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Von Neumann-Morgenstern Stable Sets in Matching Problems*

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Abstract

The following properties of the core of a one-to-one matching problem are well-known: (i) the core is non-empty; (ii) the core is a lattice; and (iii) the set of unmatched agents is identical for any two matchings belonging to the core. The literature on two-sided matching focuses almost exclusively on the core and studies extensively its properties. Our main result is the following characterization of (von Neumann-Morgenstern) stable sets in one-to-one matching problems. We show that a set of matchings is a stable set of a one-to-one matching problem only if it is a maximal set satisfying the following properties: (a) the core is a subset of the set; (b) the set is a lattice; and (c) the set of unmatched agents is identical for any two matchings belonging to the set. Furthermore, a set is a stable set if it is the unique maximal set satisfying properties (a), (b), and (c). We also show that our main result does not extend from one-to-one matching problems to many-to-one matching problems.

JEL Classification: C78, J41, J44.

Keywords: Matching Problem, Von Neumann-Morgenstern Stable Sets.

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1 Introduction

Von Neumann and Morgenstern (1944) introduced the notion of a stable set of a cooperative game.[1] The idea behind a stable set is the following (Myerson, 1991; Osborne and Rubinstein, 1994): suppose the players consider a certain set of allocations of the cooperative game to be the possible outcomes (or proposals) of the game, without knowing which one will be ultimately chosen. Then any stable set of the game is a set of possible outcomes having the following properties: (i) for any allocation in the stable set there does not exist any coalition which prefers a certain other possible (attainable) outcome to this allocation, i.e. no coalition has a credible objection to any stable outcome; and (ii) for any allocation outside of the stable set there exists a coalition which prefers a certain other possible (attainable) outcome to this allocation, i.e. any unstable outcome is credibly objected by a coalition through a stable outcome. Conditions (i) and (ii) are robustness conditions of stable sets. (i) is referred to as internal stability of a set and (ii) as external stability of a set. The core of a cooperative game is always internally stable but it may violate external stability.

Von Neumann and Morgenstern believed that stable sets should be the main solution concept for cooperative games in economic environments. Unfortunately, there is no general theory for stable sets. The theory has been prevented from being successful because it is very difficult working with it, which Aumann (1987) explains as follows: "Finding stable sets involves a new tour de force of mathematical reasoning for each game or class of games that is considered. Other than a small number of elementary truisms (e.g. that the core is contained in every stable set), there is no theory, no tools, certainly no algorithms."

These facts helped the core to become the dominant multi-valued solution concept of cooperative games. The core of a game is extensively studied and well understood

[1] Stable sets are called "solutions" in their book. We follow the convention of most of the recent literature and refer to "solutions" as stable sets.

by the literature. This led a number of papers to identify classes of games where the core is the unique stable set of the game (e.g., Shapley (1971), Peleg (1986a), Einy, Holzman, Monderer, and Shitovitz (1997) and Biswas, Parthasarathy, and Ravindran (2001)).[2]

This paper is the first study of stable sets in matching markets. In a matching market there are two disjoint sets of agents, usually called men and women or workers and firms, and we face the problem of matching agents from one side of the market with agents from the other side where each individual has the possibility of remaining unmatched. Matching problems arise in a number of important economic environments such as (entry-level) labor markets, college admissions, or school choice. The literature on two-sided matching problems focuses almost exclusively on the core.[3] However, the core may violate external stability, i.e. there may be matchings outside the core which are not blocked (or objected) by a coalition through a core matching. Those matchings are only blocked through some "hypothetical matching" which does not belong to the core. Once such a matching is proposed it is not clear why it will be replaced by an element in the core. We show that a sufficient condition for this is that at the core matching, which is optimal for one side of the market, the agents of that side can gain by reallocating their partners.

Here our purpose is not to investigate when the core is the unique stable set for a matching problem. However, the answer to this question will be a straightforward corollary of our main result. We find that any stable set shares a number of well-known and extensively studied properties of the core of a matching problem. Our main result shows that for one-to-one matching problems any stable set is a maximal set satisfying the following properties: (a) the core is a subset of the set; (b) the set is a lattice; and (c) the set of unmatched agents is identical for any two matchings belonging to the set. The converse also holds (i.e. a set is a stable set for a one-to-one

[2]Note that the core of a cooperative game is always unique.

[3]Two of the few exceptions are Klijn and Masso (2003) and Echenique and Oviedo (2004a) who apply the bargaining set of Zhou (1994) to matching problems.

matching problem) if a set is the unique maximal set satisfying the properties (a), (b), and (c). The literature on matching studied extensively when the core is a lattice and when the set of unmatched agents is identical. However, there is no such result saying that if a set possesses certain properties, then it coincides with the core. From our main result it is immediate that the core is the unique stable set if and only if it is a maximal set satisfying (b) the set is a lattice and (c) the set of unmatched agents is identical for all matchings belonging to the set. Furthermore, our main result facilitates considerably the search for stable sets in one-to-one matching problems: we just need to look at maximal sets satisfying (a), (b), and (c) (and if the maximal set is unique, then it is a stable set). We also show that the main result does not extend to many-to-one matching problems.

