turn, the second highest the second, and so on.<sup>14</sup>

### 3 Solvability: Necessary conditions for general environments

In this section we introduce three simple (and independent) necessary conditions for the solvability of an arbitrary weak priority structure. Our first condition is concerned with situations where a pair of distinct agents have equal priority for an object, for which at least one other agent has strictly higher priority.

**Definition 2** (Cyclic ties). *A tie  $i_1 \sim_o i_2$  between two distinct agents  $i_1, i_2 \in I$  is strongly cyclic, if there exist agents  $j_1, j_2 \in I \setminus \{i_1, i_2\}$  and objects  $p_1, p_2$  such that either*

(a)  $i_1 \succ_{p_1} j_1 \succ_o i_1$  and  $i_2 \succ_{p_2} j_2 \succ_o i_2$ , with  $p_1 = p_2$  if  $j_1 = j_2$ , or

(b)  $\{i_1, i_2\} \succ_{p_1} j_1 \succ_{p_2} j_2 \succ_o i_1$ .

A weak priority structure  $\succeq$  is acyclic, if it does not contain a strongly cyclic tie.

Note that HET environments do not have strongly cyclic ties, since for each object there is at most one agent (the owner) who has strictly higher priority than any other agent. In Example 1 the tie  $1 \sim_o 2$  is strongly cyclic (according to case (a) above with  $p_1 = p_2$  and  $j_1 = j_2$ ) because  $\{1, 2\} \succ_p 3 \succ_o 1$ . We illustrate the other types of strongly cyclic ties in the following example.

**Example 2** (Cyclic ties). Let  $I = \{1, 2, 3, 4\}$ ,  $O = \{o, p, q\}$ ,  $\succeq$ , and  $\succeq'$  be given by

	$\succeq_o$	$\succ_p$	$\succeq_q$		$\succeq'_o$	$\succeq'_p$	$\succeq'_q$
	3	4	3		4	4	3
(a):	4	1	2	and (b):	1, 2, 3	1	4
	1, 2	3	4			2	1
		2	1			3	2

<sup>13</sup>Strategy-proofness of the TTC for housing markets was first established by Roth (1982b).

<sup>14</sup>For a characterization of serial dictatorships, see Svensson (1999).

For  $\succ$  the tie  $1 \sim_o 2$  is strongly cyclic in the sense of (a) of Definition 2 because  $1 \succ_p 3 \succ_o 1$  and  $2 \succ_q 4 \succ_o 2$ .

For  $\succ'$  the tie  $1 \sim'_o 2$  is strongly cyclic in the sense of (b) of Definition 2 because  $\{1, 2\} \succ'_p 3 \succ'_q 4 \succ'_o 1$ .

Next, we introduce two conditions that are concerned with larger groups of agents who have identical priority for a given object. These conditions introduce two important measures of variability within the priority structure that will be seen to play a key role for determining its solvability.

**Definition 3** (Priority reversals). *A priority reversal is a seven-tuple  $(i_1, i_2, j_1, j_2, o, p_1, p_2)$  such that  $i_1 \sim_o i_2 \sim_o j_1 \sim_o j_2$ ,  $i_1 \succ_{p_1} j_1 \succ_{p_1} i_2$ , and  $i_2 \succ_{p_2} j_2 \succ_{p_2} i_1$ .*

*A priority reversal is strong if  $j_1 \neq j_2$ , and weak if  $j_1 = j_2$ .*

In the following, we always denote weak priority reversals by  $(i_1, i_2, j_1, o, p_1, p_2)$  and refer to  $j_1$  as the *in-between agent*. Before proceeding, note that HET environments have no priority reversals, since there are at most two indifference classes in the priority ranking of an object.

**Definition 4** (Inconsistent weak priority reversals). *A weak priority reversal  $(i_1, i_2, j_1, o, p_1, p_2)$  is inconsistent, if there is an agent  $j_2 \in I \setminus \{i_1, i_2, j_1\}$  such that  $\{i_1, i_2, j_1\} \sim_o j_2$ ,  $\{i_1, i_2, j_1\} \succ_{p_1} j_2$ ,  $\{i_1, i_2, j_1\} \succ_{p_2} j_2$ , and  $\{i_1, i_2\} \succ_{p_3} j_2 \succ_{p_3} j_1$  for some  $p_3 \in O \setminus \{p_1, p_2\}$ .*

We illustrate the last two definitions with the following example.

**Example 3** (Priority reversals). Let  $I = \{1, 2, 3, 4\}$ ,  $O = \{o, p, q, r\}$ ,  $\succ$ , and  $\succ'$  be given by

$\succ_o$	$\succ_p$	$\succ_q$	$\succ_r$	$\succ'_o$	$\succ'_p$	$\succ'_q$	$\succ'_r$
1, 2, 3, 4	1	2	1	1, 2, 3, 4	1	2	1
	3	4	2		3	3	2
	2	1	3		2	1	4
	4	3	4		4	4	3

For  $\succ$  there is a strong priority reversal because  $1 \succ_p 3 \succ_p 2$  and  $2 \succ_q 4 \succ_q 1$  (and  $1 \sim_o 2 \sim_o 3 \sim_o 4$ ).

For  $\succ'$  there is an inconsistent weak priority reversal because  $1 \succ'_p 3 \succ'_p 2 \succ'_p 4$ ,  $2 \succ'_q 3 \succ'_q 1 \succ'_q 4$ , and  $\{1, 2\} \succ'_r 4 \succ'_r 3$  (and  $1 \sim'_o 2 \sim'_o 3 \sim'_o 4$ ).

We are now ready to present the first main result of this paper.

**Theorem 1.** *An environment  $\succ$  is solvable only if*

- (i)  $\succ$  is acyclic,
- (ii)  $\succ$  has no strong priority reversals, and
- (iii)  $\succ$  contains no inconsistent weak priority reversal.

We briefly sketch the intuition underlying the necessity of *no strongly cyclic ties*: Suppose there are two distinct agents  $i_1, i_2$  such that  $i_1 \sim_o i_2$  for some object  $o$  and such that there exists another object  $p$  and a third agent  $j$  with  $\{i_1, i_2\} \succ_p j \succ_o i_1 \sim_o i_2$ . Let  $f$  be an arbitrary constrained efficient matching mechanism. Furthermore, suppose that when the preference profile is such that  $i_1$  and  $i_2$  are both only interested in  $o$  and  $j$  ranks  $p$  higher than  $o$ ,  $i_2$  is left unmatched by  $f$ . If  $f$  was strategy-proof, it should not assign  $i_2$  to  $o$ , when she claims instead that  $o$  is her first and  $p$  is her second choice. But since  $i_2 \succ_p j$ , the only other stable allocation in this case would be one where  $i_2$  receives  $p$  and  $j$  receives  $o$ . Since  $i_2$  and  $j$  form a stable improvement cycle, this allocation is not constrained efficient. Hence,  $f$  would have to assign  $i_2$  to  $o$ , so that  $f$  cannot be strategy-proof. The reasoning behind the other necessary conditions is related but more complicated.

We now discuss two interesting corollaries of Theorem 1. First, we consider an application to school choice. Here, distance to school is an important criterion for determining a student's priority. This is usually implemented by defining some radius around the school and then classifying all students living within this radius as belonging to that school's *walk-zone*.<sup>15</sup> Assume for simplicity that we are dealing with the *pure walk-zone priority* case, where students living within the walk-zone of a school have strictly higher priority for that school than all others, and no further priority distinctions between students are made. More formally,  $(I, O, \succeq)$  is a *pure walk-zone environment*, if for each object  $o$  there is a (possibly empty) set  $W_o \subseteq I$  such that  $W_o \succ_o I \setminus W_o$ , and  $i \sim_o j$  for all  $i, j$  such that either  $i, j \in W_o$ , or  $i, j \in I \setminus W_o$ . The following is an immediate corollary of Theorem 1.

**Corollary 1.** *Suppose  $(I, O, \succeq)$  is a pure walk-zone environment.*

*If there exist two objects  $o, p$  such that  $|W_o \setminus W_p| \geq 2$  and  $W_p \setminus W_o \neq \emptyset$ , then  $(I, O, \succeq)$  is not solvable.*

*Proof.* Take two distinct students  $i_1, i_2 \in W_o \setminus W_p$  and some student  $i_3 \in W_p \setminus W_o$ . Then we have  $i_1 \sim_o i_2 \succ_o i_3$  as well as  $i_3 \succ_p i_1 \sim_p i_2$ , so that  $i_1 \sim_p i_2$  is a strongly cyclic tie. By Theorem 1,  $\succeq$  is not solvable.  $\square$

Thus, if a school choice authority wants to impose a walk-zone priority structure, it will generally have to sacrifice either constrained efficiency or strategy-proofness. The second corollary is concerned with the following question: what is the maximum number of objects (students) such that an environment is always solvable?

**Corollary 2.** *Whenever there are at least three agents in  $I$  and at least two objects in  $O$ , there exists a weak priority structure  $\succeq$  such that  $(I, O, \succeq)$  is unsolvable.*

While one may have suspected that with two objects one of the objects could always be used to break any ties at the other object, this is not generally true since even with two objects there is a possibility of strongly cyclic ties. Thus, except for trivial environments, one always has to restrict the priority structure to guarantee solvability.

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<sup>15</sup>See e.g. Abdulkadiroglu et al. (2006) for a study of the school choice mechanism in Boston, where walk-zones play an important role.

## 4 IB Environments: Full Characterization

In this section we focus on IB environments. As discussed above, these environments generalize all previously known solvable environments. The following is the main result of this section.

**Theorem 2.** *An IB environment  $\succeq$  is solvable if and only if*

- (i)  $\succeq$  is acyclic,
- (ii)  $\succeq$  has no strong priority reversals, and
- (iii)  $\succeq$  contains no inconsistent weak priority reversals.

The necessity part of this result follows immediately from our first main result and we now provide a detailed outline of the sufficiency part. While the full details of the proof are a bit involved and relegated to the Appendix, it is relatively straightforward to describe a matching mechanism that is strategy-proof and constrained efficient for *all* solvable IB environments. We first describe the structural implications of solvability, i.e. the constraints imposed by acyclicity, no strong priority reversals, and no inconsistent weak priority reversals on the shape of the priority structure. We use these constraints to first break most of the ties exogenously and then resolve any remaining ties using an ADA-algorithm with preference-based tie-breaking.

For ease of exposition, we make one more assumption which requires additional terminology. Call a set of agents  $J \subseteq I$  *connected* (with respect to  $\succeq|_J$ ), if there is no strict subset  $\tilde{J} \subset J$  such that  $\tilde{J} \succeq_o J \setminus \tilde{J}$  for all objects  $o \in O$ . The following assumption will be maintained throughout the main body of this text.

**Assumption 1.** *The set of agents  $I$  is connected.*

As we detail in the discussion section, this is without loss of generality for our purposes. Note that for HET environments, connectedness requires that at least one of the objects is occupied (otherwise,  $i \sim_o j$  for all  $i, j \in I$ ) and that all agents own at least one of the objects (otherwise, all agents who own at least one of the objects always have weakly higher priority than everyone else). While this formally excludes e.g. the house allocation problem, our analysis can easily be modified to handle such cases, as we show in the discussion section. Before proceeding, it is instructive to compare connectedness with the following stronger notion of variability: A priority structure is *perfectly pairwise variable*, if for any pair of distinct agents  $i$  and  $j$  there exist two objects  $o$  and  $p$  such that  $i \succ_o j$  and  $j \succ_p i$ . We will see shortly that connectedness does allow for solvable priority structures that differ substantially from HET and strict environments. However, under the stronger assumption of perfect pairwise variability the set of solvable IB environments only consists of these two polar cases.

**Corollary 3.** *Let  $|I| \geq 4$  and  $\succeq$  be a perfectly pairwise variable IB environment.*

*If  $\succeq$  is solvable, then either  $\succeq$  is a HET environment or  $\succeq$  is a strict environment.*

Thus, to obtain solvable environments outside the two discovered by the previous literature one has to restrict the variability of the priority structure. The next subsections show in detail how this can be achieved.

## 4.1 Structural implications of solvability

In this subsection we derive the basic implications of the three necessary conditions for IB environments. In particular, we show that these conditions place stringent restrictions on which groups of agents can be involved in ties and how the priorities within a group of tied agents can vary across the priority structure. Some additional terminology and notation will be necessary: for all objects  $o$  and all  $k$ , let  $r_k(\succeq_o)$  denote the set of agents who have  $k$ th highest priority for  $o$ , i.e.  $|\{i \in I : i \succ_o j\}| = k - 1$  for all  $j \in r_k(\succeq_o)$ . We omit set brackets whenever  $r_k(\succeq_o)$  is a singleton. For object  $o$ , let  $K_o$  denote the largest integer  $k$  such that  $r_k(\succeq_o)$  is a singleton. Note that since  $\succeq$  is an IB environment,  $r_k(\succeq_o)$  is a singleton for all  $k \leq K_o$ . Let  $\kappa := \max_{o \in O} K_o$  be the largest integer  $k$  such that  $r_k(\succeq_o)$  is a singleton for some  $o \in O$ . Note that an IB environment  $\succeq$  is a HET environment if and only if  $\kappa = 1$ . For any  $k = 1, \dots, \kappa$ , let

$$L^k := \cup_{\{o \in O : k \leq K_o\}} r_k(\succeq_o)$$

be the set of agents who have unique  $k$ th highest priority for at least one of the objects and let

$$L_k := L^k \setminus (L^1 \cup \dots \cup L^{k-1})$$

be the set of agents who have unique  $k$ th highest priority for at least one object and always rank (weakly) below at least  $k - 1$  agents. For notational convenience, we set  $L^0 = L_0 = \emptyset$ . The sets  $L_k$  can be thought of as *level sets* of the priority structure and play an important role in our subsequent analysis. Now consider an arbitrary agent  $i \in I$ . Since  $I$  is connected, there is a unique  $k$  such that  $i \in L_k$ . We refer to this  $k$  as agent  $i$ 's *priority level* and denote it by  $k_i$ . Note that higher priority levels are associated with *lower* priorities for the objects. Finally, let  $K$  be the maximal priority level among all agents, i.e. the largest integer  $k$  such that  $L_k \neq \emptyset$ . It turns out that two particular priority levels play a key role for solvable priority structures and these are defined next.

**Definition 5.** *Let  $\succeq$  be an IB environment.*

- (i) *The threshold of  $\succeq$ , denoted by  $K_T$ , is either the largest integer  $k$  such that at least one agent  $i \in L_1 \cup \dots \cup L_k$  has  $(k + 3)$ rd highest priority for some object  $o \in O$ , or  $K_T = 0$  if no such integer exists.*
- (ii) *The floor of  $\succeq$ , denoted by  $K_F$ , is the smallest integer  $k$  greater than  $K_T$  such that  $L_k \cup \dots \cup L_K$  is connected.*

For agents with priority levels higher than the threshold, priorities vary by at most three ranks within the strict part of the priority structure. Agents with priority levels higher than the floor always have weakly lower priority than agents whose priority level is between the threshold and the floor. To get some intuition for these definitions, consider the following example.

**Example 4.** *Consider a priority-based allocation problem with eight agents  $1, \dots, 8$ , six objects  $o_1, o_2, p_1, p_2, p_3, p_4$ , and a priority structure given by the following table.*

$\succ_{o_1}$	$\succ_{o_2}$	$\succ_{p_1}$	$\succ_{p_2}$	$\succ_{p_3}$	$\succ_{p_4}$
1	2	1	2	1	2
4	3	3	3	3	3
3	4	4	4	4	4
2, 5, 6, 7, 8	1, 5, 6, 7, 8	5	1	2	1
		2	6	5	5
		6	5	7	8
		7	8	6	6
		8	7	8	7

For this example we have  $L_1 = \{1, 2\}$ ,  $L_2 = \{3, 4\}$ ,  $L_3 = \emptyset$ ,  $L_4 = \{5\}$ ,  $L_5 = \{6\}$ , and  $L_6 = \{7, 8\}$ . Note that  $K = 6$  and  $\kappa = 8$ . The threshold here is  $K_T = 2$ , since (a) 2 has unique highest priority for objects  $o_2$ ,  $p_2$ , and  $p_4$ , but ranks only 5th at object  $p_1$ , and (b) for all agents apart from 1 and 2, priorities vary by at most three ranks within the strict part of the priority structure. The floor in this example is  $K_F = 3$  since (a)  $L_3 \cup L_5 \cup L_6 = \{5, 6, 7, 8\}$  is connected, and (b)  $L_2 \cup \dots \cup L_6 = \{3, 4, 5, 6, 7, 8\}$  is not connected since 3 and 4 have strictly higher priority than agents 5,  $\dots$ , 8 for all objects.

The next lemma shows that any pair of agents whose priority levels are smaller than the floor must always be strictly ranked.

**Lemma 1.** *If  $\succeq$  is a solvable IB environment, then all agents with priority levels smaller than the floor must always be strictly ranked, that is,  $\succeq|_{L_1 \cup \dots \cup L_{(K_F-1)}}$  must be strict.*

Note that for HET environments we always have  $K_T = 0$  and  $K_F = 1$  so that the last lemma is vacuously satisfied. The last result implies that we can partition the agents into two groups as follows: agents in  $I_S = L_1 \cup \dots \cup L_{(K_F-1)}$  must always be strictly ranked, while ties can potentially occur between agents in  $I_T = I \setminus I_S$ . In the above Example 4 we have  $I_S = \{1, 2, 3, 4\}$  and  $I_T = \{5, 6, 7, 8\}$ . Note that for any agent  $i \in I_S$ , there must exist an agent  $j \in I_S$  such that  $i \succ_o j$  for some object  $o$ .<sup>16</sup> By analogy with the above definitions, let  $L_k(I_T)$  be the set of agents in  $I_T$  who have unique  $k$ th highest priority among agents in  $I_T$ , but never rank higher. For agent  $i \in I_T$ ,  $i$ 's priority level within  $I_T$  is the (unique)  $k$  such that  $i \in L_k(I_T)$  and is denoted by  $k_i(I_T)$ . Finally, let  $\bar{K}$  be the largest integer  $k$  such that  $L_k(I_T) \neq \emptyset$ . With these preparations we have the following.

**Lemma 2.** *Let  $\succeq$  be a solvable IB environment such that  $\kappa > 1$  and  $|I_T| \geq 4$ .*

*Then we must have*

(i)  $|L_1(I_T)| = 2$ ,

(ii)  $|L_k(I_T)| = 1$ , for all  $k \in \{2, \dots, \bar{K} - 1\}$ , and

<sup>16</sup>By definition, there is an object  $o$  such that  $i = r_k(\succeq_o)$  for some  $k \leq K_F - 1$ . If there is no agent  $j \in I_S$  such that  $i \succ_o j$ , it has to be the case that  $k = K_F - 1 = |I_S|$ . If  $I \setminus I_S \neq \emptyset$ , we obtain a contradiction to connectedness, since no agent in  $I \setminus I_S$  can ever rank above an agent in  $I_S$ . Since  $I = I_S$  is connected, there has to be some  $p \in O$  such that  $i \succ_p j$  for some  $j \in I$ .

(iii)  $|L_{\overline{K}}(I_T)| \in \{1, 2\}$ .

Hence, level sets below the threshold have to be *small*. Note that this lemma implies  $|I_T| \in \{\overline{K} + 1, \overline{K} + 2\}$ . In Example 4 we have  $L_1(I_T) = \{5, 6\}$ ,  $L_2(I_T) = \{7, 8\}$ , and  $\overline{K} = 2$ .

## 4.2 Exogenous tie-breaking

It is intuitively clear, that the main obstacle to constrained efficiency is that we sometimes have to break ties on the basis of agents' preferences. We now show that for a solvable IB environment that is not a HET environment, most ties in the priority structure can be broken exogenously. The main idea is very similar to that of a serial dictatorship in the sense that we want to break ties in favor of agents with lower priority levels (remember, that lower priority levels mean a higher priority for obtaining the objects). For the following discussion, we assume that  $\succsim$  is a solvable environment that is not a HET environment, i.e.  $\kappa > 1$ . Formally, we construct an almost strict priority structure  $\succsim'$  from  $\succsim$  by setting, for each object  $o \in O$ ,  $i \succ'_o j$  if and only if either  $i \succ_o j$ , or  $i \sim_o j$  and one of the following conditions is satisfied:

- (1)  $i \in I_S$  and  $j \in I_T$
- (2)  $i, j \in I_T$  and  $k_i(I_T) < k_j(I_T)$
- (3)  $i, j \in I_T$ ,  $|I_T| \geq 3$ ,  $k_i(I_T) = k_j(I_T) = 1$ , and  $i \in r_1(\succsim_p |_{I_T}) \cup r_2(\succsim_p |_{I_T})$  for all  $p \in O^{17}$
- (4)  $\{i, j\} = I_T$ ,  $I_S \neq \emptyset$ , and  $I_S \succ_p I_T \setminus \{i\}$  for all  $p \in O$

To illustrate the exogenous tie-breaking procedure we calculate the resulting semi-strict priority structure for a simple example.

**Example 5** (Example 4 continued). *In this case we have that  $I_S = \{1, 2, 3, 4\}$  and  $I_T = \{5, 6, 7, 8\}$ . Furthermore,  $L_1(I_T) = \{5, 6\}$ ,  $L_2(I_T) = \{7, 8\}$ , and 5 always ranks at least second among agents in  $I_T$ . Hence, the exogenous tie-breaking procedure would yield (we only report priority rankings of objects which initially had ties):*

$$1 \succ'_{o_1} 4 \succ'_{o_1} 3 \succ'_{o_1} 2 \succ'_{o_1} 5 \succ'_{o_1} 6 \succ'_{o_1} 7 \sim'_{o_1} 8$$

and

$$2 \succ'_{o_2} 3 \succ'_{o_2} 4 \succ'_{o_2} 1 \succ'_{o_2} 5 \succ'_{o_2} 6 \succ'_{o_2} 7 \sim'_{o_2} 8.$$

We now investigate the conditions under which two distinct agents can remain with equal priorities for some of the objects after the exogenous tie-breaking stage. Such ties will then be broken by the procedure discussed in the next subsection.

**Lemma 3.** *Let  $\succsim$  be a solvable IB environment such that  $\kappa > 1$  and  $|I| \geq 4$ . Suppose  $i_1$  and  $i_2$  are two distinct agents such that  $i_1 \sim'_o i_2$  for some object  $o \in O$ .*

*Then there is a unique agent  $j_{i_1, i_2} \in I \setminus \{i_1, i_2\}$  for whom there exist two objects  $p_1, p_2 \in O$  such that  $i_1 \succ_{p_1} j_{i_1, i_2}$  and  $i_2 \succ_{p_2} j_{i_1, i_2}$ . Furthermore, for all objects  $q \in O$ ,*

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<sup>17</sup>Lemma 6 (b) in Appendix B shows that this rule can never result in two self contradictory tie-breaking recommendations.

- (1) if  $i_1 \succ_q j_{i_1, i_2}$ , then  $j_{i_1, i_2} \succ_q i_2$ ,
- (2) if  $i_2 \succ_q j_{i_1, i_2}$ , then  $j_{i_1, i_2} \succ_q i_1$ , and
- (3)  $I \setminus \{j_{i_1, i_2}\} \succeq_q \{i_1, i_2\}$ .

This lemma has important implications for the set of agents who can remain tied after our exogenous tie-breaking procedure. In the following, we denote by  $I^0$  the set of agents who are always strictly ranked by  $\succeq'$  and by  $I^1$  the set of agents who have equal priority for at least one of the objects, i.e.  $I^0 := \{i \in I : i \succ'_o j \text{ for all } j \in I \setminus \{i\}, o \in O\}$  and  $I^1 = I \setminus I^0$ .

**Remark 2.** The above results have strong implications for the structure of  $I^0$  and  $I^1$  that will be helpful for the analysis below.

- (1) For HET environments we have  $I^0 = \emptyset$  and  $I^1 = I$ , since no ties are broken exogenously.
- (2) For solvable IB environments with  $\kappa > 1$ , our exogenous tie-breaking procedure guarantees that whenever  $|I| \geq 4$  at most two agents can be involved in ties with respect to  $\succeq'$ , i.e.  $|I^1| \leq 2$ :
  - For the case of  $|I_T| \geq 4$ , Lemma 3 implies that if there is an object  $o$  such that  $i \sim_o j$  for the two agents  $i, j \in L_1(I_T)$ , at least one of them must always have at least second highest priority among agents in  $I_T$ . Hence, we would have set  $i \succ'_o j$  so that we obtain  $I^0 = I \setminus L_{\overline{K}}(I_T)$  as well as  $I^1 = L_{\overline{K}}(I_T)$ . By Lemma 2, we know that  $|L_{\overline{K}}(I_T)| \leq 2$  whenever  $|I_T| \geq 4$ .
  - If  $|I| \geq 4$  and  $|I_T| = 3$ , there must either be an agent  $i \in I_T$  who always has at least second highest priority among agents in  $I_T$  or we must have  $i \succ_o j$  for all  $i, j \in I_T$ . To see this, suppose to the contrary that  $i_1 \sim_o i_2$  for some object  $o$  and two distinct agents  $i_1, i_2 \in I_T$ , even though each agent in  $I_T$  has third highest priority for at least one of the objects with respect to  $\succeq|_{I_T}$ . By the rules for exogenous tie-breaking introduced above, we must have  $i_1 \sim'_o i_2$ . Let  $j \in I_T \setminus \{i_1, i_2\}$  be the third agent in  $I_T$ . Since every agent in  $I_T$  has third highest priority for at least one of the objects with respect to  $\succeq|_{I_T}$ , there is an object  $p$  such that  $\{i_1, i_2\} \succ_p j$ . Since  $i_1 \sim'_o i_2$  and  $|I| \geq 4$ , this is a contradiction to Lemma 3.

### 4.3 ADA with endogenous tie-breaking

We are now ready to describe a matching mechanism which is constrained efficient and (group) strategy-proof whenever the priority structure satisfies conditions (i)-(iii) of Theorem 2. The mechanism collects a (strict) preference profile  $R$  from the agents and computes an outcome using the *agent-proposing deferred acceptance algorithm with endogenous tie-breaking* (ADA-ETB) described below. This algorithm combines the ADA for strict priority structures with a tie-breaking routine, where temporary assignments are used to break remaining ties. For the following discussion, fix a weak priority structure  $\succeq$  that is weakly acyclic, has no strong priority reversals and contains no inconsistent weak priority reversal. Let  $\succeq'$  be the priority

structure obtained by the exogenous tie-breaking procedure described in the last subsection and remember that  $I^0$  denotes the set of agents who are always strictly ranked according to  $\succeq'$  and that  $I^1 = I \setminus I^0$ . We now describe how the ADA-ETB proceeds for an arbitrary strict preference profile  $R$ .

**Algorithm 3: ADA-ETB**

**Round 0:** Apply the ADA to the problem  $(I^0, \succeq' |_{I^0}, R_{I^0})$ . Let  $\mu^0$  be the resulting matching and proceed to Round 1.

**Round 1:** Each agent in  $I^1$  applies to her most preferred object.

Each object  $o$  compares the set of new applicants in this round with the set of applicants temporarily matched to it under  $\mu^0$  (if any), temporarily admits all highest priority agents with respect to  $\succeq'_o$  and rejects all others.

Let  $\mu^1$  be the resulting temporary assignment.

If one of the rejected agents has not yet applied to all acceptable objects, go to Round 2.

If all rejected agents have applied to all acceptable objects and  $\mu^1$  is a matching, stop.

In any other case, use routine  $TB(\mu^0, \mu^1)$  to determine a rejection and go to Round 2.

⋮

**Round  $t$ :** Each agent  $i \in I$  who was rejected by some object in Round  $t - 1$  applies to her next most preferred acceptable object (if any).

Each object  $o$  compares the set of new applicants in this round with the set of applicants temporarily matched to it under  $\mu^{t-1}$  (if any), temporarily admits all highest priority agents with respect to  $\succeq'_o$  and rejects all others.

Let  $\mu^t$  be the resulting assignment.

If one of the rejected agents has not yet applied to all acceptable objects, go to Round  $t + 1$ .

If all rejected agents have applied to all acceptable objects and  $\mu^t$  is a matching, stop.

In any other case, use routine  $TB(\mu^0, \dots, \mu^t)$  to determine a rejection and go to Round  $t + 1$ .

⋮

If there is no other way for the main algorithm to proceed, a tie-breaking stage is invoked. For the description of this procedure remember that we assume  $I \subset \mathbb{N}$  throughout. We say that agent  $i$  is rejected by  $o$  in favor of agent  $j$  in round  $t$ , if  $i$  is rejected by  $o$  in round  $t$ ,  $i \notin \mu^t(o)$ ,

and  $j \in \mu^t(o)$ . Note that this notion of being rejected in favor of another agent only refers to rejections that are issued *outside* the tie-breaking routine. Next, we describe the tie-breaking routine.

**Tie-breaking Routine**  $TB(\mu^0, \dots, \mu^t)$ :

If  $\kappa > 1$ , go to (TB.1). Otherwise go to (TB.2).

**(TB.1)** Let  $o$  be an object such that  $|\mu^t(o)| \geq 2$ .

If  $|\mu^t(o)| = 3$ ,  $o$  rejects the agent with the lowest index among those temporarily matched to it.

If  $\mu^t(o) = \{i_1, i_2\}$  for two distinct  $i_1, i_2 \in I$  with  $i_1 > i_2$ ,  $o$  rejects  $i_1$  if either

- (a) there is a round  $s < t$  in which  $j_{i_1, i_2}$  was rejected by an object in favor of  $i_2$ , or
- (b)  $\mu^t(j_{i_1, i_2}) \in O$  and  $\{i_2, j_{i_1, i_2}\} \succ_{\mu^t(j_{i_1, i_2})} i_1$ ,

and  $o$  rejects  $i_2$  in any other case.

**(TB.2)** Let  $J \subseteq I$  be the set of agents  $i$  such that  $\mu^t(i) \in O$  and  $Q \subseteq O$  be the set of objects  $o$  such that  $\mu^t(o) \neq \emptyset$ , and go to (1).

(1) Construct a directed graph on  $J \cup Q$  by letting

- each  $i \in J$  such that  $\mu^t(i) \in Q$  point to  $\mu^t(i)$ ,
- each  $o \in Q$  whose owner is in  $J$  point to its owner, and
- each  $o \in Q$  whose owner is not in  $J$  point to the highest indexed agent in  $J$ .<sup>18</sup>

Go to (2).

(2) If there is no cycle, go to (3).

Otherwise, let  $C = \{j_1, p_1, \dots, j_N, p_N\}$  be a cycle,<sup>19</sup> set  $J := J \setminus \{j_1, \dots, j_N\}$  and  $Q := Q \setminus \{p_1, \dots, p_N\}$ , and go back to (1).

(3) Each object  $o \notin Q$  rejects all agents  $i \in J$  such that  $\mu^t(i) = o$ .

Given a preference profile  $R$  and a solvable IB environment  $\succeq$ , let  $AT^{\succeq}(R)$  be the matching chosen by the ADA-ETB algorithm. Since  $\succeq'$  may still contain ties, the ADA-ETB allows for temporary violations of capacity constraints in the sense that it allows two or more equal priority agents to be temporarily matched to the same object. Only if there is no other way for the algorithm to proceed, is an equal priority (with respect to  $\succeq'$ ) agent rejected in favor of another equal priority agent in the tie-breaking routine. In case of a HET environment, the decision who to reject is determined by (TB.2). Here, all agents in  $J$  point to their (temporarily) assigned object in the first iteration of (1). If  $\mu^t(i) = o$  is contained in a cycle  $C$  that does not contain  $i$ , then  $i$  does not point anywhere in subsequent iterations of (1) and is rejected by  $o$  once the procedure reaches (3) (and stops). If the environment is *not* a HET environment, the decision who to reject is determined by (TB.1) on basis of the priority structure, temporary assignments, and the history of rejections in the course of the main algorithm. Note that our earlier results imply that  $|\mu^t(o)| = 3$  is possible only when  $|I| = 3$ . If  $\mu^t(o) = \{i_1, i_2\}$  for two

<sup>18</sup>Note that this trivially includes all unowned objects.

<sup>19</sup>This means that for all  $n \in \{1, \dots, N\}$ , agent  $j_n$  points to object  $p_n$  and object  $p_n$  points to agent  $j_{n+1}$ , where  $N + 1 := 1$ .

distinct agents  $i_1$  and  $i_2$  such that  $i_1 > i_2$ , preference is given to the higher indexed agent unless either (a)  $j_{i_1, i_2}$  was rejected in favor of  $i_2$  in some earlier round, or (b)  $j_{i_1, i_2}$  and  $i_2$  have *strictly* higher priority for the temporary assignment of  $j_{i_1, i_2}$  than  $i_1$  (w.r.t. the original priority structure  $\succ$ ). To illustrate the procedure, we now determine the chosen allocation for a simple example.

**Example 6** (Example 4 continued). *Suppose agents' preferences are as follows*

$R_1$	$R_2$	$R_3$	$R_4$	$R_5$	$R_6$	$R_7$	$R_8$
$o_1$	$o_1$	$p_1$	$p_1$	$p_2$	$p_3$	$p_3$	$o_2$
				$p_3$	$p_2$	$o_2$	

For this example, we have  $I^0 = \{1, \dots, 6\}$  and  $I^1 = \{7, 8\}$ . In Round 0 of the ADA-ETB, a temporary matching for agents 1 through 6 is determined using the ADA for strict priority structures. The ADA ends after the first step and produces the matching

$$\mu^0 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ o_1 & 2 & p_1 & 4 & p_2 & p_3 \end{pmatrix}.$$

In Round 1 of the ADA-ETB, agents 7 and 8 apply to their first choice objects  $p_3$  and  $o_2$ , respectively. Since  $7 \succ'_{p_3} 6$ , object  $p_3$  rejects 6 in favor of 7. In Round 2, 6 then applies to object  $p_2$ . Since  $6 \succ_{p_2} 5$ , this leads to the rejection of 5, who then applies to  $p_3$  in Round 3. As  $5 \succ_{p_3} 7$ , 7 is rejected by  $p_3$  and applies to  $o_2$  in Round 4 of the algorithm. Since  $7 \sim'_{o_2} 8$ , we obtain the following temporary assignment

$$\mu^4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ o_1 & 2 & p_1 & 4 & p_3 & p_2 & o_2 & o_2 \end{pmatrix}.$$

Since no agent is rejected in Round 4 and  $\mu^4$  is not a matching, the algorithm proceeds to the tie-breaking routine. Since  $\kappa > 1$ , the algorithm moves to subroutine (TB.1). Now note that  $j_{7,8} = 6$  and recall that 6 was rejected by  $p_3$  in favor of 7 in Round 1 of the ADA-ETB. Hence, 8 is rejected by  $o_2$  and, given that this is her only acceptable object, subsequently remains unmatched. We thus obtain the following outcome of the ADA-ETB procedure

$$AT^{\succ}(R) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ o_1 & 2 & p_1 & 4 & p_3 & p_2 & o_2 & 8 \end{pmatrix}.$$

We now discuss the allocative and incentive properties of the proposed procedure.

**Theorem 3.** *Let  $\succ$  be a solvable IB environment.*

*Then  $AT^{\succ}$  is constrained efficient.*

Given that Erdil and Ergin's (2008) stable improvement cycles procedure finds a constrained efficient matching for *all* priority-based allocation problems and that unsolvable environments exist, there are situations in which their mechanism succeeds in finding a constrained efficient

matching, while the ADA-ETB mechanism is not applicable. As our next result shows, the major advantage of our new mechanism is that for all solvable IB environments it is not only guaranteed to find a constrained efficient matching, but it also provides agents with dominant strategy incentives to submit preferences truthfully. This is surprising given that we sometimes rely on agents' preferences to break ties.

**Theorem 4.** *Let  $\succeq$  be a solvable IB environment.*

*Then  $AT^{\succeq}$  is group strategy-proof.*

Theorems 3 and 4 complete the proof of the sufficiency part of Theorem 2. Before proceeding, we briefly comment on the proof of Theorem 4. Proofs of the (group) strategy-proofness of the ADA procedure for strict environments usually rely on properties of the set of stable matchings.<sup>20</sup> Since those properties do not generally hold for matching models with indifferences, we cannot extrapolate from previous proof techniques here. Next, note that Theorem 4 yields the following interesting corollary.

**Corollary 4.** *An IB environment  $\succeq$  is solvable if and only if there exists a group strategy-proof and constrained efficient matching mechanism.*

Thus, imposing the logically stronger incentive compatibility requirement of group strategy-proofness does not necessitate additional restrictions of the priority structures, at least when attention is restricted to IB environments.<sup>21</sup> We conclude this section by discussing the design of the algorithm and some important special cases.

First, we discuss the importance of taking the history of rejections into account when breaking ties between equal priority agents (with respect to  $\succeq'$ ). As we show in Appendix A.2, whenever the tie-breaking routine is invoked in a round  $t$  where  $\{i_1, i_2\} = \mu^t(o)$  for two distinct agents  $i_1, i_2$  with  $i_1 > i_2$  and  $p := \mu^t(j_{i_1, i_2})$  is such that  $i_2 \succ_p j_{i_1, i_2} \succ_p i_1$ , any constrained efficient and strategy-proof mechanism must award object  $o$  to  $i_2$ . Now consider the situation in which, all else being equal,  $i_2$  strictly prefers  $p$  over  $o$ . In this case she would cause the rejection of  $j_{i_1, i_2}$  by  $p$  who would then continue to propose down her preference list in subsequent rounds. This could initiate a rejection chain which ultimately leads to the rejection of  $i_2$  by  $p$ . Strategy-proofness requires that in this case  $i_2$  must still receive a place at  $o$ , as otherwise she would be strictly better off concealing her preference for  $p$ . In particular, whenever  $i_2$  has caused a rejection of  $j_{i_1, i_2}$  and subsequently competes with  $i_1$  for an object  $o$  such that  $i_1 \sim_o i_2$ , we have to ensure that  $i_2$  obtains object  $o$ .

Next, we discuss the relationship of the ADA-ETB with existing mechanisms. If  $\succeq$  is strict, then the tie-breaking stage is never invoked and  $AT^{\succeq} = ADA^{\succeq}$ . For HET environments, we have  $AT^{\succeq} = TTC^{\succeq}$ : the first time the tie-breaking routine is invoked, all agents who do

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<sup>20</sup>For example, the most general result on the strategy-proofness of the ADA for agents on the proposing side is the one by Hatfield and Milgrom (2005). Their proof relies heavily on the so called rural hospitals theorem (cf. Roth, 1986).

<sup>21</sup>For general social choice problems, Barberà et al. (2010) recently characterized preference domains for which individual implies group strategy-proofness. Since their domain restriction is never satisfied for matching problems with more than one object, the above corollary is not implied by their result.

not belong to one of the top-trading cycles are rejected by their first choices. It is relatively straightforward to establish that an agent belonging to a top-trading cycle is not subsequently rejected in the course of the ADA-ETB (this is formally established in the proof of Theorem 3 in the appendix). Iterative application of these arguments establishes the equivalence of  $AT^\succeq$  and  $TTC^\succeq$  if  $\succeq$  is a HET environment. The main difference to previous presentations of the TTC procedure is that above we basically reinterpret the TTC as a tie-breaking subroutine.

## 5 Discussion and Conclusion

In this paper we introduced a simple set of necessary conditions for the existence of strategy-proof and constrained efficient mechanisms in priority-based allocation problems with weak priorities. Furthermore, we showed that these conditions are also sufficient in a large class of environments that includes all previously known solvable environments. An interesting direction for future research is to extend our results to a full characterization of all solvable environments and the case where multiple copies of some or all objects are available. Given the strong restrictions we have shown to be necessary for the solvability of IB environments, our conditions have immediate implications for more general environments when there is only one copy of each object. If multiple copies of some or all of the objects are available, the conditions for solvability clearly become more permissive. At the extreme end, where each object can accommodate all agents, no restrictions on the shape of the priority structure are necessary in order to guarantee solvability. However, exactly how much leverage is gained by increasing capacities is an interesting question that we leave for future research.<sup>22</sup> In the remainder of this section, we discuss some other important issues.

### 5.1 Relaxing connectedness

We have claimed above that the assumption of connectedness is without loss of generality for our analysis. If this assumption is violated, we will now show how the set of all agents can be partitioned into a sequence of subsets to which our results apply.

To identify the first set in this sequence, we search for a *minimally connected subset* of  $I$  (with respect to  $\succeq$ ), i.e. a minimal set  $J \subseteq I$  such that  $J \succeq_o I \setminus J$  for all objects  $o$ . If there is a unique minimally connected subset  $J$  of  $I$ , set  $J_1 := J$ . Otherwise, we must have  $i \sim_o j$  for all  $i, j \in I$  and we let  $J_1$  be a singleton set consisting of only the highest indexed agent in  $I$ .<sup>23</sup> Proceeding iteratively, suppose we have defined minimally connected subsets  $J_1, \dots, J_t$  and  $I_{t+1} := I \setminus (J_1, \dots, J_t) \neq \emptyset$ . If there is a unique minimally connected subset  $J \subseteq I_{t+1}$  with respect to  $\succeq|_{I_{t+1}}$ , we set  $J_{t+1} := J$ . Otherwise, let  $J_{t+1}$  be the singleton set consisting of the

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<sup>22</sup>One problem in obtaining a characterization for such cases is that now some objects can be assigned to several agents. This makes the construction of counterexamples needed to demonstrate necessity much more difficult, as now one has to keep track of which *subset* of agents out of an equal priority group gets an object.

<sup>23</sup>To see this, note that if there are two sets  $J, J'$  such that  $J \succeq_o J'$  and  $J' \succeq_o J$  for all objects  $o$  then  $J \sim_o J'$  for all objects  $o$ . Furthermore,  $J \neq J'$  implies  $|J \cup J'| \geq 2$ . Hence, if there are two agents  $i, j \in I$  such that  $i \succ_p j$  for some object  $p$ ,  $J \sim_p J' \succeq_p i \succ_p j$  and  $\succeq$  cannot be an IB environment.

highest indexed agent in  $I_{t+1}$ . In the following, we will refer to the just defined partition  $(J_t)_t$  as the *minimal top-set partition* of  $I$  with respect to  $\succeq$ . We obtain the following immediate corollary of Theorem 2.

**Corollary 5.** *An IB environment  $\succeq$  is solvable if and only if, for all  $t$ ,*

- (i)  $\succeq|_{J_t}$  is acyclic,
- (ii)  $\succeq|_{J_t}$  has no strong priority reversals, and
- (iii)  $\succeq|_{J_t}$  contains no inconsistent weak priority reversals.

To describe a group strategy-proof and constrained efficient mechanism, we slightly extend the domain of the ADA-ETB mechanism as follows: Given a preference profile  $R$  of agents in  $I$ , a subset of agents  $J \subseteq I$ , and a subset of objects  $Q \subseteq O$ , let  $AT^{\succeq}(R, J, Q)$  be the outcome chosen by the ADA-ETB procedure when the set of agents is  $J$ , the set of available objects is  $Q$ , and preferences of agents in  $J$  are given by  $R_J$ . Now suppose that (i)-(iii) are satisfied for all  $t$ . Given some preference profile  $R$ , consider the following iterative procedure for allocating the available goods among the agents: in the first step, calculate  $AT^{\succeq}(R, J_1, O)$ , let  $O_1$  be the set of objects assigned to some agent in  $J_1$  under this matching, and  $Q_1 := O \setminus O_1$  be the set of remaining objects. Now fix some  $t \geq 2$  and suppose that  $O_1, Q_1, \dots, O_{t-1}, Q_{t-1}$  have already been defined. In the  $t$ th step, calculate  $AT^{\succeq}(R, J_t, Q_{t-1})$ , let  $O_t$  be the set of objects assigned to some agent in  $J_t$  under this matching, and  $Q_t := Q_{t-1} \setminus O_t$  be the set of remaining objects. Let  $\overline{AT}^{\succeq}(R)$  be the final matching of objects to agents induced by this iterative procedure. By Theorem 3 and the assumed validity of (i)-(iii) for all  $t$ ,  $AT^{\succeq}(R, J_t, Q_{t-1})$  is a constrained efficient matching of objects in  $Q_{t-1}$  to agents in  $J_t$ . By construction of the iterative procedure and the definition of the minimal top-set partition, this guarantees that  $\overline{AT}^{\succeq}(R)$  is a constrained efficient matching of objects in  $O$  to agents in  $I$ : for all  $s, t$  such that  $s < t$ , an agent in  $J_s$  can never envy an agent in  $J_t$  for her assignment since (a) *all* agents in  $J_s$  have weakly higher priority for *all* objects than *all* agents in  $J_t$  and (b) receive preferred treatment in the sequential mechanism described above. Hence, a stable improvement cycle would have to consist exclusively of agents in  $J_t$  and objects in  $O_t$  for some  $t$ , which is not possible given the properties of the ADA-ETB. Finally,  $\overline{AT}^{\succeq}(\cdot)$  is group strategy-proof if conditions (i)-(iii) are satisfied for all  $t$ . This follows from Theorem 4, since the reports of agents in  $J_t$  do not have any influence on the set of objects  $Q_{t-1}$  available to them. Hence, just as we claimed above, the assumption of connectedness is without loss of generality.

For the house allocation problem, the above iterative procedure boils down to a serial dictatorship (with the right to choose determined by the indexing of agents). Note also that depending on the priority structure, it could be the case that there is a group strategy-proof and constrained efficient mechanism for only a strict subset of the agents. For example, it could be that the minimal top-set partition of  $I$  is, say,  $(J_1, J_2)$ , that  $\succeq|_{J_1}$  is solvable, but  $\succeq|_{J_2}$  is not.

## 5.2 Exogenous versus endogenous tie-breaking

The ADA-ETB relies on agents' preferences to break (some of the) ties in the priority structure. In contrast, Abdulkadiroglu et al. (2009) focus on exogenous, or fixed, tie-breaking, where ties are broken without reference to student preferences. They show that for an arbitrary weak priority structure no strategy-proof mechanism can Pareto dominate the SDA mechanism resulting from *single tie-breaking*, which refers to an exogenous tie-breaking procedure in which ties are resolved in the same way across all objects. Given the importance of exogenous tie-breaking for the existing literature, it is instructive to characterize environments for which there exists some exogenous tie-breaking procedure guaranteeing constrained efficiency. We first formally define the notion of solvability by exogenous tie-breaking.

**Definition 6.** *An environment  $\succeq$  is solvable by exogenous tie-breaking, if there exists a strict priority structure  $\succeq^S$  such that  $i \succ_o^S j$  whenever  $i \succ_o j$  and such that for all preference profiles  $R$ ,  $ADA^{\succeq^S}(R) \in CE^{\succeq}(R)$ .*

Note that it is without loss of generality to restrict attention to the ADA algorithm here, since it is the unique constrained efficient mechanism for strict environments. We now characterize the class of IB environments that are solvable by exogenous tie-breaking. For this, we need the following definition.

**Definition 7.** *A tie  $i_1 \sim_o i_2$  is weakly cyclic, if there exist two agents  $j_1, j_2 \in I \setminus \{i_1, i_2\}$  and objects such that either*

- (a)  $i_1 \succ_{p_1} j_1 \succ_o i_1$  and  $i_2 \succ_{p_2} j_2 \succ_o i_2$ , or
- (b)  $\{i_1, i_2\} \succ_{p_1} j_1 \succ_{p_2} j_2 \succ_o i_1$ .

*A weak priority structure is strongly acyclic if it contains no weakly cyclic ties.*

From the definition, it is immediate that any strongly cyclic tie is weakly cyclic. The converse is not true. However, the only type of weakly cyclic tie  $i_1 \sim_o i_2$  which is not strongly cyclic, is the one where case (a) above applies only with  $j_1 = j_2$  and  $p_1 \neq p_2$ . We are now ready to present our characterization result.

**Theorem 5.** *A connected IB environment  $\succeq$  with  $|I| \geq 4$  is solvable by exogenous tie-breaking if and only if it is strongly acyclic and has no weak priority reversals.*

*Furthermore, if  $\succeq$  is solvable by exogenous tie-breaking, it is solvable by single tie-breaking.*

Comparing Theorems 5 and 2, we see that the differences between the class of solvable IB environments and the class of IB environments solvable by exogenous tie-breaking is small. While this can be seen as further support for the view that not much is lost by focusing on exogenous tie-breaking, one should take note that HET environments satisfy weak, but not strong acyclicity. Hence, if we were to limit attention to exogenous tie-breaking, we would lose this practically important class of environments. The above characterization of IB environments solvable by exogenous tie-breaking does not extend to environments with  $|I| = 3$ , where, as we detail in Appendix B.3, slightly stronger conditions are needed.

### 5.3 Relation to other characterizations

In an influential paper, Ergin (2002) characterized strict environments for which constrained efficiency is equivalent to full efficiency.<sup>24</sup> If there is only one copy of each object, as we assume throughout, his condition is that there should not be three distinct agents  $i_1, i_2, i_3$  and two objects  $o, p$  such that  $i_1 \succ_o i_2 \succ_o i_3$  and  $i_3 \succ_p i_1$ . If a priority structure satisfies this requirement, it is called *Ergin-acyclic*. Ergin (2002) showed that for strict environments  $\succeq$  the following statements are equivalent: (i)  $\succeq$  is Ergin-acyclic, (ii)  $ADA^\succeq$  is efficient, and (iii)  $ADA^\succeq$  is strongly group strategy-proof.

While Ergin's characterization applies only to strict environments, it is easy to extend it to our class of connected IB environments. The reason is that a connected IB environment  $\succeq$  is Ergin-acyclic if and only if it is a HET environment: suppose  $\succeq$  is a connected IB environment such that for some object  $o$ ,  $i_1 \succ_o i_2 \succ_o i_3$ . By Ergin-acyclicity, no agent  $j$  such that  $i_3 \succeq_o j$  can ever have strictly higher priority for one of the objects than  $i_1$ . But then  $\succeq$  can be connected only if there exists an agent  $j$  such that  $i_3 \succeq_o j$  and  $j \succ_p i_2$  for some  $p$ . Since  $\succeq$  is an IB environment, we must have  $i_1 \succ_p j \succ_p i_2$ . By Ergin-acyclicity,  $i_2$  can never have strictly higher priority for one of the objects than  $i_1$  and we obtain a contradiction to connectedness.

This implies in particular, that Ergin-acyclicity is more demanding than the conditions for the solvability of IB environments. Furthermore, there is no logical relationship between Ergin's condition and the possibility of solving a weak priority structure by exogenous tie-breaking. Finally, note that the above observation yields the following extension of Ergin's characterization for strict environments.

**Theorem 6.** *Let  $\succeq$  be a connected IB environment. Then the following are equivalent:*

- (i)  $\succeq$  is Ergin-acyclic
- (ii)  $AT^\succeq$  is efficient
- (iii)  $AT^\succeq$  is strongly group strategy-proof.

To see that this result holds note that, as explained above, the Ergin-acyclicity of  $\succeq$  implies that  $\succeq$  is a HET environment, so that  $AT^\succeq(\cdot)$  reduces to the TTC mechanism. For the other direction, note that whenever  $\succeq$  has an Ergin-cycle, the counterexamples from Ergin (2002) apply, irrespective of whether  $\succeq$  is strict or not.

### 5.4 Insufficiency of existing mechanisms

Given the nature of this investigation it is important to know whether some of the existing mechanisms would work in any solvable environments. We now provide an example showing that all pre-existing mechanisms may fail to be either strategy-proof or constrained efficient even though an environment is solvable.

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<sup>24</sup>Kesten (2006) characterizes strict environments for which TTC and ADA coincide by means of a similar acyclicity condition (which is equivalent to Ergin's condition for the case of unit capacities). Ehlers and Erdil (2010) characterize general priority environments such that all constrained efficient matchings are always efficient. Their condition is stronger than Ergin's.

**Example 7.** *There are three agents  $i_1, i_2, i_3$  and seven objects  $o, p_1, \dots, p_6$ . The priority structure is given by*

$\succ$	$\succ_o$	$\succ_{p_1}$	$\succ_{p_2}$	$\succ_{p_3}$	$\succ_{p_4}$	$\succ_{p_5}$	$\succ_{p_6}$
	$i_1, i_2, i_3$	$i_1$	$i_1$	$i_2$	$i_2$	$i_3$	$i_3$
		$i_2$	$i_3$	$i_1$	$i_3$	$i_1$	$i_2$
		$i_3$	$i_2$	$i_3$	$i_1$	$i_2$	$i_1$

*Note that since there are only three agents and no ties below the top of the priority structure, Theorem 2 implies that the above environment is solvable. Since there are several weak priority reversals, the environment is not solvable by exogenous tie-breaking by Theorem 5. Furthermore, Ergin-acyclicity is not satisfied so that in particular no efficient mechanism is stable. Finally, we argue that the stable improvement cycles procedure by Erdil and Ergin (2008) cannot be used. In order to implement this procedure, we would need to first resolve indifferences at object  $o$ , then run the ADA-algorithm using this modified fixed priority structure, and finally eliminate all stable improvement cycles.*

*Given the symmetries of the example it is easy to see that we can assume without loss of generality that  $i_1 \succ'_o i_2 \succ'_o i_3$ . Consider first the preference profile given by  $R_{i_1} = p_5, o$ ,  $R_{i_2} = o$ , and  $R_{i_3} = o$ . The ADA with the just mentioned strict priority structure would leave  $i_3$  unmatched, assign  $p_5$  to  $i_1$  and  $o$  to  $i_2$ . Since there is no stable improvement cycle, this is also the final outcome of the stable improvement cycles procedure. Suppose now that  $i_3$  changed her preferences to  $R'_{i_3} = o, p_5$ , while all other agents leave their preferences unchanged. Now the ADA procedure would leave  $i_2$  unmatched, assign  $o$  to  $i_1$  and  $p_5$  to  $i_3$ . In particular, there is a (unique) stable improvement cycle since  $i_1$  and  $i_3$  mutually prefer each other's assignment and  $i_2$  cannot veto this exchange. Hence, the stable improvement cycles procedure would assign  $o$  to  $i_3$  at the profile  $(R_{i_1}, R_{i_2}, R'_{i_3})$  and  $i_3$  has an incentive to manipulate this procedure at  $R$ .*

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## A Proof of Theorem 1

### A.1 Necessity of Acyclicity

We first introduce a logically weaker notion of strongly cyclic ties for general environments.

**Definition 8.** (i) An  $(i, o)$ -cycle is a sequence of  $L$  distinct agents  $i^1, \dots, i^L = i^0 = i$  and a sequence of  $L$  distinct objects  $o^1, \dots, o^L = o$  such that for all  $l \in \{1, \dots, L\}$ ,  $i^{l-1} \succ_{o^l} i^l$ .

(ii) A tie  $i \sim_o j$  between two distinct agents  $i, j \in I$  is strongly cyclic, if there is an  $(i, o)$ -cycle  $(i^l, o^l)_{l=1}^L$  and a  $(j, o)$ -cycle  $(j^m, p^m)_{m=1}^M$  such that either

(a')  $i^l \neq j^m$  for all  $l, m$  and  $o^l \neq p^m$  for all  $l \in \{1, \dots, L-1\}, m \in \{1, \dots, M-1\}$ , or

(b') there are  $l, m$  such that  $\{i^1, o^1, \dots, i^{l-1}, o^{l-1}\} \cap \{j^1, p^1, \dots, j^{m-1}, p^{m-1}\} = \emptyset$  and  $(i^{l'}, o^{l'}) = (j^{m'}, p^{m'})$  for all  $l' \geq l$  and  $m' \geq m$ .

A tie  $i \sim_o j$  is thus strongly cyclic, if there is an  $(i, o)$ -cycle and a  $(j, o)$ -cycle such that the two cycles are either completely disjoint (case (a')) or “meet” exactly once and then coincide (case (b')).

**Proposition 1.** If  $\succeq$  contains a strongly cyclic tie in the sense of Definition 8, then  $\succeq$  is not solvable.

*Proof.* Suppose  $\succeq$  contains a strongly cyclic tie  $i \sim_o j$  with  $(i, o)$ -cycle  $(i^l, o^l)_l^L$  and  $(j, o)$ -cycle  $(j^m, p^m)_m^M$  that satisfy either condition (a') or (b') of Definition 8. Now consider a preference profile  $R$  such that all agents in  $I \setminus \{i^1, \dots, i^L, j^1, \dots, j^M\}$  rank objects in  $\{o, o^1, \dots, o^{L-1}, p^1, \dots, p^{M-1}\}$  as unacceptable,

$$R_i = o,$$

$$R_j = o,$$

$$R_{i^l} = o^l, o^{l+1} \text{ for all } l \in \{1, \dots, L-1\}, \text{ and}$$

$$R_{j^m} = p^m, p^{m+1} \text{ for all } m \in \{1, \dots, M-1\}.$$

This preference profile is well defined since either condition (a') or (b') of Definition 28 is satisfied. It is easy to see that there are precisely two constrained efficient matchings for this problem (we only specify the relevant part of the matchings):

$$\mu = \begin{pmatrix} i & j & i^l(l < L) & j^m(m < M) \\ o & j & o^l & p^m \end{pmatrix} \text{ and } \nu = \begin{pmatrix} i & j & i^l(l < L) & j^m(m < M) \\ i & o & o^l & p^m \end{pmatrix}.$$

That these two assignments are matchings follows since the cycles satisfy (a') or (b') of Definition 8. Now assume that, contrary to what we want to show, there exists a strategy-proof and constrained efficient matching mechanism  $f$ . Assume first that  $f(R) = \nu$ . Suppose  $i$  claims that her preferences are actually  $R'_i = o, o^1$  and consider the preference profile  $R' = (R'_i, R_{-i})$ . By strategy-proofness, we must have  $f_i(R') \neq o$ . But then stability can only be satisfied when  $f_{i^l}(R') = o^{l+1}$  for all  $l \leq L$  (with  $o^{L+1} := o^1$ ), and  $i^1, \dots, i^L$  form a stable improvement cycle of  $f(R')$  at  $R'$ , a contradiction. If  $f(R) = \mu$ , we can similarly derive a contradiction by letting  $j$  deviate to  $R'_j = o, p^1$ . This completes the proof.  $\square$

While it is immediate that a strongly cyclic tie in the sense of Definition 2 (a) or (b) is also strongly cyclic in the sense of Definition 8, the converse does not necessarily hold. However, the next result shows that for IB environments acyclicity in the sense of Definition 2 and the absence of strong priority reversals jointly imply that there are no strongly cyclic ties in the sense of Definition 8.

**Lemma 4.** *Let  $\succeq$  be an arbitrary IB-environment and suppose  $i \sim_o j$  is a strongly cyclic tie in the sense of Definition 8. Then  $\succeq$  either contains a strongly cyclic tie in the sense of Definition 2 or  $\succeq$  has a strong priority reversal.*

*Proof.* Suppose  $\succeq$  contains a strongly cyclic tie  $i \sim_o j$  with cycles  $(i^l, o^l)_{l=1}^L$  and  $(j^m, p^m)_{m=1}^M$  that are either completely disjoint (case (a')) or meet exactly once and then coincide (case (b')). We assume that the cycles are minimal in the sense that no strict subset of  $\{i^1, o^1, \dots, i^L, o^L, j^1, p^1, \dots, j^M, p^M\}$  contains a strongly cyclic tie (that is, contains a strongly cyclic tie and the corresponding cycles).

Suppose first that case (a') is satisfied. If  $\max\{L, M\} \geq 3$ , we obtain an immediate contradiction to the assumed minimality: If  $L \geq 3$ , we must have  $i \succeq_o i^1$ , since if  $i^1 \succ_o i$ ,  $(i^1, o^1, i, o)$  is an  $(i, o)$ -cycle of length 2, contradicting the assumed minimality. Since  $\succeq$  is an IB environment and  $i \sim_o j$ , this implies  $i \sim_o i^1$ . But then  $i^1 \sim_o j$  is strongly cyclic with cycles  $(i^l, o^l)_{l=2}^L$  and  $(j^m, p^m)_{m=1}^M$ , so that we still obtain a contradiction to the assumed minimality. Hence, we must have  $L = M = 2$  and  $i \sim_o j$  is strongly cyclic in the sense of Definition 2.

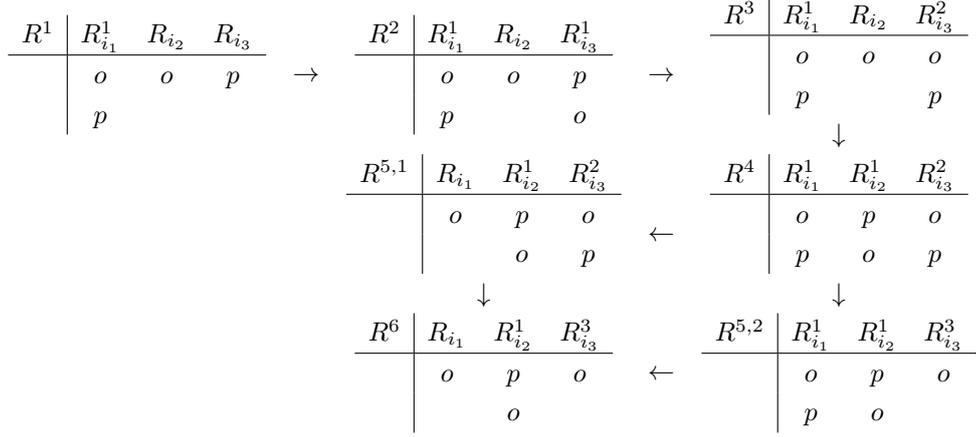
Next, suppose that case (b') is satisfied and let  $l, m$  be the point where the two cycles meet. A straightforward variation of the above arguments shows that minimality requires  $l = m = 1$  and hence,  $L = M$ . We now show that  $L \geq 3$  implies that  $\succeq$  contains a strong priority reversal. Since  $\succeq$  is an IB environment and  $i \sim_o j$ , the assumed minimality of the cycle implies  $i \sim_o j \sim_o i^1 \sim_o i^2$ . We can assume w.l.o.g. that  $i \succ_{o^1} j \succ_{o^1} i^1$ . Minimality of the cycles implies that  $i^2 \succ_{o^1} i^1$  since otherwise we would have  $\{i, j\} \succ_{o^1} i^1 \succeq_{o^1} i^2$  so that there is a joint cycle  $(i^l, o^l)_{l=2}^L$  of length  $L - 1$ . Since  $i \sim_o i^1$ , a similar argument shows that minimality implies  $i^2 \succeq_{o^2} i$ . If  $i^2 \sim_{o^2} i$ , we obtain a contradiction to minimality since  $\{i, i^2\} \succ_{o^1} i^1 \succ_{o^2} i$ . Hence, we must have  $i^1 \succ_{o^2} i^2 \succ_{o^2} i$ . Since  $i \succ_{o^1} j \succ_{o^1} i^1$  and  $i^1 \sim_o i^2 \sim_o i \sim_o j$ , we obtain a strong priority reversal as claimed.  $\square$

## A.2 Tie-breaking lemma

**Lemma 5.** *Let  $i_1, i_2, i_3$  be three distinct agents and suppose there are two distinct objects  $o$  and  $p$  such that  $i_1 \sim_o i_3 \sim_o i_2$  and  $i_1 \succ_p i_3 \succ_p i_2$ . Consider a preference profile  $R$  such that  $R_{i_1} = o$ ,  $R_{i_2} = o$ ,  $R_{i_3} = p$ , and  $jP_j\{o, p\}$  for all  $j \in I \setminus \{i_1, i_2, i_3\}$  such that  $j \succeq_o i_1$  or  $j \succeq_p i_2$ .*

*If  $f$  is a constrained efficient and strategy-proof mechanism, then  $f_{i_1}(R) = o$ .*

*Proof.* Suppose to the contrary that under the conditions of the lemma, we have  $f_{i_1}(R) \neq o$ . Then by constrained efficiency,  $f_{i_2}(R) = o$ . Throughout the proof we only consider preferences and matchings for agents  $i_1, i_2, i_3$  with the understanding that the preferences of other agents are fixed at some  $R_{-\{i_1, i_2, i_3\}}$  which satisfies the conditions of Lemma 5. To derive a contradiction, we consider seven preference profiles that are summarized in the following diagram. Arrows indicate how we move between the profiles.



Consider first the profile  $R^1$ . By strategy-proofness, we must have  $f_{i_1}(R^1) \neq o$  since  $R^1$  is obtained from  $R$  by adding  $p$  to  $i_1$ 's list of acceptable objects. Since  $i_1 \succ_p i_3$ , stability requires that  $f_{i_1}(R^1) = p$  and consequently  $f_{i_3}(R^1) = i_3$ . Next, consider the profile  $R^2$ . Strategy-proofness requires that  $f_{i_3}(R^2) \neq p$  since  $R^2$  is obtained from  $R^1$  by adding  $o$  to  $i_3$ 's list. By non-wastefulness, we must have  $f_{i_1}(R^2) = p$ . But then we cannot have  $f_{i_3}(R^2) = o$  since otherwise  $i_1$  and  $i_3$  would form a stable improvement cycle of  $f(R^2)$  at  $R^2$ . Hence,  $f_{i_2}(R^2) = o$ . Since  $R^3$  is obtained from  $R^2$  by letting  $i_3$  reshuffle her list of acceptable objects, strategy-proofness requires that  $f_{i_3}(R^3) = i_3$ . But then non-wastefulness requires that  $f(R^3) = f(R^2)$ . Now suppose  $i_2$  changes her ranking to  $R_{i_2}^1 = p, o$  leading to the preference profile  $R^4$ . By stability,  $f_{i_2}(R^4) \neq p$  since either  $i_1$  or  $i_3$  will have to be rejected by  $o$  and both have strictly higher priority for  $p$  than  $i_2$ . Strategy-proofness requires that  $f_{i_2}(R^4) = o$ , and stability implies  $f_{i_1}(R^4) = p$  and  $f_{i_3}(R^4) = i_3$ .

Now suppose first that  $i_1$  declares  $p$  unacceptable, leading to the profile  $R^{5,1}$ . By strategy-proofness, we must have  $f_{i_1}(R^{5,1}) = i_1$ . By constrained efficiency we must have  $f_{i_2}(R^{5,1}) = p$  and  $f_{i_3}(R^{5,1}) = o$ . Starting from  $R^{5,1}$  suppose  $i_3$  declared  $p$  unacceptable, leading to the profile  $R^6$ . By strategy-proofness, we must have  $f_{i_3}(R^6) = o$

Next, suppose that starting from  $R^4$ ,  $i_3$  declares  $p$  unacceptable, leading to the profile  $R^{5,2}$ . By strategy-proofness,  $f_{i_3}(R^{5,2}) = i_3$ . By constrained efficiency,  $f_{i_1}(R^{5,2}) = o$  and  $f_{i_2}(R^{5,2}) = p$ . Starting from  $R^{5,2}$  suppose  $i_1$  declared  $p$  unacceptable, leading to  $R^6$ . By strategy-proofness, we must have  $f_{i_1}(R^6) = o$ . This is a contradiction since we have shown above that  $i_3$  must also obtain  $o$  at  $R^6$ .  $\square$

### A.3 Necessity of no strong priority reversals

Suppose to the contrary that  $\succeq$  contains a strong priority reversal  $(i_1, i_2, j_1, j_2, o, p_1, p_2)$  (i.e.  $i_1 \succ_{p_1} j_1 \succ_{p_1} i_2$ ,  $i_2 \succ_{p_2} j_2 \succ_{p_2} i_1$ ,  $i_1 \sim_o i_2 \sim_o j_1 \sim_o j_2$ , and  $j_1 \neq j_2$ ) but there is a constrained efficient and strategy-proof mechanism  $f$ . Consider a preference profile  $R$  such that all agents in  $I \setminus \{i_1, i_2, j_1, j_2\}$  rank objects in  $\{o, p_1, p_2\}$  as unacceptable,

$$\begin{aligned} R_{i_1} &= o, \\ R_{i_2} &= o, \\ R_{j_1} &= p_1, \text{ and} \\ R_{j_2} &= p_2. \end{aligned}$$

The tie-breaking lemma implies  $f_{i_1}(R) = o$  since  $i_1 \succ_{p_1} j_1 \succ_{p_1} i_2$ . However, since  $i_2 \succ_{p_2} j_2 \succ_{p_2} i_1$ , the same lemma also implies  $f_{i_2}(R) = o$ . Since  $o$  can only be assigned to one agent, this is a

contradiction and completes the proof.  $\square$

## A.4 Necessity of no inconsistent weak priority reversals

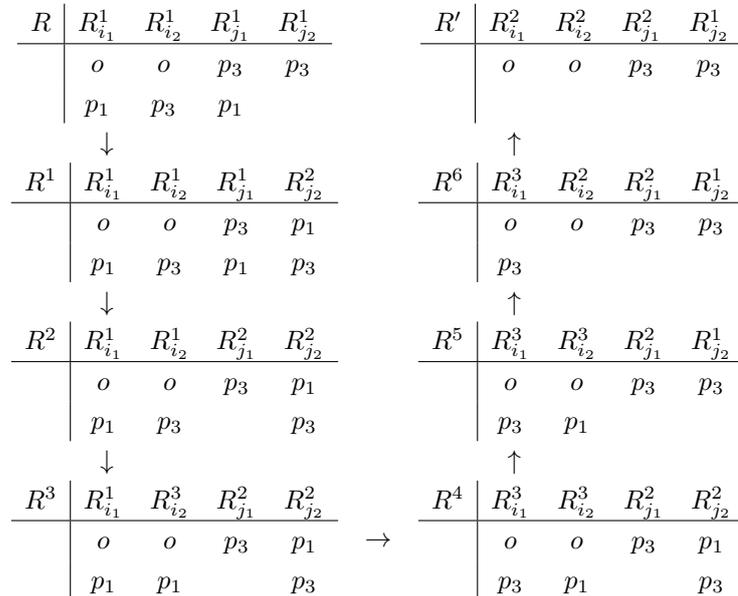
Suppose  $\succeq$  contains an inconsistent weak priority reversal  $(i_1, i_2, j_1, o, p_1, p_2)$  (i.e.  $i_1 \sim_o i_2 \sim_o j_1$ ,  $i_1 \succ_{p_1} j_1 \succ_{p_1} i_2$  and  $i_2 \succ_{p_2} j_1 \succ_{p_2} i_1$ ). Let  $j_2 \in I \setminus \{i_1, i_2, j_1\}$  be an agent such that  $\{i_1, i_2, j_1\} \sim_o j_2$ ,  $\{i_1, i_2, j_1\} \succ_{p_1} j_2$ ,  $\{i_1, i_2, j_1\} \succ_{p_2} j_2$ , and  $\{i_1, i_2\} \succ_{p_3} j_2 \succ_{p_3} j_1$  for some object  $p_3 \in O \setminus \{o, p_1, p_2\}$ . Suppose that contrary to what we want to show there exists a constrained efficient and strategy-proof mechanism  $f$ . Throughout the proof we assume that all agents in  $I \setminus \{i_1, i_2, j_1, j_2\}$  rank all objects in  $\{o, p_1, p_2, p_3\}$  as unacceptable and will not specify those agents' preferences.

**Claim 1:**

- (a) Let  $R_{i_1}^1 = o, p_1, R_{i_2}^1 = o, p_3, R_{j_1}^1 = p_3, p_1, R_{j_2}^1 = p_3$ , and consider the profile  $R = (R_{i_1}^1, R_{i_2}^1, R_{j_1}^1, R_{j_2}^1)$ . If  $f$  is strategy-proof and constrained efficient, we must have  $f_{i_1}(R) = o$ .
- (b) Let  $R_{i_1}^2 = o, R_{i_2}^2 = o, R_{j_1}^2 = p_3, R_{j_2}^2 = p_3$ , and consider the profile  $R' = (R_{i_1}^2, R_{i_2}^2, R_{j_1}^2, R_{j_2}^2)$ . If  $f$  is strategy-proof and constrained efficient, then  $f_{i_1}(R) = o$  implies  $f_{i_1}(R') = o$ .

*Proof.* (a) Suppose to the contrary that  $f_{i_2}(R) = o$ . Constrained efficiency then implies  $f_{i_1}(R) = p_1$ ,  $f_{j_1}(R) = j_1$ , and  $f_{j_2}(R) = p_3$ . Now suppose that  $j_1$  unilaterally changes her ranking to  $R_{j_1}^3 = p_1$  and consider the resulting preference profile  $\tilde{R} = (R_{i_1}^1, R_{i_2}^1, R_{j_1}^3, R_{j_2}^1)$ . By strategy-proofness, we must have  $f_{j_1}(\tilde{R}) = j_1$ . But this is compatible with constrained efficiency only when  $f_{i_2}(\tilde{R}) = o$  and thus  $f(\tilde{R}) = f(R)$ . Next, suppose that  $i_1$  unilaterally deviates to  $R_{i_1}^2 = o$  and consider the resulting preference profile  $\tilde{R}' = (R_{i_1}^2, R_{i_2}^1, R_{j_1}^3, R_{j_2}^1)$ . By strategy-proofness we must have  $f_{i_1}(\tilde{R}') = i_1$ . Non-wastefulness implies  $f_{i_2}(\tilde{R}') = o$ ,  $f_{j_1}(\tilde{R}') = p_1$ , and  $f_{j_2}(\tilde{R}') = p_3$ . Finally, assume that  $i_2$  unilaterally deviates to  $R_{i_2}^2 = o$  and consider the resulting preference profile  $\tilde{R}'' = (R_{i_1}^2, R_{i_2}^2, R_{j_1}^3, R_{j_2}^1)$ . By strategy-proofness, we must have  $f_{i_2}(\tilde{R}'') = o$  and constrained efficiency implies  $f(\tilde{R}'') = f(\tilde{R}')$ . But this is a violation of the tie-breaking lemma, since  $i_1 \succ_{p_1} j_1 \succ_{p_1} i_2$ ,  $R_{i_1}^2 = o, R_{i_2}^2 = o, R_{j_1}^3 = p_1$ , and all agents in  $I \setminus \{i_1, i_2, j_1\}$  rank objects  $o$  and  $p_1$  as unacceptable.

- (b) The following diagram summarizes the preference profiles used in the proof and also indicates how we move between those profiles.



By part (a) of Claim 1, we have  $f_{i_1}(R) = o$ ,  $f_{i_2}(R) = p_3$ ,  $f_{j_1}(R) = p_1$ , and  $f_{j_2}(R) = j_2$ . Consider a unilateral deviation of  $j_2$  to  $R_{j_2}^2 = p_1, p_3$  and the resulting preference profile  $R^1$ . By strategy-proofness, we must have  $f_{j_2}(R^1) \neq p_3$ . This is compatible with constrained efficiency only if  $f_{i_2}(R^1) = p_3$  and  $f(R^1) = f(R)$ . Next, suppose  $j_1$  unilaterally deviates to  $R_{j_1}^2 = p_3$  and consider the resulting preference profile  $R^2$ .

We claim that we must have  $f_{i_1}(R^2) = o$ : otherwise  $f_{i_2}(R^2) = o$ . By strategy-proofness,  $i_2$  would still have to obtain  $o$  if she unilaterally deviated to  $R_{i_2}^2 = o$ , leading to the profile  $\hat{R} = (R_{i_1}^1, R_{i_2}^2, R_{j_1}^2, R_{j_2}^2)$ . But then constrained efficiency implies  $f_{i_1}(\hat{R}) = p_1$ ,  $f_{j_1}(\hat{R}) = j_1$ , and  $f_{j_2}(\hat{R}) = p_3$ . Now consider a unilateral deviation of  $j_1$  to  $R_{j_1}^1$  and the resulting preference profile  $\hat{R}' = (R_{i_1}^1, R_{i_2}^2, R_{j_1}^1, R_{j_2}^2)$ . By strategy-proofness, we must have  $f_{j_1}(\hat{R}') \neq p_3$ . This is compatible with constrained efficiency only when  $f_{i_2}(\hat{R}') = o$ . But then  $f$  cannot be strategy-proof since  $i_2$  can obtain  $\hat{R}'$  from  $R^1$  by a unilateral deviation from  $R_{i_2}^1$  to  $R_{i_2}^2$  and  $f_{i_2}(R^1) \neq o$ , as we have already established. Hence, it has to be the case that  $f_{i_1}(R^2) = o$  and constrained efficiency implies  $f_{i_2}(R^2) = p_3$ ,  $f_{j_1}(R^2) = j_1$ , and  $f_{j_2}(R^2) = p_1$ .

By strategy-proofness,  $i_2$  cannot obtain  $o$  by unilaterally deviating to  $R_{i_2}^3 = o, p_1$ . Constrained efficiency implies  $f_{i_1}(R^3) = o$ ,  $f_{i_2}(R^3) = p_1$ ,  $f_{j_1}(R^3) = j_1$ , and  $f_{j_2}(R^3) = p_3$ . Next, consider a unilateral deviation of  $i_1$  to  $R_{i_1}^3 = o, p_3$ , leading to the preference profile  $R^4$ . By strategy-proofness,  $f_{i_1}(R^4) = o$  and constrained efficiency implies  $f(R^4) = f(R^3)$ . Strategy-proofness implies that  $j_2$  must still get  $p_3$  when she deviates to  $R_{j_2}^1 = p_3$ . Since  $i_1 \succ_{p_3} j_2$ , this is compatible with constrained efficiency only when  $f_{i_1}(R^5) = o$  and  $f(R^5) = f(R^4)$ . Now consider a unilateral deviation of  $i_2$  to  $R_{i_2}^2 = o$ , leading to the profile  $R^6$ . By strategy-proofness, we must have  $f_{i_2}(R^6) = i_2$  and constrained efficiency implies  $f_{i_1}(R^6) = o$ . Since  $R'$  can be obtained from  $R^6$  via a unilateral deviation of  $i_1$  to  $R_{i_1}^3 = o$ , we must have  $f_{i_1}(R') = o$  as claimed.  $\square$

Observe that for the proof of part (a) of the claim, we only need to assume that the priority structure is such that  $i_1 \succ_{p_1} j_1 \succ_{p_1} i_2$  and  $j_2 \succ_{p_3} j_1$ . Now let  $R_{i_1}^3 = o, p_3$ ,  $R_{i_2}^4 = o, p_2$ ,  $R_{j_1}^3 = p_3, p_2$ ,  $R_{j_2}^1 = p_3$ , and consider the preference profile  $R'' = (R_{i_1}^3, R_{i_2}^4, R_{j_1}^3, R_{j_2}^1)$ . Since  $i_2 \succ_{p_2} j_1 \succ_{p_2} i_1$ , we can simply exchange the roles of agents  $i_1$  and  $i_2$  and of objects  $p_1$  and  $p_2$  in the proof of part (a) to conclude that a strategy-proof and constrained efficient matching mechanism  $f$  has to satisfy  $f_{i_2}(R'') = o$ . Similarly, for the proof of part (b), we only need to assume that the priority structure is such that  $i_1 \succ_{p_1} j_1 \succ_{p_1} i_2$  and  $\{i_1, i_2\} \succ_{p_3} j_2 \succ_{p_3} j_1$ . Since  $i_2 \succ_{p_2} j_1 \succ_{p_2} i_1$ , we can again simply exchange the roles of  $i_1$  and  $i_2$ , and of  $p_1$  and  $p_2$  in the proof to conclude that if  $f$  is strategy-proof and constrained efficient, then  $f_{i_2}(R'') = o$  implies  $f_{i_2}(R') = o$ . Thus, we obtain that  $f_{i_1}(R') = f_{i_2}(R') = o$ , which is a contradiction since there is only one copy of object  $o$ . This completes the proof.  $\square$

## B Proofs for Section 4

For the proofs in this section, we fix a solvable IB environment  $\succeq$  throughout. Theorem 1 implies that  $\succeq$  must be acyclic, has no strong priority reversals, and contains no inconsistent weak priority reversals.

### Proof of Lemma 1

If  $K_T = 0$ , then  $K_F = 1$  and  $L_1 \cup \dots \cup L_{(K_F-1)} = \emptyset$  since  $I$  is connected, i.e. there is nothing to show. For the following, let  $K_T > 0$ .

We show first that all agents with a priority level smaller than or equal to the threshold must be strictly ranked. Let  $i_1 \in L_1 \cup \dots \cup L_{K_T}$  and  $o_1 \in O$  be such that  $i_1 \in r_{[K_T+3]}(\succeq_{o_1})$ .

We show first that  $\succeq_{o_1} |_{(L_1 \cup \dots \cup L_{K_T})}$  must be strict. Suppose to the contrary that there are two distinct  $j_1, j_2 \subseteq L_1 \cup \dots \cup L_{K_T}$  such that  $j_1 \sim_{o_1} j_2$ . By the definition of  $K_T$ , no agent in  $L_1 \cup \dots \cup L_{K_T}$  can have strictly lower priority for  $o_1$  than agent  $i_1$ . Since  $\succeq$  is an IB environment, we must have  $\{j_1, j_2\} \subseteq r_{[K_T+3]}(\succeq_{o_1})$ . Let  $p_1, p_2$  be such that  $r_{[K_T-l_1]}(\succeq_{p_1}) = j_1$  and  $r_{[K_T-l_2]}(\succeq_{p_2}) = j_2$  for some  $l_1, l_2 \geq 0$  (note that such objects must exist by the definition of  $L_1 \cup \dots \cup L_{K_T}$ ). Since  $\{j_1, j_2\} \subseteq r_{[K_T+3]}(\succeq_{o_1})$ , there are  $K_T + 2$  agents distinct from  $i_1, j_1, j_2$ , who have strictly higher priority for  $o_1$  than  $j_1$  and  $j_2$ . Since at  $p_1$  and  $p_2$  there are at most  $K_T - 1$  agents with strictly higher priority than  $j_1$  and  $j_2$ , respectively, there must be two distinct agents  $j_3, j_4$  such that  $j_1 \succ_{p_1} j_3$ ,  $j_2 \succ_{p_2} j_4$ , and  $\{j_3, j_4\} \succ_{o_1} j_1 \sim_{o_1} j_2$ , contradicting the acyclicity of  $\succeq$ . Thus,  $\succeq_{o_1} |_{(L_1 \cup \dots \cup L_{K_T})}$  is strict.

Since  $i_1 \in L_1 \cup \dots \cup L_{K_T}$ , there must exist some  $o_2 \in O$  such that  $r_{[K_T-l]}(\succeq_{o_2}) = i_1$  for some  $l \geq 0$ . We claim that  $\succeq_{o_2} |_{(L_1 \cup \dots \cup L_{K_T})}$  must be strict. Suppose to the contrary that  $j_1 \sim_{o_2} j_2$  for two distinct  $j_1, j_2 \in L_1 \cup \dots \cup L_{K_T}$ . Since  $\succeq$  is an IB environment, we must have  $i_1 \succ_{o_2} j_1 \sim_{o_2} j_2$  in this case. Since  $\succeq_{o_1} |_{(L_1 \cup \dots \cup L_{K_T})}$  is strict and, by the definition of  $K_T$ , no agent in  $L_1 \cup \dots \cup L_{K_T}$  can have strictly lower priority than  $i_1$  for object  $o_1$ , we must have  $\{j_1, j_2\} \succ_{o_1} i_1$ . But then the tie  $j_1 \sim_{o_2} j_2$  is strongly cyclic, a contradiction. Thus,  $\succeq_{o_2} |_{(L_1 \cup \dots \cup L_{K_T})}$  is strict.

Next, suppose there exists some  $o_3 \in O \setminus \{o_1, o_2\}$  such that  $j_1 \sim_{o_3} j_2$  for two distinct  $j_1, j_2 \in L_1 \cup \dots \cup L_{K_T}$ . Since  $\succeq$  is an IB environment and  $\succeq_{o_1} |_{(L_1 \cup \dots \cup L_{K_T})}$  is strict, this tie is strongly cyclic unless  $i_1 \sim_{o_3} j_1 \sim_{o_3} j_2$  as well. Let  $i_2$  be the agent with the lowest priority for object  $o_2$  among agents in  $L_1 \cup \dots \cup L_{K_T}$ . Then if  $i_2 \succ_{o_3} j_1$ , the tie  $j_1 \sim_{o_3} j_2$  is strongly cyclic since  $\{j_1, j_2\} \succ_{o_2} i_2$ . Hence,  $i_1 \sim_{o_3} i_2 \sim_{o_3} j_1 \sim_{o_3} j_2$ . Since  $i_1 \in r_{[K_T+3]}(\succeq_{o_1})$  and  $i_1 = r_{[K_T-l]}(\succeq_{o_2})$  for some  $l \geq 0$ , there must be two distinct agents  $i_3, i_4 \in I \setminus \{i_1, i_2\}$  such that  $\{i_3, i_4\} \succ_{o_1} i_1 \succ_{o_2} \{i_3, i_4\}$ . Suppose first that  $i_3 \succ_{o_3} i_1 \sim_{o_3} i_2$ . Since  $i_2 \in L_1 \cup \dots \cup L_{K_T}$ , there must exist an object  $o_4$  such that  $i_2 = r_{[K_T-l']}(\succeq_{o_4})$  for some  $l' \geq 0$ . As  $i_1 \in r_{[K_T+3]}(\succeq_{o_1})$ , there are  $K_T + 2$  agents with strictly higher priority for  $o_1$  than  $i_1$ . Of these agents, at least two must have strictly lower priority for  $o_4$  than agent  $i_2$ . In particular, there is an agent  $j_3 \in I \setminus \{i_1, i_2, i_3\}$  such that  $i_2 \succ_{o_4} j_3$  and  $j_3 \succ_{o_1} i_1$ . If  $j_3 \succ_{o_3} i_2$ , the tie  $i_1 \sim_{o_3} i_2$  is strongly cyclic since  $i_1 \succ_{o_2} i_3 \succ_{o_3} i_1$  and  $i_2 \succ_{o_4} j_3 \succ_{o_3} i_2$ , a contradiction. But then the tie  $i_2 \sim_{o_3} j_3$  is strongly cyclic, since  $\{i_2, j_3\} \succ_{o_1} i_1 \succ_{o_2} i_3 \succ_{o_3} i_2$ . Hence, we must have  $i_3 \sim_{o_3} i_1$  and a symmetric argument shows that  $i_4 \sim_{o_3} i_1$  as well. Combining the above, we obtain that  $i_1 \succ_{o_2} \{i_2, i_3, i_4\}$ ,  $\{i_2, i_3, i_4\} \succ_{o_1} i_1$ , and  $i_1 \sim_{o_3} i_2 \sim_{o_3} i_3 \sim_{o_3} i_4$ . It is then easy to see that no matter how agents  $i_2, i_3, i_4$  are ranked relative to each other at  $o_1$  and  $o_2$ , we must obtain a strong priority reversal, a contradiction. This completes the proof that two agents in  $L_1 \cup \dots \cup L_{K_T}$  can never have equal priority for one of the objects.

Now suppose there exist two distinct agents  $i_1, i_2 \in L_1 \cup \dots \cup L_{K_F-1}$  such that  $i_1 \sim_{o_1} i_2$  for some object  $o_1 \in O$ . By the above, there are two cases left to consider (a)  $i_1, i_2 \in L_{K_T+1} \cup \dots \cup L_{K_F-1}$ , and (b)  $i_1 \in L_{K_T+1} \cup \dots \cup L_{K_F-1}$  and  $i_2 \in L_1 \cup \dots \cup L_{K_T}$ .

We consider case (a) first. We must have  $L_{K_F} \cup \dots \cup L_K \neq \emptyset$  since  $K_F$  is the smallest integer  $k \geq K_T$  such that  $L_k \cup \dots \cup L_K$  is connected. Thus, for all  $i \in L_{K_T+1} \cup \dots \cup L_{K_F-1}$  and all  $o \in O$ ,  $i \succeq_o (L_{K_F} \cup \dots \cup L_K)$ . Now, since  $I$  is connected there must exist an object  $o_2 \neq o_1$ , an agent

$i_3 \in L_{K_F} \cup \dots \cup L_K$ , and an agent  $i_4 \in L_1 \cup \dots \cup L_{K_T}$ , such that  $i_3 \succ_{o_2} i_4$ . By definition of  $K_F$ , we must have  $\{i_1, i_2\} \succ_{o_2} i_3 \succ_{o_2} i_4$  so that we obtain a strongly cyclic tie if either  $i_3 \succ_{o_1} i_1$  or  $i_4 \succ_{o_1} i_1$ . Since  $\succeq$  is an IB environment, we must thus have  $i_1 \sim_{o_1} i_2 \sim_{o_1} i_3 \sim_{o_1} i_4$ . However, since  $i_4 \in L_1 \cup \dots \cup L_{K_T}$  there is an object  $o_3$  and an agent  $i_5 \in L_1 \cup \dots \cup L_{K_T}$  such that  $i_4 \succ_{o_3} i_5$ . As shown above,  $\succeq|_{(L_1 \cup \dots \cup L_{K_T})}$  must be strict, and by  $i_4, i_5 \in L_1 \cup \dots \cup L_{K_T}$ , it has to be the case that  $i_5 \succ_{o_1} i_4 \sim_{o_1} i_1 \sim_{o_1} i_2$ . Since  $\{i_1, i_2\} \succ_{o_2} i_4 \succ_{o_3} i_5 \succ_{o_1} \{i_1, i_2\}$ , the tie  $i_1 \sim_{o_1} i_2$  is strongly cyclic, a contradiction.

Next, we consider case (b). By the above,  $\succeq|_{(L_1 \cup \dots \cup L_{K_T})}$  is strict and because  $\succeq$  is an IB environment, we must have  $(L_1 \cup \dots \cup L_{K_T}) \setminus \{i_2\} \succ_{o_1} i_2$ . By the definition of  $K_F$  and  $i_1 \in L_{K_T+1} \cup \dots \cup L_{K_F-1}$ , we must have  $(L_{K_F} \cup \dots \cup L_K) \sim_{o_1} i_1$ . Since  $I$  is connected and  $(L_{K_T+1} \cup \dots \cup L_{K_F-1}) \succeq_o (L_{K_F} \cup \dots \cup L_K)$  for all objects  $o$ , there must exist an agent  $i_3 \in L_{K_F} \cup \dots \cup L_K$  and an agent  $i_4 \in L_1 \cup \dots \cup L_{K_T}$  such that  $i_3 \succ_{o_2} i_4$  for some object  $o_2 \in O \setminus \{o_1\}$ . By definition of  $K_F$ , we must have  $i_1 \succ_{o_2} i_4$  as well. If  $i_4 \neq i_2$ , the tie  $i_1 \sim_{o_1} i_3$  is strongly cyclic since  $i_4 \succ_{o_1} i_2 \sim_{o_1} i_1$  as  $\succeq|_{(L_1 \cup \dots \cup L_{K_T})}$  is strict and  $\{i_1, i_3\} \succ_{o_2} i_4$ . If  $i_4 = i_2$ , there must exist an agent  $i_5 \in L_1 \cup \dots \cup L_{K_T}$  such that  $i_2 \succ_{o_3} i_5$  for some object  $o_3$  since  $i_2 \in L_1 \cup \dots \cup L_{K_T}$ . Since  $\{i_1, i_3\} \succ_{o_2} i_2 \succ_{o_3} i_5 \succ_{o_1} i_2$  (by  $i_5 \in (L_1 \cup \dots \cup L_{K_T}) \setminus \{i_2\}$ ), the tie  $i_1 \sim_{o_1} i_3$  is strongly cyclic, a contradiction.  $\square$

## Proof of Lemma 2

Note that by the definition of  $K_F$ ,  $I_T = I \setminus (L_1 \cup \dots \cup L_{(K_F-1)})$  must be connected (with respect to  $\succeq|_{I_T}$ ). Hence,  $|L_1(I_T)| \geq 2$  if  $|I_T| \geq 4$ . Now assume  $|L_1(I_T)| \geq 3$  and let  $i_1, i_2, i_3 \in L_1(I_T)$  be three distinct agents. If  $\succeq|_{I_T}$  is not a HET environment and  $|I_T| \geq 4$ , there must exist an object  $o_1 \in O$  such that (w.l.o.g.)  $i_1 \succ_{o_1} i_4 \succ_{o_1} \{i_2, i_3\}$  for some  $i_4 \in I_T$  (who may or may not be in  $L_1(I_T)$ ). By definition of  $K_T$  and  $L_1(I_T)$ , we cannot have  $i_2 \succ_{o_1} i_3$  or  $i_3 \succ_{o_1} i_2$ , as otherwise either  $i_2$  or  $i_3$  must have at most fourth highest priority among agents in  $I_T$  for  $o_1$ . Hence,  $i_2 \sim_{o_1} i_3$  and since  $i_2, i_3 \in L_1(I_T)$ , this tie is strongly cyclic: there are two distinct objects for which  $i_2$  and  $i_3$  have the unique highest priority among agents in  $I_T$ , so that in particular both of them have strictly higher priorities for these objects than  $i_1$  and  $i_4$ . Now suppose that  $\succeq|_{I_T}$  is a HET environment, so that  $\overline{K} = 1$ . Since  $I$  is connected and  $\succeq$  is not a HET environment, there must exist an agent  $i_5 \in I \setminus I_T$  such that (w.l.o.g.)  $i_1 \succ_{o_2} i_5 \succ_{o_2} (L_1(I_T) \setminus \{i_1\})$  for some  $o_2 \in O$ . Let  $o_3 \in O$  be such that  $i_2 \succ_{o_3} i_1$ . A tie  $i_5 \sim_{o_3} i_1$  is strongly cyclic and hence we must have  $i_5 \succ_{o_3} i_1$  as well (since  $\succeq|_{I_T}$  is a HET environment, we must have  $i_1 \sim_{o_3} i_3$ , so that  $i_1 \succ_{o_3} i_5$  is impossible). Since  $i_3 \in L_1(I_T)$ , there is an object  $o_4 \in O \setminus \{o_2, o_3\}$  such that  $i_3 \succ_{o_4} i_2$ . But then the tie (remember that we assumed  $\succeq|_{I_T}$  is a HET environment)  $i_1 \sim_{o_3} i_3$  is strongly cyclic (by  $i_1 \succ_{o_2} i_5 \succ_{o_3} i_1$  and  $i_3 \succ_{o_4} i_2 \succ_{o_3} i_3$ ), a contradiction to solvability. Hence, we must have  $|L_1(I_T)| = 2$ , which is (i). It is straightforward to adapt the arguments above to show that  $|L_k(I_T)| \leq 2$  for all  $k \in \{1, \dots, \overline{K}\}$ , which also establishes part (iii) of the lemma.

Hence, we are left to establish part (ii). Suppose to the contrary that there is some smallest  $k \in \{2, \dots, \overline{K} - 1\}$  such that  $|L_k(I_T)| = 2$ . Let  $i_1, i_2 \in L_k(I_T)$  be two distinct agents. Since  $k \leq \overline{K} - 1$ , there must be a third agent  $i_3 \in L_{k+1}(I_T)$  such that  $r_{[k+1]}(\succeq_{o_1}|_{I_T}) = i_3$  for some  $o_1 \in O$ . Since  $k$  is the smallest integer greater or equal than two such that  $|L_k(I_T)| \geq 2$  and  $|L_1(I_T)| = 2$ , we must have  $|L_1(I_T) \cup \dots \cup L_{k-1}(I_T)| = k$ . Furthermore, priorities in the strict part of  $\succeq|_{I_T}$  vary by at most three ranks, so that  $\{i_1, i_2\} \subseteq r_{[k+2]}(\succeq_{o_1}|_{I_T})$  and in particular  $i_1 \sim_{o_1} i_2$ . Since  $i_1, i_2 \in L_k(I_T)$ , there exist objects  $o_2, o_3 \in O$  such that  $r_k(\succeq_{o_2}|_{I_T}) = i_1$  and  $r_k(\succeq_{o_3}|_{I_T}) = i_2$ . Note that  $i_3 \in L_{k+1}(I_T)$

implies that  $i_1 \succ_{o_2} i_3$  and  $i_2 \succ_{o_3} i_3$ . Furthermore, there must be an agent  $i_4 \in L_{k-1}(I_T)$  such that  $i_1 \succ_{o_2} i_4$ . Since  $i_3 \in L_{k+1}(I_T)$  and  $i_3 \succ_{o_1} i_1 \sim_{o_1} i_2$ , we must have  $i_4 \succ_{o_1} i_3$  given that  $i_4 \in r_{k-1}(\succeq_{o_1} | I_T) \cup r_k(\succeq_{o_1} | I_T) \cup r_{k+1}(\succeq_{o_1} | I_T)$ . But then the tie  $i_1 \sim_{o_1} i_2$  is strongly cyclic since  $i_1 \succ_{o_2} i_4 \succ_{o_1} i_1$  and  $i_2 \succ_{o_3} i_3 \succ_{o_1} i_2$ . This is a contradiction and establishes (ii).  $\square$

The following lemma contains some important additional properties of the exogenous tie-breaking procedure, which will be used in the proofs below.

**Lemma 6.** *Let  $\succeq$  be a solvable IB environment such that  $\kappa > 1$ .*

- (a) *If there are three distinct agents  $i_1, i_2, j \in I$  and two objects  $o_1, o_2 \in O$  such that  $i_1 \succ_{o_1} j \succeq_{o_1} i_2$  and  $i_2 \succ_{o_2} j \succeq_{o_2} i_1$ , then  $j \succ_{o_1} i_2$  and  $j \succ_{o_2} i_1$ .*
- (b) *If  $|I_T| \geq 3$ , then there is at most one agent  $i \in L_1(I_T)$  such that  $i \in \{r_1(\succeq_o | I_T), r_2(\succeq_o | I_T)\}$  for all objects  $o \in O$ .*
- (c) *If there are two distinct agents  $i_1$  and  $i_2$  such that for some object  $o \in O$ , both  $i_1 \sim_o i_2$  and  $i_1 \succ'_o i_2$ , then there does not exist an agent  $j \in I \setminus \{i_1, i_2\}$  and an object  $p$  such that  $i_2 \succ_p j \succeq_p i_1$ .*

### Proof of Lemma 6

- (a) Acyclicity requires that either  $[j \sim_{o_1} i_2 \text{ and } j \sim_{o_2} i_1]$ , or  $[j \succ_{o_1} i_2 \text{ and } j \succ_{o_2} i_1]$ : If  $j \succ_{o_1} i_2$ , a tie  $j \sim_{o_2} i_1$  is strongly cyclic since  $\{i_1, j\} \succ_{o_1} i_2$ . If  $j \succ_{o_2} i_1$ , a tie  $j \sim_{o_1} i_2$  is strongly cyclic since  $\{i_2, j\} \succ_{o_2} i_1$ .

We will now show that  $i_1 \succ_{o_1} j \sim_{o_1} i_2$  and  $i_2 \succ_{o_2} j \sim_{o_2} i_1$  is impossible. Let  $j' \in I \setminus \{i_1, i_2, j\}$  be arbitrary. By connectedness, there exists a sequence of  $n$  distinct agents  $j^1, \dots, j^n = j'$  and a sequence of  $n$  distinct objects  $p^1, \dots, p^n$  such that  $j \succ_{p^1} j^1 \succ_{p^2} \dots \succ_{p^n} j^n = j'$ .

If  $j^1 \notin \{i_1, i_2\}$ , the tie  $i_2 \sim_{o_1} j$  is strongly cyclic if  $j^1 \succ_{o_1} j$  since  $j \succ_{p^1} j^1 \succ_{o_1} j$  and  $i_2 \succ_{o_2} i_1 \succ_{o_1} i_2$ . Similarly, the tie  $i_1 \sim_{o_2} j$  is strongly cyclic if  $j^1 \succ_{o_2} j$ . Hence, we must have  $j^1 \sim_{o_1} j$  and  $j^1 \sim_{o_2} j$ . If  $j^k \notin \{i_1, i_2\}$  for all  $k \leq n$ , an iterative application of these arguments yields  $j^k \sim_{o_1} j$  and  $j^k \sim_{o_2} j$  for all  $k \leq n$ , so that in particular  $j' = j^n \sim_{o_1} j$  and  $j' = j^n \sim_{o_2} j$ . Next, suppose there is a  $k$  such that  $j^k = i_1$  and  $j^l \neq i_2$  for all  $l < k$  (the arguments in case there is a  $k$  such that  $j^k = i_2$  and  $j^l \neq i_1$  for all  $l < k$  are completely analogous). Iterative application of the above arguments yields  $j^l \sim_{o_1} j$  and  $j^l \sim_{o_2} j$  for all  $l < k$ . Since  $i_1 \succ_{o_1} j^{k-1} \sim_{o_1} i_2$  and  $j^{k-1} \succ_{p^k} j^k = i_1$ , the tie  $j^{k-1} \sim_{o_1} i_2$  is strongly cyclic unless  $i_1 \succeq_{p^k} i_2$ . However, if  $j^{k-1} \succ_{p^k} i_1 \succ_{p^k} i_2$ , the tie  $i_1 \sim_{o_2} j^{k-1}$  is strongly cyclic, so that we must have  $j^{k-1} \succ_{p^k} i_1 \sim_{p^k} i_2$ . If  $j^{k+1} \neq i_2$ ,  $i_1 \sim_{p^k} i_2$  is strongly cyclic if  $j^{k+1} \succ_{p^k} i_1$  since  $i_1 = j^k \succ_{p^{k+1}} j^{k+1} \succ_{p^k} i_1$  and  $i_2 \succ_{o_2} j^{k-1} \succ_{p^k} j^k = i_1$ . Hence, we must have  $j^{k+1} \sim_{p^k} i_1 \sim_{p^k} i_2$  so that in particular  $j^{k-1} \succ_{p^k} j^{k+1}$ . If  $j^{k+1} \succ_{o_1} i_2$ , the tie  $j^{k-1} \sim_{o_1} i_2$  is strongly cyclic since  $j^{k-1} \succ_{p^k} j^{k+1} \succ_{o_1} j^{k-1}$  and  $i_2 \succ_{o_2} i_1 \succ_{o_1} i_2$ . Hence,  $j^{k+1} \sim_{o_1} i_2$ . Similarly, the acyclicity of  $i_1 \sim_{o_2} j^{k-1}$  requires that  $j^{k+1} \sim_{o_2} i_1$ . Note that these arguments trivially apply in case of  $j^1 = i_1$  (set  $j^0 := j$  in the previous arguments). If  $j^l \neq i_2$  for all  $l > k$ , iterative application of these arguments yields  $j^l \sim_{o_1} i_1$  and  $j^l \sim_{o_2} i_2$  for all  $l > k$ , so that in particular  $j' \sim_{o_1} i_1$  and  $j' \sim_{o_2} i_2$ .

So suppose there is a  $k' > k$  such that  $j^{k'} = i_2$  (and  $j^k = i_1$ ). If  $k' > k + 1$ , then using the same arguments as above, we find that acyclicity requires  $j^l \sim_{o_1} i_2$  and  $j^l \sim_{o_2} i_1$  for all  $l \in \{1, \dots, k-1, k+1, \dots, k'-1\}$ . But then, similar arguments to above show that we must have  $j^{k'-1} \succ_{p^{k'}} i_1 \sim_{p^{k'}} i_2$ . The tie  $i_1 \sim_{p^{k'}} i_2$  is strongly cyclic if  $j^{k'+1} \succ_{p^{k'}} i_1$  since  $i_2 = j^{k'} \succ_{p^{k'+1}} j^{k'+1} \succ_{p^{k'}} i_1$  and  $i_1 \succ_{o_1} j^{k'-1} \succ_{p^{k'}} j^{k'} = i_2 \sim_{p^{k'}} i_1$ . Hence, we must have  $i_1 \sim_{p^{k'}} i_2 \sim_{p^{k'}} j^{k'+1}$ ,

so that in particular  $j^{k'-1} \succ_{p^{k'}} j^{k'+1}$ . But then the acyclicity of  $i_2 \sim_{o_1} j^{k'-1}$  requires that  $j^{k'+1} \sim_{o_1} i_2$  and the acyclicity of  $i_1 \sim_{o_2} j^{k'-1}$  requires that  $j^{k'+1} \sim_{o_2} i_1$ . If  $k' = k + 1$ , the tie  $i_2 \sim_{o_1} j^{k-1}$  (where  $j^0 := j$ ) is strongly cyclic if  $j^{k+2} \succ_{o_1} i_2$  since  $i_2 = j^{k+1} \succ_{p^{k+2}} j^{k+2} \succ_{o_1} i_2$  and  $j^{k-1} \succ_{p^k} j^k = i_1 \succ_{o_1} j^{k-1}$ . But the tie  $i_2 \sim_{o_1} j^{k+2}$  would also be strongly cyclic if  $\{j^{k+2}, i_2\} \succ_{o_2} i_1$ , so that we must have  $j^{k+2} \sim_{o_2} i_1$  as well. Thus, no matter whether  $k' = k + 1$  or  $k' > k + 1$ , iterative application of these arguments shows that we must have  $j^l \sim_{o_1} i_2$  and  $j^l \sim_{o_2} i_1$  for all  $l > k'$ . In particular,  $j' \sim_{o_1} i_2$  and  $j' \sim_{o_2} i_1$ . Since  $j'$  was arbitrary, we obtain that  $r_2(\succeq_{o_1}) = I \setminus \{i_1\}$  and  $r_2(\succeq_{o_2}) = I \setminus \{i_2\}$ .

Since  $|I| \geq 3$ , connectedness also implies that there must be some agent  $i_3 \in I \setminus \{i_1, i_2\}$  such that (w.l.o.g.)  $i_3 \succ_{o_3} i_1$  for some object  $o_3$  (otherwise,  $\{i_1, i_2\} \succeq_o I \setminus \{i_1, i_2\}$  for all objects  $o$ ). Acyclicity of the ties  $i_1 \sim_{o_2} i_3$  and  $i_2 \sim_{o_1} i_3$  requires that  $i_1 \sim_{o_3} i_2$ . Using the same arguments as above, we obtain  $r_2(\succeq_{o_3}) = I \setminus \{i_3\}$ . However, since  $\kappa > 1$  there must exist an object  $o_4$  such that  $r_3(\succeq_{o_4}) \neq \emptyset$ . Since at least one of the agents  $i_1, i_2, i_3$  must have at most third highest priority for  $o_4$ , at least one of the ties at  $o_1, o_2, o_3$  must be strongly cyclic, a contradiction.

- (b) Suppose to the contrary that there are two distinct  $i_1, i_2 \in L_1(I_T)$  such that  $\{i_1, i_2\} \subseteq \{r_1(\succeq_o |_{I_T}), r_2(\succeq_o |_{I_T})\}$  for all objects  $o \in O$  even though  $|I_T| \geq 3$ . Let  $o_1$  and  $o_2$  be two objects such that  $\{i_1\} = r_1(\succeq_{o_1})$  and  $\{i_2\} = r_1(\succeq_{o_2})$ . Since  $|I_T| \geq 3$  and  $I_T$  is connected, there must be a third agent  $i_3$  and an object  $o_3$  such that (w.l.o.g.)  $i_3 \succ_{o_3} i_1$ . Since neither  $i_1$  nor  $i_2$  can have third highest priority for an object among agents in  $I_T$ , we must have  $i_1 \sim_{o_3} i_2$ . This tie is acyclic only if  $i_1 \succ_{o_1} i_2 \sim_{o_1} i_3$  and  $i_2 \succ_{o_2} i_1 \sim_{o_2} i_3$ . If there exists an object  $p$  such that  $|r_2(\succeq_p |_{I_T})| = 1$ , we must have  $\{i_1, i_2\} = r_1(\succeq_p |_{I_T}) \cup r_2(\succeq_p |_{I_T})$  and hence  $\{i_1, i_2\} \succ_p i_3$  so that  $i_1 \sim_{o_3} i_2$  is strongly cyclic. This implies that  $\succeq |_{I_T}$  is a HET environment. Since  $\succeq$  is a solvable IB environment such that  $\kappa > 1$ , we must have  $I_S \neq \emptyset$ . By assumption,  $I$  is connected so that there must be an agent  $j \in I_S$  such that (w.l.o.g.)  $i_1 \succ_{o_1} j$ . Since  $j \in I_S$ , there is an agent  $j' \in I_S$  such that  $j \succ_q j'$  for some object  $q$ . A tie  $j \sim_{o_1} i_2$  is strongly cyclic, since  $j' \succ_{o_1} j$  by Lemma 1 and  $i_2 \succ_{o_2} i_1 \succ_{o_1} i_2$ . Hence, we must have  $j \succ_{o_1} i_2 \sim_{o_1} i_3$ . But then  $j \succ_{o_2} i_1 \sim_{o_2} i_3$ , since a tie  $j \sim_{o_2} i_1$  would be strongly cyclic given that  $\{i_1, j\} \succ_{o_1} i_2 \succ_{o_2} i_1$ . Since  $i_1 \succ_{o_1} j \succ_{o_2} i_1$  and  $i_3 \succ_{o_3} i_2 \succ_{o_2} i_3$ , the tie  $i_1 \sim_{o_2} i_3$  must be strongly cyclic, a contradiction.

- (c) Let  $i_1$  and  $i_2$  be two distinct agents such that  $i_1 \sim_o i_2$  and  $i_1 \succ'_o i_2$ . Suppose to the contrary that there is an agent  $j \in I \setminus \{i_1, i_2\}$  and an object  $p$  such that  $i_2 \succ_p j \succeq_p i_1$ . For the following note that  $\{i_1, i_2\} \subseteq I_S$  is impossible by Lemma 1.

Consider first the case of  $i_1 \in I_S$  and  $i_2 \in I_T$ . We show first that if  $j \in I_S \setminus \{i_1\}$ , there cannot be an object  $q$  such that  $i_1 \succ_q j$ : otherwise, Lemma 1 would imply that  $j \succ_o i_1 \sim_o i_2$ , so that by acyclicity  $j \succeq_q i_2$ . If either  $i_1 \succ_q j \sim_q i_2$  or  $i_2 \succ_p j \sim_p i_1$ , we obtain a contradiction to the just proved part (a) of this lemma. Hence, we must have  $j \succ_q i_2$  as well as  $j \succ_p i_1$ . Now let  $\tilde{q}$  be such that  $i_1 = r_{k_{i_1}}(\succeq_{\tilde{q}})$ . Note that  $i_1 \in I_S$  and  $i_2 \in I_T$  imply  $k_{i_1} < k_{i_2}$  and that  $i_2 \succ_p i_1$  implies  $\tilde{q} \neq p$ . By definition, there must be at least  $k_{i_2} - 1$  agents with strictly higher priority for  $p$  than agent  $i_2$ . As there are only  $k_{i_1} - 1 < k_{i_2} - 1$  agents with strictly higher priority for  $\tilde{q}$  than  $i_1$ , there must be at least one agent  $j' \in I \setminus \{i_1, i_2\}$  such that  $i_1 \succ_{\tilde{q}} j' \succ_p i_2$ . Since  $i_2 \succ_p j$ , we must have  $j \neq j'$ . If  $j' \succ_o i_1$ , the tie  $i_1 \sim_o i_2$  is strongly cyclic since  $i_1 \succ_{\tilde{q}} j' \succ_o i_1$  and  $i_2 \succ_p j \succ_o i_2$ . However, given that  $\{i_2, j'\} \succ_p i_1 \succ_q j \succ_o i_2$ , a tie  $i_2 \sim_o j'$  must be strongly cyclic. Hence, if  $j \in I_S \setminus \{i_1\}$ , there cannot be an object  $q$  such that  $i_1 \succ_q j$ .

Since  $i_1 \in I_S$ , there is an agent  $j' \in I_S$  and an object  $q$  such that  $i_1 \succ_q j'$ . By the above and

Lemma 1, we must have  $j \neq j'$  as well as  $j' \succ_o i_1 \sim_o i_2$ . If  $j \succ_o i_1 \sim_o i_2$ , we obtain an immediate contradiction to acyclicity. Hence,  $j \sim_o i_1 \sim_o i_2$ . But then we must also have  $i_2 \succ_p j \sim_p i_1$ , since otherwise  $j \sim_o i_2$  would be strongly cyclic given that  $\{j, i_2\} \succ_p i_1 \succ_q j' \succ_o j$ . Since  $j \sim_p i_1$  and  $\succeq$  is an IB environment, we must have  $j' \succ_p i_1$  by Lemma 1. Now note that since  $I$  is connected, there must exist a sequence of  $n$  distinct agents  $j^1, \dots, j^n = i_2$  and a sequence of objects  $p^1, \dots, p^n$  such that  $j \succ_{p^1} j^1 \succ_{p^2} j^2 \dots \succ_{p^n} j^n = i_2$ . If  $j^1 = i_2$ , the tie  $j \sim_p i_1$  is strongly cyclic since  $j \succ_{p^1} i_2 \succ_p j$  and  $i_1 \succ_q j' \succ_p i_1$ . If  $j^1 = j'$ ,  $j \sim_p i_1$  is strongly cyclic, since  $i_1$  must have strictly higher priority than  $i_2$  for at least one of the objects given that  $k_{i_1} < k_{i_2}$ . Hence,  $j^1 \notin \{i_2, j'\}$  and acyclicity implies  $j^1 \sim_p i_1$  as well. We can obviously repeat all of the above arguments with  $j^1$  in place of  $i_1$ . Proceeding iteratively, we must eventually obtain a contradiction to acyclicity.

Next, consider the case of  $i_1, i_2 \in I_T$ . If  $|I_T| = 2$ , there is nothing to show, since  $i_1 \sim_o i_2$  and  $i_1 \succ'_o i_2$  means that only  $i_1$  can ever have strictly higher priority for one of the objects than any agent in  $I_S$ . So suppose that  $|I_T| = 3$  and let  $i_3$  be the third agent in  $I_T$ . Note that we must have  $i_1 \in L_1(I_T)$  and that  $i_1$  always has at least second highest priority among agents in  $I_T$ , as otherwise we would not have set  $i_1 \succ'_o i_2$ . Suppose first that  $i_2 \succ_p j \succ_p i_1$ . Since  $i_1$  always ranks at least second among agents in  $I_T$ , this is only possible when  $j \in I_S$  and  $i_1 \succeq_p i_3$ . In particular,  $j \succ_p i_3$ . Since  $i_1$  is always ranked at least second among agents in  $I_T$ , part (b) of this Lemma implies that there exists an object  $q$  such that  $\{i_1, i_3\} \succ_q i_2$ . Since a tie  $i_1 \sim_p i_3$  is strongly cyclic, we must have  $i_1 \succ_p i_3$ . If  $i_3 \succ_o i_1$ , the tie  $i_1 \sim_o i_2$  is then strongly cyclic since  $\{i_1, i_2\} \succ_p i_3$ . If  $j \succ_o i_1$ , the tie  $i_1 \sim_o i_3$  is strongly cyclic since  $\{i_1, i_3\} \succ_q i_2 \succ_p j \succ_o i_1$ . But in the only remaining possibility of  $i_1 \sim_o i_2 \sim_o i_3 \sim_o j$ , we obtain a strong priority reversal given that  $\{i_1, i_3\} \succ_q i_2$  and  $i_2 \succ_p j \succ_p i_1 \succ_p i_3$ . Hence, we must have  $i_2 \succ_p j \sim_p i_1$ . If  $j = i_3$ , the tie  $i_1 \sim_p i_3$  is strongly cyclic since  $\{i_1, i_3\} \succ_q i_2$ . Hence,  $j \in I_S$  and the acyclicity of  $i_1 \sim_p j$  requires that  $i_1 \succ_q i_2 \succeq_q j$ . Summarizing our findings, we have  $j \in I_S$ ,  $i_1 \in I_T$ ,  $j \sim_p i_1$ ,  $j \succ'_p i_1$ , and  $i_1 \succ_q i_2 \succeq_q j$  in this case. As we have already established above, this is impossible.

Hence, we are left to discuss the case of  $|I_T| \geq 4$ . Assume that  $i_1 \in L_k(I_T)$  and  $i_2 \in L_{k'}(I_T)$ . By  $i_1 \succ'_o i_2$ , we have  $k \leq k'$ . Note that  $k' \leq k + 1$  since agents with priority levels  $k + 2$  or above can never have strictly higher priority than  $i_1$  for one of the objects.

Suppose first that  $k' = k + 1$ . If  $i_2 \succ_p j \succ_p i_1$ , we must have  $j \in I_S$  since otherwise we cannot have  $i_1 \in r_k(\succeq_p |I_T) \cup r_{k+1}(\succeq_p |I_T) \cup r_{k+2}(\succeq_p |I_T)$ . However, there has to exist an agent  $j' \in L_{k-1}(I_T) \cup L_k(I_T)$  such that  $j' \succ_p i_2$  and  $i_1 \succ_q \{i_2, j'\}$  for some  $q \neq p$ . Acyclicity is easily seen to require that  $i_1 \sim_o i_2 \sim_o j \sim_o j'$ . If  $i_1 \succ_q j' \succ_q i_2$ , we obtain a strong priority reversal between  $i_1$  and  $i_2$ . Since  $j' \in L_{k-1}(I_T) \cup L_k(I_T)$  and  $i_2 \in L_{k+1}(I_T)$ , the only remaining possibility is that  $i_1 \succ_q i_2 \sim_q j'$ . However, since  $\{i_2, j'\} \succ_p i_1$ , the tie  $i_2 \sim_q j'$  is strongly cyclic. Hence, we must have  $i_2 \succ_p j \sim_p i_1$ . If  $j \in I_S$ , we obtain a contradiction to acyclicity since for any object  $p'$  such that  $i_1 \succ_{p'} i_2$  (such an object must exist given that  $i_1 \in L_k(I_T)$  and  $i_2 \in L_{k+1}(I_T)$ ), our arguments above show that  $j \succ_{p'} i_2$  as well, so that  $j \sim_p i_1$  would be strongly cyclic. Hence, we must have  $j \in I_T$  as well. If there is an object  $q$  such that  $j \succ_q i_2$ , acyclicity of  $i_1 \sim_p j$  requires  $i_2 \succeq_q i_1$ . Since  $i_2 \sim_q i_1$  implies a contradiction to part (a) of this Lemma (as  $i_2 \succ_p i_1 \sim_p j$  and  $j \succ_q i_1 \sim_q i_2$ ), we must have  $i_2 \succ_q i_1$ . This implies  $j \in L_k(I_T)$  and, by Lemma 2,  $k = 1$  as well as  $L_1(I_T) = \{i_1, j\}$ . But since  $i_2 \succ_p i_1 \sim_p j$ , this contradicts  $|I_T| \geq 4$  (which, by Lemma 2 is only possible if  $|L_1(I_T)| = 2$ ). Hence,  $j$  must always have weakly

lower priority than agent  $i_2$ . This implies that  $j$  must always have weakly lower priority than  $i_1$  as well. Since  $I_T$  is connected, there must exist a sequence of  $n$  distinct agents  $j^1, \dots, j^n = i_2$  in  $I_T$  and a sequence of objects  $p^1, \dots, p^n$  such that  $j \succ_{p^1} j^1 \succ_{p^2} j^2 \dots \succ_{p^n} j^n = i_2$ . By the previous discussion we must have  $j^1 \in I_T \setminus \{i_1, i_2\}$ . If  $j^1 \succ_p j$ , the tie  $i_1 \sim_p j$  is strongly cyclic since  $\{i_1, j\} \succ_{p^1} j^1 \succ_p j$ . Hence, acyclicity implies  $j^1 \sim_p j$  and we can repeat all of the above arguments with  $j^1$  in place of  $j$ . Proceeding iteratively, we must eventually obtain a contradiction to acyclicity.

Hence, we must have  $k = k'$ . Since  $i_1 \succ'_o i_2$ ,  $k = 1$  and  $i_1$  always has at least second highest priority among agents in  $I_T$ . Since  $|I_T| \geq 4$ , part (b) of this Lemma implies that there is an agent  $i_3 \in L_2(I_T)$  and an object  $q \in O$  such that  $\{i_1, i_3\} \succ_q i_2$ . If  $i_2 \succ_p j \sim_p i_1$ , acyclicity requires  $i_3 \succ_p i_1$ , so that  $j \neq i_3$ . But then  $i_1$  has at most third highest priority for  $p$  among agents in  $I_T$ , contradicting the above. Hence,  $i_2 \succ_p j \succ_p i_1 \succ_p i_3$  and  $j \in I_S$ . If  $i_3 \succ_o i_1$ , the tie  $i_1 \sim_o i_2$  is strongly cyclic so that we must have  $i_3 \sim_o i_1 \sim_o i_2$ . But then if  $j \succ_o i_1$  the tie  $i_1 \sim_o i_3$  is strongly cyclic since  $\{i_1, i_3\} \succ_q i_2 \succ_p j \succ_o i_1$ . Hence,  $i_1 \sim_o i_2 \sim_o i_3 \sim_o j$  and we obtain a strong priority reversal as  $\{i_1, i_3\} \succ_q i_2$  and  $i_2 \succ_p j \succ_p i_1 \succ_p i_3$ . This contradiction completes the proof.  $\square$

### Proof of Lemma 3

Suppose first that  $i_1, i_2 \in I_T$  are two distinct agents such that  $i_1 \sim_o i_2$ , but not necessarily  $i_1 \sim'_o i_2$ , and  $i_2 \succ_p i_1$  for two distinct objects  $o$  and  $p$ . We now show that whenever  $i_1 \succ_q j$  for some agent  $j \in I_S$  and an object  $q$ , then  $j \succ_q i_2$ . Since  $j \in I_S$ , there must exist an agent  $j' \in I_S$  such that  $j \succ_r j'$  for some object  $r$ . But by Lemma 1 we must have  $j \succ_o j'$  or  $j' \succ_o j$ , so that either  $j \succ_o i_1$  or  $j' \succ_o i_1$ . Hence, the tie  $i_1 \sim_o i_2$  is strongly cyclic if  $\{i_1, i_2\} \succ_q j$ . Now suppose that  $i_1 \succ_q j \sim_q i_2$ . The tie  $j \sim_q i_2$  is strongly cyclic unless  $i_1 \succeq_p j$ . But  $i_2 \in I_T$  and  $j \in I_S$  imply that  $j \succ'_q i_2$ . Since  $i_2 \succ_p i_1 \succeq_p j$ , we obtain a contradiction to Lemma 6 part (c).

For the remainder of this proof let  $i_1$  and  $i_2$  be two distinct agents such that  $i_1 \sim'_o i_2$  for some object  $o \in O$ . Note that by our exogenous tie-breaking procedure, we must have  $i_1 \sim_o i_2$  and  $\{i_1, i_2\} \subseteq I_T$ . Furthermore, there must exist objects  $p$  and  $q$  such that  $i_1 \succ_p i_2$  and  $i_2 \succ_q i_1$ .

We first establish the Lemma for the case of  $I_T = \{i_1, i_2\}$ . Since  $I$  is connected and  $|I| \geq 4$ , at least one of the agents in  $I_T$ , say  $i_1$ , must have strictly higher priority than some agent in  $I_S$  for at least one of the objects. By the rules for exogenous tie-breaking,  $i_1 \sim'_o i_2$  then implies that the same must be true for  $i_2$ . Hence, there are agents  $j_1, j_2 \in I_S$  and objects  $p_1, p_2$  such that  $i_1 \succ_{p_1} j_1$  and  $i_2 \succ_{p_2} j_2$ . By the above, we must have  $j_1 \succ_{p_1} i_2$  and  $j_2 \succ_{p_2} i_1$ . Now suppose that contrary to what we want to show  $j_1 \neq j_2$ . By acyclicity, we cannot have  $\{j_1, j_2\} \succ_o i_1$ . If  $j_1 \succ_o i_1 \sim_o i_2 \sim_o j_2$ , the tie  $i_2 \sim_o j_2$  is strongly cyclic, since  $\{i_2, j_2\} \succ_{p_2} i_1 \succ_{p_1} j_1 \succ_o i_2$ . If  $j_2 \succ_o i_1 \sim_o i_2 \sim_o j_1$ , the tie  $i_1 \sim_o j_1$  is strongly cyclic, since  $\{i_1, j_1\} \succ_{p_1} i_2 \succ_{p_2} j_2 \succ_o i_1$ . By Lemma 1 and  $\{j_1, j_2\} \subseteq I_S$ , these are the only remaining possibilities so that  $j_1 \neq j_2$  necessarily leads to a contradiction. Hence, we must have  $j_1 = j_2$  and we can set  $j_{i_1, i_2} := j_1 = j_2$ . The above arguments establish that  $i_1 \succ_q j_{i_1, i_2}$  implies  $j_{i_1, i_2} \succ_q i_2$  and  $i_2 \succ_q j_{i_1, i_2}$  implies  $j_{i_1, i_2} \succ_q i_1$ , so that conditions (1) - (3) of Lemma 3 are satisfied.

Next, suppose that  $|I_T| = 3$  and let  $j_1$  be such that  $\{i_1, i_2, j_1\} = I_T$ . By the connectedness of  $I_T$  and the rules for exogenous tie-breaking, we must have  $\{i_1, i_2\} \subseteq L_1(I_T)$  and that both,  $i_1$  and  $i_2$ , have the lowest priority among agents in  $I_T$  for at least one of the objects. Hence, there exist objects  $p_1, p_2$  such that  $\{i_1, j_1\} \succ_{p_1} i_2$  and  $\{i_2, j_1\} \succ_{p_2} i_1$ . We will now argue that there cannot be an object  $q$  such

that  $i_1 \succ_q j_2$  for some  $j_2 \in I_S$ . Suppose the contrary. By the above, we must have  $i_1 \succ_q j_2 \succ_q i_2$ . Since  $i_2 \in L_1(I_T)$ , there exists an object  $r$  such that  $i_2 \succ_r \{i_1, j_1\}$ . If  $\{j_1, j_2\} \succ_o i_1$ , the tie  $i_1 \sim_o i_2$  is strongly cyclic. If  $j_2 \succ_o i_1 \sim_o i_2 \sim_o j_1$ , the tie  $i_2 \sim_o j_1$  is strongly cyclic since  $\{i_2, j_1\} \succ_{p_2} i_1 \succ_q j_2 \succ_o i_2$ . If  $j_1 \succ_o i_1 \sim_o i_2 \sim_o j_2$ , the tie  $i_1 \sim_o j_2$  is strongly cyclic since  $\{i_1, j_2\} \succ_q i_2 \succ_r j_1 \succ_o i_1$ . Hence, we must have  $i_1 \sim_o i_2 \sim_o j_1 \sim_o j_2$ . Given that  $i_1, j_1 \in I_T$ ,  $j_1 \succ_{p_2} i_1$ , and  $i_1 \sim_o j_1$ , the above implies  $i_1 \succ_q j_2 \succ_q \{i_2, j_1\}$ . Since  $\{i_2, j_1\} \succ_{p_2} i_1$  we obtain a strong priority reversal. A completely analogous argument shows that  $i_2$  can also never have strictly higher priority for one of the objects than any agent in  $I_S$ . Hence,  $I_S \succeq_p \{i_1, i_2\}$  for all  $p \in O$ . Since  $I$  is connected, this implies that  $j_1$  must have strictly higher priority than one of the agents in  $I_S$  for at least one of the objects and that  $j_1 \in L_1(I_T)$  as well. Now let  $j_{i_1, i_2} := j_1$  and  $q_1$  and  $q_2$  be two objects such that  $i_1 \succ_{q_1} j_{i_1, i_2}$  and  $i_2 \succ_{q_2} j_{i_1, i_2}$ . To complete the proof in case of  $|I_T| = 3$ , we will now show that if  $i_1 \succ_q j_{i_1, i_2}$  for some object  $q$ , then  $j_{i_1, i_2} \succ_q i_2$ . Note first that given the assumed connectedness of  $I$  and  $|I| \geq 4$ , there must exist an object  $q'$  such that  $j_{i_1, i_2} \succ_{q'} j_3$  for some  $j_3 \in I_S$ . If  $\{i_1, i_2\} \succ_q j_{i_1, i_2}$ , the tie  $i_1 \sim_o i_2$  would be strongly cyclic unless  $j_{i_1, i_2} \sim_o j_3 \sim_o i_1 \sim_o i_2$ . By the above, we must then have  $j_{i_1, i_2} \succ_{q'} j_3 \succ_{q'} \{i_1, i_2\}$ . Since  $\{i_1, i_2\} \succ_q j_{i_1, i_2}$  and  $j_{i_1, i_2} \succ_{q'} j' \succ_{q'} \{i_1, i_2\}$ , we obtain a strong priority reversal. If  $i_1 \succ_q j_{i_1, i_2} \sim_q i_2$ , we obtain a contradiction to acyclicity since  $\{i_2, j_{i_1, i_2}\} \succ_{p_2} i_1$ . A completely analogous argument shows that if  $i_2 \succ_q j_{i_1, i_2}$  for some  $q$ , then  $j_{i_1, i_2} \succ_q i_1$ . This completes the proof in case  $|I_T| = 3$ .

Hence, we are left to consider the case of  $|I_T| \geq 4$ . Here, we show first that  $\{i_1, i_2\} = L_1(I_T)$  is impossible. Suppose the contrary. Note that our rules for exogenous tie-breaking imply that both  $i_1$  and  $i_2$  have third highest priority among agents in  $I_T$  for at least one of the objects. Hence, if there are two distinct agents  $i_3, i_4 \in L_2(I_T)$ , there must exist two objects  $p_1, p_2$  such that  $i_1 \succ_{p_1} i_3 \succ_{p_1} i_2$  and  $i_2 \succ_{p_2} i_4 \succ_{p_2} i_1$ . If either  $i_3 \succ_o i_1$  or  $i_4 \succ_o i_2$ , we obtain a contradiction to acyclicity. If  $i_1 \sim_o i_2 \sim_o i_3 \sim_o i_4$ , we obtain a strong priority reversal  $(i_1, i_2, i_3, i_4, o, p_1, p_2)$ . So suppose that  $\{i_3\} = L_2(I_T)$  and let  $i_4 \in L_3(I_T)$  be arbitrary. As before,  $i_1 \sim'_o i_2$  implies that there exist two objects  $p_1$  and  $p_2$  such that  $i_1 \succ_{p_1} i_3 \succ_{p_1} i_2$  and  $i_2 \succ_{p_2} i_3 \succ_{p_2} i_1$ . Furthermore, since  $i_4 \in L_3(I_T)$  and  $\{i_1, i_2, i_3, i_4\} \subseteq I_T$ , there must exist an object  $p_3$  such that  $\{i_1, i_2\} \succ_{p_3} i_4 \succ_{p_3} i_3$ . Since  $i_4 \in L_3(I_T)$ , it has to be the case that  $i_2 \succeq_{p_1} i_4$  and  $i_1 \succeq_{p_2} i_4$ . Since  $\{i_1, i_2, i_4\} \succ_{p_3} i_3$ , acyclicity requires  $i_2 \succ_{p_1} i_4$  and  $i_1 \succ_{p_2} i_4$ . But then the tie  $i_1 \sim_o i_2$  is strongly cyclic unless  $i_1 \sim_o i_4$  as well. However, in this case the weak priority reversal  $(i_1, i_2, i_3, o, p_1, p_2)$  would not be consistent. This shows that  $\{i_1, i_2\} = L_1(I_T)$  is impossible when  $|I_T| \geq 4$ . We are hence left to consider the case of  $\{i_1, i_2\} = L_{\bar{K}}(I_T)$  for some  $\bar{K} \geq 2$ . Let  $j \in I \setminus \{i_1, i_2\}$  and  $p$  be such that  $i_1 \succ_p j$ . If  $j \in I_S$ , the above arguments imply that  $j \succ_p i_2$ . If  $j \in I_T$ , the definition of  $I_T$  implies that  $j \in L_{\bar{K}-1}(I_T)$  and  $j \succeq_p i_2$ , since otherwise  $j$  would have at most  $\bar{K} + 2$ nd highest priority for  $p$ . Now suppose that  $j \sim_p i_2$ . Since  $\{i_1, i_2\} = L_{\bar{K}}(I_T)$ , there is an object  $q$  such that  $i_2 \succ_q i_1$ . Since  $j \sim_p i_2$  would otherwise be strongly cyclic, we must have  $i_1 \succeq_q j$ . But then we obtain a contradiction to Lemma 6 (c) since  $i_2 \sim_p j$ ,  $j \succ'_p i_2$ , and  $i_2 \succ_q i_1 \succeq_q j$ . This shows that whenever  $i_1 \succ_p j$  for some  $j \in I \setminus \{i_1, i_2\}$ , we must have  $j \succ_p i_2$ . An analogous argument shows that if  $i_2 \succ_p j$  for some  $j \in I \setminus \{i_1, i_2\}$ , then  $j \succ_p i_1$ . Now by the connectedness of  $I$  and  $I_T$ , there must be two agents  $j_1, j_2 \in I \setminus \{i_1, i_2\}$  and two objects  $p_1, p_2$  such that  $i_1 \succ_{p_1} j_1 \succ_{p_1} i_2$  and  $i_2 \succ_{p_2} j_2 \succ_{p_2} i_1$ . If  $j_1 \neq j_2$ , acyclicity would require that  $i_1 \sim_o i_2 \sim_o j_1 \sim_o j_2$ . But then we obtain a strong priority reversal  $(i_1, i_2, j_1, j_2, o, p_1, p_2)$ . Hence, there must be a unique agent  $j_{i_1, i_2} \in L_{\bar{K}-1}(I_T)$  who can have strictly lower priority for some of the objects than either  $i_1$  or  $i_2$  and this agent can never rank below  $i_1$  and  $i_2$ . This completes the proof.  $\square$

### Proof of Corollary 3

Assume to the contrary that  $\succeq$  is neither a HET environment nor strict.

Note first that under the conditions of the corollary, there cannot be a pair of distinct agents  $i_1, i_2$  such that  $i_1 \sim'_o i_2$  for some  $o$ : Perfect pairwise variability and  $|I| \geq 4$  imply that there exist two (not necessarily distinct) objects  $p_1, p_2$  and two distinct agents  $j_1, j_2$  such that  $i_1 \succ_{p_1} j_1$  and  $i_2 \succ_{p_2} j_2$ . Hence, we obtain a contradiction to Lemma 3 if  $i_1 \sim'_o i_2$ .

Now suppose that there are two distinct agents  $i_1, i_2$  such that  $i_1 \sim_o i_2$  and  $i_1 \succ'_o i_2$ . If there is an agent  $i_3$  such that  $i_3 \succ_o i_1$ , we must obtain a contradiction to either Lemma 6 (c) or acyclicity: by perfect pairwise variability there must exist an object  $p$  such that  $i_2 \succ_p i_3$  and the tie  $i_1 \sim_o i_2$  is strongly cyclic unless  $i_2 \succ_p i_3 \succeq_p i_1$ . Hence, given that  $|I| \geq 4$ , we can assume w.l.o.g. that there are four distinct agents  $i_1, i_2, i_3, i_4$  such that  $i_1 \sim_o i_2 \sim_o i_3 \sim_o i_4$  and  $i_1 \succ'_o i_2 \succ'_o i_3 \succ'_o i_4$ . By perfect pairwise variability there exists an object  $q$  such that  $i_4 \succ_q i_1$ . Since  $\{i_1, i_2, i_3\} \sim_o i_4$  and  $\{i_1, i_2, i_3\} \succ'_o i_4$ , Lemma 6 (c) implies that  $\{i_2, i_3\} \succ_q i_4 \succ_q i_1$ . But since  $\{i_2, i_3\} \sim_o i_1$  and  $i_1 \succ'_o \{i_2, i_3\}$  we still obtain a contradiction to Lemma 6 (c) and this completes the proof.  $\square$

## B.1 Proof of Theorem 3

It is immediate that the outcome of the ADA-ETB is always stable, since an agent is rejected by some object only if some other agent with at least weakly higher priority also applies it. Now suppose to the contrary that for some profile  $R$ ,  $AT^{\succeq}(R)$  admits a SIC  $i_1, \dots, i_L$ . We assume without loss of generality that the SIC is minimal in the sense that no strict subset of  $\{i_1, \dots, i_L\}$  forms a SIC. Set  $AT^{\succeq}_{i_l}(R) := o_l$  for all  $l \in \{1, \dots, L\}$  and let  $\{\mu^t\}_t$  denote the sequence of temporary assignments in the ADA-ETB.

Let  $t_0$  be the round in which  $i_L$  is rejected by  $o_1$  and  $J_0$  be the set of agents temporarily matched to  $o_1$  by the end of round  $t_0$ . We now gather some preliminary observations about  $t_0$  and  $J_0$ : first, since the labeling of agents in a stable improvement cycle is immaterial, we can assume without any loss of generality that there is no round  $t < t_0$  in which some agent  $i_l \in \{i_1, \dots, i_L\}$  is rejected by  $o_{l+1}$ . Second, since  $i_L$  has to be one of the highest priority agents desiring  $o_1$  at  $AT^{\succeq}(R)$ , we must have  $J_0 \sim_{o_1} i_L$ : if there was some  $j \in J_0$  such that  $j \succ_{o_1} i_L$ , we would have  $|J_0| = 1$  since  $\succeq$  is an IB environment. Furthermore, the definition of  $t_0$  implies that  $j \neq i_1$ . Since  $o_1 \neq AT^{\succeq}_j(R)$ ,  $i_L$  cannot be one of the highest priority agents among those who desire  $o_1$  at  $AT^{\succeq}(R)$ . This implies in particular that there cannot be a SIC for strict environments. Third, the definition of  $t_0$  implies that  $i_1 \notin J_0$ . But then we must have  $|J_0| = 1$ , since the rules of the ADA-ETB imply that  $|J_0| > 1$  is possible only if  $|I| = 3$ . But in this case  $|J_0| \geq 2$  implies  $i_1 \in J_0$ , contradicting our definition of  $t_0$ . In the following, let  $i_0$  be the unique agent in  $J_0$ . Finally, note that we must have  $i_0 \notin \{i_1, \dots, i_L\}$  since otherwise agents in some strict subset of  $\{i_1, \dots, i_L\}$  would form a SIC, contradicting the assumed minimality.

We now first derive a contradiction in case  $\succeq$  is not a HET environment and  $|I| = 3$ . In this case we must have  $L = 2$  and  $\mu^{t_0}(i_1) = o_2$ . Since  $AT^{\succeq}_{i_2}(R) = o_2$ , stability requires  $i_2 \succeq_{o_2} i_1$ . But then we must also have  $i_0 \succeq_{o_2} i_1$ , since otherwise  $\{i_1, i_2\} \succ_{o_2} i_0$  given that  $\succeq$  is an IB environment, so that  $i_2$  could not have been rejected by  $o_2$  in favor of  $i_0$ . Hence, we are left to discuss three possible cases: (a)  $\{i_0, i_2\} \succ_{o_2} i_1$ , (b)  $i_0 \succ_{o_2} i_1 \sim_{o_2} i_2$ , or (c)  $i_0 \sim_{o_2} i_1 \sim_{o_2} i_2$ .<sup>25</sup> Note also that  $i_2$  must be temporarily matched to object  $o_2$  when  $i_1$  applies to  $o_1$ . In case (a), acyclicity requires that  $i_1 \sim_{o_1} i_0$ . Since  $i_1$  has

<sup>25</sup>Given that  $AT^{\succeq}_{i_1}(R) = o_1$  and  $|I| = 3$ , it could clearly not have been the case that  $i_1$  was rejected by some object in favor of  $i_0$  prior to round  $t_0$ .

the lowest priority for object  $o_2$ , we cannot have  $i_1 \succ'_{o_1} i_0$ . Since  $i_2 = j_{i_0, i_1}$  and  $\{i_0, i_2\} \succ_{o_2} i_1$ ,  $i_0$  would not have been rejected by  $o_1$  in favor of  $i_1$ . In case (b),  $i_1 \sim_{o_2} i_2$  is strongly cyclic unless  $i_0$  always has at least second highest priority. By Lemma 6 (b), both  $i_1$  and  $i_2$  must have third highest priority for some of the objects. Acyclicity then requires  $i_1 \sim_{o_1} i_0$  and the rules for exogenous tie-breaking yield  $i_0 \succ'_{o_1} i_1$ , so that  $i_0$  could not have been rejected by  $o_1$  in favor of  $i_1$ . In case (c), neither  $i_1$  nor  $i_2$  can always have at least second highest priority for all objects given that  $i_1$  is rejected by  $o_2$  in favor of  $i_2$  and  $i_2$  is rejected by  $o_1$  in favor of  $i_0$ . But then the tie  $i_0 \sim_{o_1} i_2$  would be strongly cyclic unless  $i_1 \sim_{o_1} i_0$  as well. Furthermore, we must have  $i_2 \succeq'_{o_2} i_1$  or  $i_2 > i_1$ , and  $i_0 \succeq'_{o_1} i_2$  or  $i_0 > i_2$ , so that again  $i_0$  could not have been rejected by  $o_1$  in favor of  $i_1$ , contradicting  $AT_{i_1}^{\succeq}(R) = o_1$ .

For the remaining cases we now show first that  $i_0 \sim'_{o_1} i_L$  is impossible.

Case 1:  $\succeq$  is a HET environment

Since  $i_L$  is rejected by  $o_1$  in favor of an equal priority agent  $i_0$ , there must be a cycle in the tie-breaking routine of round  $t_0$  that included  $i_0$  and  $o_1$ .

We now show more generally that whenever a cycle  $C = (j_1, p_1, \dots, j_N, p_N)$  forms in the tie-breaking routine of some round  $t$ , then no agent in this cycle is subsequently rejected by the object she points to. This implies that  $i_0$  could not have been subsequently rejected by  $o_1$  and completes the proof in Case 1.

Suppose the contrary and let  $t' > t$  be the first round of the ADA-ETB where such a rejection occurs. This implies that there must be some  $n \in \{1, \dots, N\}$  such that  $j_{n+1}$  is not the owner of  $p_n$  (where  $j_{N+1} := j_1$ ): a cycle in which each agent points to the object owned by the next agent in the cycle will form in each subsequent tie-breaking stage, so that no agent in the cycle would subsequently be rejected by the object she points to. Furthermore, since in each iteration of the tie-breaking routine, all objects that do not point to their owners point to the same agent,  $p_n$  can be the only object in the cycle that is not owned by  $j_{n+1}$ . If  $j_n$  is rejected by  $p_n$  in round  $t'$  since its owner applied to it, we obtain a contradiction to the definition of  $t'$ : the owner of  $p_n$  must have been part of a cycle  $C'$  that was formed in the tie-breaking routine of round  $t$ , since a cycle containing an object but not its owner can only be formed once the owner is removed from the set of agents to be considered in the tie-breaking routine. But then the owner of  $p_n$  must have been rejected by the object she pointed to in  $C'$  in some round  $t'' < t'$ . This is a contradiction to the definition of  $t'$ . Hence, the first rejection of an agent from  $C$  must occur in the tie-breaking routine of round  $t'$ . Every arrow from  $C$  except potentially the arrow from  $p_n$  to  $j_{n+1}$  will be present at the beginning of the tie-breaking routine. If the arrow from  $p_n$  to  $j_{n+1}$  does not form, there must be a cycle  $C^1 = (j_1^1, p_1^1, \dots, j_{N_1}^1, p_{N_1}^1)$  such that (a)  $C^1$  formed in (the tie-breaking routine of Round)  $t$  before  $C$  and does not form in  $t'$ , and (b) one of the agents in  $C^1$  either owns  $p_n$  or has a higher index than  $j_{n+1}$ . By the definition of  $t'$ , all arrows from agents in  $C^1$  to objects in  $C^1$  are present at the beginning of the tie-breaking routine of round  $t'$ . As above, there must thus be an  $n_1$  such that  $j_{n_1+1}^1$  is not the owner of  $p_{n_1}^1$  and we can repeat all of the above arguments with  $j_{n_1+1}^1$  in place of  $j_{n+1}$ . Given the finiteness of the problem it is clear, that we must eventually obtain a contradiction.

Case 2:  $\succeq$  is not a HET environment and  $|I| \geq 4$

In this case  $i_0 \sim'_{o_1} i_L$  implies that  $\{i_0, i_L\} \subseteq L_{\overline{K}}(I_T)$  and that  $\succeq' |_{I \setminus \{i_0, i_L\}}$  is strict. Furthermore, as argued above it has to be the case that  $i_0 \neq i_{L-1}$ . By the rules of the ADA-ETB, assignments

for  $i_0$  and  $i_L$  are determined only after objects have been temporarily assigned among agents in  $I \setminus \{i_0, i_L\}$ . Since  $t_0$  is the first round of the ADA-ETB in which an agent is rejected by the final assignment of her successor in the SIC, we must thus have  $\mu^{t_0}(i_{L-1}) = o_L$ . By the construction of  $\succeq'$ , we can only have  $AT_{i_L}^{\succeq}(R) = o_L$  if  $i_L \succ_{o_L} i_{L-1}$ . By Lemma 3, we must have  $i_{L-1} \succ_{o_L} i_0$ . But then  $i_0$  would have been rejected by  $o_1$  in favor of  $i_L$  unless there was a round  $s < t_0$  in which  $i_{L-1}$  was rejected by some object  $p \neq o_1$  in favor of  $i_0$ . By Lemma 3, this is possible only if  $i_o \succ_p i_{L-1} \succ_p i_L$ . Furthermore,  $i_0 \in \mu^{t_0}(o_1)$  is possible only if there is a round  $t' \in \{s+1, \dots, t_0-1\}$  in which  $i_0$  is rejected by  $p$ . However, this is possible only if there is a round  $t \in \{s+1, \dots, t'-1\}$  in which some agent  $j \in I \setminus \{i_0, i_{L-1}, i_L\}$  is rejected by  $o_L$  in favor of  $i_{L-1}$ . In particular,  $i_{L-1} \succeq_{o_L} j$  so that, by Lemma 3, we must have  $i_{L-1} \succeq_{o_L} \{i_0, i_L\}$ , which is a contradiction.

Since no ties are broken exogenously in a HET environment, the above already establishes constrained efficiency of the ADA-ETB in this case. So suppose that  $\succeq$  is not a HET environment and  $|I| \geq 4$ . The above implies that  $i_0 \succ'_{o_1} i_L$ . Now assume that for some  $l \in \{1, \dots, L\}$  we have shown  $i_0 \sim_{o_1} i_{l'}$  and  $i_0 \succ'_{o_1} i_{l'}$  for all  $l' \in \{l, \dots, L\}$ . We show that  $i_0 \sim_{o_1} i_{l-1}$  and  $i_0 \succ'_{o_1} i_{l-1}$  as well. Since  $o_l P_{i_{l-1}} o_{l-1}$ , there must exist some object  $p \in O$  such that  $i_l \succ_p i_{l-1}$ . Otherwise  $i_{l-1}$  could never envy  $i_l$ . By the inductive assumptions of  $i_l \sim_{o_1} i_0$  and  $i_0 \succ'_{o_1} i_l$ , and by Lemma 6 (c), we must have  $i_0 \succ_p i_{l-1}$  as well. Acyclicity immediately implies  $i_{l-1} \sim_{o_1} i_0$ . If  $i_0 \succ_p i_l \succ_p i_{l-1}$ , Lemma 6 (c) together with  $i_0 \sim_{o_1} i_{l-1}$  implies  $i_0 \succeq'_{o_1} i_{l-1}$ . Since  $i_0 \succ'_{o_1} i_l$  and  $i_0 \sim_{o_1} i_l$ , we cannot have  $i_0 \sim'_{o_1} i_{l-1}$ , so that  $i_0 \succ'_{o_1} i_{l-1}$ . For the only remaining case of  $i_l \succ_p i_0 \succ_p i_{l-1}$ , Lemma 6 (c) implies  $i_l \succeq'_{o_1} i_{l-1}$ , so that  $i_0 \succ'_{o_1} i_{l-1}$  by transitivity. This inductive argument completes the proof since it implies that  $i_0$  could not have been rejected by  $o_1$  in favor of  $i_1$ , contradicting  $AT_{i_1}^{\succeq}(R) = o_1$ .  $\square$

## B.2 Proof of Theorem 4

For this proof we assume that  $\succeq$  is solvable but *not* a HET environment, since group strategy-proofness is known in that case (see Papai (2001)).

Suppose to the contrary that a coalition  $J \subseteq I$  can manipulate at some profile  $R$  by submitting some joint manipulation  $\tilde{R}_J$ . Let  $\tilde{R} = (\tilde{R}_J, R_{-J})$  and let  $\{\mu^t\}_{t=0}^T$  and  $\{\nu^t\}_{t=0}^{T'}$  be the sequences of temporary assignments in the ADA-ETB under  $R$  and  $\tilde{R}$ , respectively. Since the ADA for strict priority structures is group strategy-proof, there has to be at least one different tie-breaking decision in the ADA-ETBs associated with  $R$  and  $\tilde{R}$ . So there must be two distinct  $i_1, i_2 \in I$  such that for some  $o \in O$  with  $i_1 \sim'_o i_2$ , (a)  $i_1$  is rejected by  $o$  in favor of  $i_2$  in (the tie-breaking routine of) round  $t$  of the ADA-ETB for the preference profile  $R$ , and (b)  $i_2$  is rejected by  $o$  in favor of  $i_1$  in (the tie-breaking routine of) round  $t'$  of the ADA-ETB for the preference profile  $\tilde{R}$ . We cover the two cases  $|I| \geq 4$  and  $|I| \leq 3$  separately.

### Case 1: $|I| \geq 4$

In this case our tie-breaking rules and Lemma 3 imply that there is never a rejection subsequently to a tie-breaking decision between  $i_1$  and  $i_2$ . This follows since for all objects at most one of  $i_1$  and  $i_2$  can rank above one of the other agents. But then we must have  $J \cap \{i_1, i_2\} \neq \emptyset$ , as otherwise agents in  $J$  could also manipulate the ADA with strict tie-breaking if there were two copies of  $o$  (note that no agent  $i \in I \setminus \{i_1, i_2\}$  could have applied to  $o$  under either  $R$  or  $\tilde{R}$  given the construction of  $\succeq'$ ). In the following let  $p$  be the object  $i_2$  obtains under  $\tilde{R}$ .

Suppose first that  $i_2 \in J$  and note that since the ADA-ETB is individually rational, we must have  $pP_{i_2}i_2$ . Since there are no rejections subsequently to tie-breaking between  $i_1$  and  $i_2$ ,  $\tilde{\mu}^t(p) = \emptyset$ . Now consider the joint manipulation  $\hat{R}_J$  obtained from  $\tilde{R}_J$  by letting  $i_2$  exchange the preference ranking of  $o$  and  $p$  and keeping everything else the same. Clearly,  $\hat{R}_J$  must lead to the same outcome as  $\tilde{R}_J$ . But then since  $i_2$  never applies to  $o$  under  $\hat{R}$ , coalition  $J$  could also manipulate the ADA if we always broke the tie  $i_1 \sim_o i_2$  in favor of  $i_2$ , a contradiction. Hence, we must have  $i_2 \notin J$  and  $i_1 \in J$ .

Now suppose that  $|J| \geq 2$ . Since  $i_1$  is rejected by  $o$  in favor of  $i_2$  when the profile of submitted preferences is  $R$ , there cannot be a round  $s$  of the ADA-ETB under  $R$  in which an agent in  $J \setminus \{i_1\}$  was rejected by an object  $p$  in favor of  $i_1$ . But then the coalition  $J \setminus \{i_1\}$  could still profitably manipulate the ADA at  $R$  if we always break all ties in favor of  $i_2$ , i.e. used a strict priority structure  $\succeq''$  such that  $i_2 \succ_q'' i_1$  for all objects  $q$  such that  $i_1 \sim_q i_2$ . To see this consider the preference profile  $\hat{R} = (\hat{R}_{J \setminus \{i_1\}}, R_{(I \setminus J) \cup \{i_1\}})$  and let  $\hat{\mu} = ADA^{\succeq''}(\hat{R})$ . If one of the agents  $j \in J \setminus \{i_1\}$  is negatively affected, he must have been rejected by  $p := AT_j^{\succeq}(\hat{R})$  in favor of  $i_1$ . However, since  $j \in J$  we must have  $pP_jAT_j^{\succeq}(R)$ . Since all agents in  $J \setminus \{i_1\}$  submit preferences according to  $\tilde{R}$ , this is only possible if  $j$  is rejected by  $p$  in favor of  $i_1$  under the profile  $R$ , a contradiction.

The only remaining possibility is  $J = \{i_1\}$ . By our rules for tie-breaking and Lemma 3, there can not be an agent  $j \in J \setminus \{i_1, i_2\}$  such that either (a)  $j$  was rejected by some object in favor of  $i_1$  in some round  $s < t$  of the ADA-ETB for the preference profile  $R$ , or (b)  $\mu^t(j) \in O$  and  $i_1 \succ_{\mu^t(j)} j$ . This, however, implies that  $i_1$  could not have affected the temporary assignment prior to the tie-breaking decision at  $o$ . But then  $i_2$  could not have been rejected by  $o$  in favor of  $i_1$  in round  $t'$  of the ADA-ETB under  $\tilde{R}$ , a contradiction.

**Case 2:**  $|I| = 3$

Let  $j$  be the third agent in  $I$  and set  $p_1 := \mu^t(j)$ ,  $p_2 := \nu^{t'}(j)$ . Note that we must have  $p_1 \neq p_2$ . The rules for exogenous tie-breaking imply that either  $\{i_1, j\} \succ_{p_2} i_2$ , or  $\{i_2, j\} \succ_{p_1} i_1$ . Otherwise the tie  $i_1 \sim_o i_2$  would have been broken in the same way under  $R$  and  $\tilde{R}$ . We now consider the two possible cases in turn.

**Case 2.1:**  $\{i_1, j\} \succ_{p_2} i_2$

It has to be the case that  $p_1P_jp_2$  since only  $i_1$  could have caused a rejection of  $j$  at  $p_2$  and we would hence have  $\mu^t(i_1) = p_2$  otherwise.

We now show that  $j \notin J$ . To see this note that  $j \in J$  implies  $p_2P_j\mu^T(j)$ . This is possible only if  $i_1 \succ_{p_2} j \succ_{p_2} i_2$  and  $i_1$  applied to  $p_2$  in some round  $t_1 > t$  of the ADA-ETB under  $R$ . Furthermore, it has to be the case that  $p_1 = o$ : if  $p_1 \neq o$ ,  $j$  must have been rejected by  $p_1$  in favor of  $i_1$  prior to applying to  $p_2$ . But then  $j$  could not have been rejected by  $p_2$  in the ADA-ETB under  $R$  subsequently given that  $j \succ_{p_2} i_2$ , contradicting  $\mu^T(j)P_jp_2$ . Let  $t_2 > t$  be the round of the ADA-ETB under  $R$  in which  $j$  is rejected by  $o$  in favor of  $i_2$ . By the above, we must have  $\mu^{t_2}(i_1) = p_2$ . But then,  $i_2$  would have been rejected by  $o$  in favor of  $j$  since  $\{i_1, j\} \succ_{p_2} i_2$ . In particular,  $\mu^T(j) = p_1$ , which contradicts  $p_2P_j\mu^T(j)$ . Hence, we must have  $j \notin J$ .

Since  $j \notin J$ , there must be a round  $t_1 < t'$  of the ADA-ETB under  $\tilde{R}$  in which  $j$  is rejected by  $p_1$ . It is easy to see that we must have  $p_1 = o$ , as otherwise  $\nu^{t'}(i_1) = \nu^{t'}(i_2) = o$  is impossible. Since  $j$  is rejected by  $o$  prior to round  $t'$  and  $i_1 \sim_o i_2$ , we must have  $i_1 \sim_o i_2 \sim_o j$  given that  $\succeq$  is an IB environment. Since  $i_1$  was rejected by  $o$  under  $R$  when all three agents applied to  $o$ , it has to be the case that  $i_1 < i_2$  and  $i_1 < j$ , as well as  $\{i_2, j\} \succeq'_o i_1$ . Let  $t_2 < t'$  be the round of the ADA-ETB in which  $j$  is rejected by  $o$  under  $\tilde{R}$ , let  $q_1 := \nu^{t_2}(i_1)$  and  $q_2 := \nu^{t_2}(i_2)$ . If  $q_1 = o$ ,  $\{i_1, i_2\} \succ_{q_2} j$  as otherwise  $i_1$  would have been rejected by  $o$  in favor of  $j$  in round  $t_2$  given the above. But then  $i_2$  could not have applied to  $o$  subsequently, contradicting  $\nu^{t'}(i_2) = o$ . If  $q_2 = o$ ,  $\{i_1, i_2\} \succ_{q_1} j$  as otherwise  $i_1$  would have been rejected by  $o$  in round  $t_2$  in favor of  $j$ . But then  $i_1$  could not have applied to  $o$  subsequently, contradicting  $\nu^{t'}(i_1) = o$ . Since we must have either  $q_1 = o$  or  $q_2 = o$  given that  $j$  is rejected by  $o$  in round  $t_2$  of the ADA-ETB under  $\tilde{R}$ , this shows that  $\{i_1, j\} \succ_{p_2} i_2$  is impossible.

**Case 2.2:**  $\{i_2, j\} \succ_{p_1} i_1$

Assume first that  $p_2 P_j p_1$ . This is possible only if  $p_2 = o$ : agent  $j$  must have been rejected by  $p_2$  prior to round  $t$  of the ADA-ETB under  $R$ . If  $p_2 \neq o$ , this is possible only if either  $i_1$  or  $i_2$  is assigned to  $p_2$  at this point. But then either  $i_1$  or  $i_2$  must be matched to  $p_2$  in round  $t$  of the ADA-ETB under  $R$ , a contradiction. Hence, we must have  $\nu^{t'}(i_1) = \nu^{t'}(i_2) = \nu^{t'}(j) = o$ . Since  $i_2$  is rejected by  $o$  in round  $t'$  of the ADA-ETB under  $\tilde{R}$ , it has to be the case that  $i_2 < i_1$  and  $i_2 < j$ , as well as  $\{i_1, j\} \succeq'_o i_1$ . This implies that it cannot be that all three agents applied to  $o$  simultaneously in the ADA-ETB under  $R$  (as otherwise  $i_2$  would have been rejected by  $o$  given that she is the minimal agent). If  $j$  was rejected by  $o$  in favor of  $i_2$  in some round  $t_1 < t$  of the ADA-ETB under  $R$ , we must have  $\mu^{t_1}(i_1) = p_1$ , as otherwise  $i_1$  could not have subsequently applied to  $o$  (remember that  $p_1 \neq o$ ). Given that  $\{j, i_2\} \succ_{p_1} i_1$ , the tie  $i_2 \sim_o j$  would have been broken in favor of the higher indexed agent  $j$  in round  $t_1$ , contradicting  $i_2 \in \mu^t(o)$ . If  $j$  was rejected by  $o$  in favor of  $i_1$  in some round  $t_2 < t$  of the ADA-ETB under  $R$ ,  $i_2$  must have been subsequently rejected by  $\mu^{t_2}(i_2)$  given that  $i_2 \in \mu^t(o)$ . But this is possible only if  $\mu^{t_2}(i_2) = p_1$ . We obtain a contradiction since  $\{i_2, j\} \succ_{p_1} i_1$  implies that  $i_1$  would have been rejected by  $o$  in favor of  $j$  in round  $t_2$ , a contradiction.

Hence, we are left to consider the case of  $p_1 P_j p_2$ . Since  $i_1$  is rejected by  $o$  in favor of  $i_2$  in the ADA-ETB under  $R$  and  $j \succ_{p_1} i_1$ , we must have  $\mu^T(j) = p_1$ . This implies  $j \notin J$  so that  $\tilde{R}_j = R_j$ . Since  $\nu^{t'}(j) = p_2$ ,  $j$  must have been rejected by  $p_1$  in favor of  $i_2$  in some round  $t_1 < t'$  of the ADA-ETB under  $\tilde{R}$  and it has to be the case that  $i_2 \succ_{p_1} j \succ_{p_1} i_1$ . But then  $i_2$  could not have applied to  $o$  subsequently in the ADA-ETB under  $\tilde{R}$ . This contradicts  $\nu^{t'}(i_2) = o$  and completes the proof.  $\square$

### B.3 Proof of Theorem 5

#### Necessity

In the following, we show that if one of the conditions is violated, the ADA fails to choose a constrained efficient matching for at least one problem. Note that the restriction to the ADA is

without loss of generality since it is the unique constrained efficient mechanism for strict priority structures.

If one of the necessary conditions for the solvability of a weak priority structure is violated, strategy-proofness of the ADA for strict priority structures implies that the priority structure is not solvable by exogenous tie-breaking. We now consider the remaining cases one by one.

Suppose first, there is a weak priority reversal  $(i_1, i_2, j_1, o, p_1, p_2)$ . Consider an arbitrary strict transformation  $\succeq'$  of  $\succeq$ . Given symmetry, we can assume w.l.o.g. that  $i_1 \succ'_o i_2$  (in the other case one just needs to exchange the roles of  $i_1, i_2$  and  $p_1, p_2$ ). In the following we will always assume that all agents in  $I \setminus \{i_1, i_2, j_1\}$  rank  $o$  and  $p_2$  as unacceptable. If  $i_1 \succ'_o i_2 \succ'_o j_1$ , consider a preference profile  $R^1$  such that  $R^1_{i_1} = p_2, o$ ,  $R^1_{i_2} = o$ , and  $R^1_{j_1} = o, p_2$ . In this case we obtain  $ADA^{\succeq'}_{i_1}(R^1) = o$  and  $ADA^{\succeq'}_{j_1}(R^1) = p_2$  so that  $i_1$  and  $j_1$  form a stable improvement cycle. Next, suppose that  $j_1 \succ'_o i_1 \succ'_o i_2$  and consider a preference profile  $R^2$  such that  $R^2_{i_1} = o$ ,  $R^2_{i_2} = o, p_2$ , and  $R^2_{j_1} = p_2, o$ . In this case we obtain  $ADA^{\succeq'}_{i_2}(R^2) = p_2$  and  $ADA^{\succeq'}_{j_1}(R^2) = o$  so that  $i_2$  and  $j_1$  form a stable improvement cycle. Finally, assume that  $i_1 \succ'_o j_1 \succ'_o i_2$  and consider a preference profile  $R^3$  with  $R^3_{i_1} = p_2, o$ ,  $R^3_{i_2} = o, p_2$ , and  $R^3_{j_1} = o$ . In this case we obtain  $ADA^{\succeq'}_{i_1}(R^3) = o$  and  $ADA^{\succeq'}_{i_2}(R^3) = p_2$  so that  $i_1$  and  $i_2$  form a stable improvement cycle.

Next, suppose there is a weakly cyclic tie  $i_1 \sim_o i_2$ , that is not strongly cyclic. Let agent  $j$  and the two distinct objects  $p_1, p_2$  be such that  $i_1 \succ_{p_1} j \succ_o i_1$  and  $i_2 \succ_{p_2} j \succ_o i_2$ . Given symmetry, we can assume w.l.o.g. that  $i_1 \succ'_o i_2$ . Consider a preference profile  $R$  such that  $R_{i_1} = o$ ,  $R_{i_2} = o, p_2$ , and  $R_j = p_2, o$ . Suppose that all agents in  $I \setminus \{i_1, i_2, j\}$  rank objects  $o$  and  $p_2$  as unacceptable. For this profile we obtain  $ADA^{\succeq'}_{i_2}(R) = p_2$  and  $ADA^{\succeq'}_j(R) = o$  so that  $i_2$  and  $j$  form a stable improvement cycle. This completes the proof of necessity.

### Sufficiency

Note first that strong acyclicity and the absence of weak priority reversals imply that  $\succeq$  is solvable by Theorem 2. Furthermore, note also that the connectedness of  $I$  implies that  $\succeq$  cannot be a HET environment in this case, since every agent in  $I$  must be an owner and  $|I| \geq 3$ . Now let  $\succeq'$  be the weak priority structure after the exogenous tie-breaking stage. If  $\succeq'$  is strict, the solvability of  $\succeq$  by exogenous tie-breaking follows from Theorem 3. Now suppose there is a pair of distinct agents  $i_1, i_2$  such that  $i_1 \sim'_o i_2$  for some object  $o$ . By Lemma 3, there is a unique agent  $j_{i_1, i_2} \in I \setminus \{i_1, i_2\}$  such that  $i_1 \succ_{p_1} j_{i_1, i_2} \succ_{p_1} i_2$  and  $i_2 \succ_{p_2} j_{i_1, i_2} \succ_{p_2} i_1$  for two distinct objects  $p_1, p_2$ . If  $j_{i_1, i_2} \sim_o i_1$ , we obtain a weak priority reversal. If  $j_{i_1, i_2} \succ_o i_1$ , we obtain a weakly cyclic tie. Since  $\succeq$  is an IB environment, these are the only possible cases and we thus obtain a contradiction.  $\square$

We now provide a characterization of IB environments with  $|I| = 3$  that are solvable by exogenous tie-breaking.

**Theorem 7.** *Suppose  $\succeq$  is an IB environment with  $|I| = 3$  that is not strict.*

*Then  $\succeq$  is solvable by exogenous tie-breaking if and only if*

- (a)  $|L_1| = 2$ , and
- (b) *there is exactly one agent in  $L_1$  who has third highest priority for one of the objects.*

**Proof:** The sufficiency of these conditions follows immediately from Theorem 3 and the rules for exogenous tie-breaking. We now show how to extend the necessity result of Theorem 5 to this case.

Suppose first that  $L_1 = I = \{i_1, i_2, i_3\}$  and that  $i_1 \sim_o i_2$  for some object  $o$ . Since there exist two objects  $p_1, p_2$  such that  $i_1 = r_1(\succeq_{p_1})$  and  $i_2 = r_1(\succeq_{p_2})$ , the tie  $i_1 \sim_o i_2$  would be weakly cyclic if  $i_3 \succ_o i_1$ . Since the above implies that  $\succeq$  is not solvable by exogenous tie-breaking in this case, we must have  $i_3 \sim_o i_1$  as well. Now let  $p_3$  be some object such that  $i_3 = r_1(\succeq_{p_3})$ . Since for all  $j \in \{1, 2, 3\}$  a tie below the top of  $\succeq_{p_j}$  would be weakly cyclic,  $\succeq_{p_j}$  has to be strict for all  $j \in \{1, 2, 3\}$ . Given symmetry, we can assume w.l.o.g. that  $i_1 \succ_{p_1} i_2 \succ_{p_1} i_3$ . Since, as demonstrated above,  $\succeq$  cannot contain weak priority reversals if it is solvable by exogenous tie-breaking, we must have  $i_3 \succ_{p_3} i_1 \succ_{p_3} i_2$ . But then the absence of weak priority reversals requires that  $i_2 \succ_{p_2} i_3 \succ_{p_2} i_1$ . Now suppose that contrary to what we want to show,  $\succeq$  is solvable by the strict priority structure  $\succeq'$ . Since  $i_1 \succ_{p_1} i_2 \succ_{p_1} i_3$  and  $i_1 \sim_o i_2 \sim_o i_3$ , the tie-breaking lemma implies that we must have  $i_1 \succeq'_o i_3$ . Since  $\succeq'$  is assumed strict, we must have  $i_1 \succ'_o i_3$ . Since  $i_3 \succ_{p_3} i_1 \succ_{p_3} i_2$  and  $i_1 \sim_o i_2 \sim_o i_3$ , the tie-breaking lemma similarly implies that we must have  $i_3 \succ'_o i_2$ . Hence, by transitivity,  $i_1 \succ'_o i_2$ . But then we obtain a contradiction to the tie-breaking lemma: since  $i_2 \succ_{p_2} i_3 \succ_{p_2} i_1$  and  $i_1 \sim_o i_2 \sim_o i_3$ , the tie at  $o$  has to be broken in favor of  $i_2$  at the profile  $R_{i_1} = o, R_{i_2} = o, R_{i_3} = p_2$ , a contradiction.

Now suppose that  $L_1 = \{i_1, i_2\}$  and  $L_2 = \{i_3\}$ , but that there exist  $p_1, p_2$  such that  $i_1 \succ_{p_1} i_3 \succ_{p_1} i_2$  and  $i_2 \succ_{p_2} i_3 \succ_{p_2} i_1$ . As shown above, solvability by exogenous tie-breaking requires the absence of weakly cyclic ties. It is easy to see that if  $\succeq$  is not strict, there has to exist an object  $o$  such that  $i_1 \sim_o i_2 \sim_o i_3$ . But in this case we obtain a weak priority reversal, so that  $\succeq$  cannot be solvable by exogenous tie-breaking.  $\square$