Two papers in the literature on stable sets contain some similar features as our paper. One is Einy, Holzman, Monderer, and Shitovitz (1996) who study (non-atomic) glove games with a continuum of agents. They show that the core is the unique stable set of any glove game where the mass of agents holding left hand and right hand gloves is identical. Glove games are a special case of assignment games where there are two disjoint sets of buyers and sellers and each buyer-seller pair obtains a certain surplus from exchanging the good owned by the seller. Note that their result requires a continuum of agents, an equal mass of sellers and traders, and each seller's good has the same value for all buyers. Our main result does not impose any restriction on the one-to-one matching problem under consideration. The other paper is Einy and Shitovitz (2003) who study neoclassical pure exchange economies with a finite set of agents or with a continuum of agents. They show that the set of symmetric and Pareto-optimal allocations is the unique symmetric stable set. Their result holds in the continuum case without any restriction and in the finite case with the restriction that any endowment is owned by an identical number of agents and the agents owning the same endowment have identical preferences. The spirit of their result is similar as ours in the sense of determining properties of stable sets and showing that any set satisfying these properties is a stable set. Note, however, that they focus on symmetric stable sets only and for the finite case the result only holds if any endowment is owned by an identical number of agents who have identical preferences.[4] Symmetry is not meaningful in matching problems because no pair of agents is identical.

The paper is organized as follows. Section 2 introduces one-to-one matching problems. Section 3 defines stable sets and states some helpful insights. Section 4 contains the main result for one-to-one matching problems. It characterizes stable sets in terms of well-known properties of the core. Section 5 shows that this characterization does not extend to many-to-one matching problems. Section 6 concludes.

2 One-To-One Matching Problems

A one-to-one matching problem is a triple (M, W, R) where M is a finite set of men, W is a finite set of women, and R is a preference profile specifying for each man $m \in M$ a strict preference relation R_m over $W \cup \{m\}$ and for each woman $w \in W$ a strict preference relation R_w over $M \cup \{w\}$. Then vR_iv' means that v is weakly preferred to v' under R_i, and vP_iv' means v is strictly preferred to v' under R_i. Strictness of a preference relation R_i means that vR_iv' implies $v = v'$ or vP_iv'. We will keep M and W fixed and thus, a matching problem is completely described by R. Let $\mathcal{R}$ denote the set of all profiles. We will call $N = M \cup W$ the set of agents. Given R_m and $S \subseteq W$, let $R_m|S$ denote the restriction of R_m to S. Furthermore, let $A(R_m)$ denote the set of women who are acceptable for man m under R_m, i.e. $A(R_m) = \{w \in W | wP_mm\}$. Similarly we define $R_w|S$ (where $S \subseteq M$) and $A(R_w)$.

A *matching* is a function $\mu : N \to N$ satisfying the following properties: (i) for all $m \in M$, $\mu(m) \in W \cup \{m\}$; (ii) for all $w \in W$, $\mu(w) \in M \cup \{w\}$; and (iii) for all $i \in N$, $\mu(\mu(i)) = i$. Let $\mathcal{M}$ denote the set of all matchings. We say that an agent i is unmatched at matching μ if $\mu(i) = i$. Let $U(\mu)$ denote the set of agents who are

[4]This assumption is similar to the one of Einy, Holzman, Monderer, and Shitovitz (1996) that an equal mass of agents holds left hand and right hand gloves.

unmatched at μ. Given a profile R, a matching μ is called *individually rational* if for all $i \in N$, $\mu(i) R_i i$; and μ is called *Pareto-optimal* if there is no matching $\mu' \neq \mu$ such that $\mu'(i) R_i \mu(i)$ for all $i \in N$ with strict preference holding for at least one agent. Given a coalition $S \subseteq N$, we say that matching μ' *Pareto dominates for* S matching μ if $\mu'(i) R_i \mu(i)$ for all $i \in S$ with strict preference holding for at least one agent in S. We say that μ is *Pareto-optimal for* S if there is no matching $\mu' \neq \mu$ such that matching μ' Pareto dominates for S matching μ. Furthermore, we say that matching μ is *attainable* for coalition S if $\mu(S) = S$ (where $\mu(S) = \{\mu(i) | i \in S\}$).

Let R be a profile. Given two matchings μ, μ' and a coalition $S \subseteq N$, we say that μ *dominates* μ' *via* S *under* R, denoted by $\mu \succ_S^R \mu'$, if (i) $\mu(S) = S$ and (ii) for all $i \in S$, $\mu(i) P_i \mu'(i)$. We say that S *blocks* μ' if $\mu \succ_S^R \mu'$ for some matching μ. We say that μ *dominates* μ' *under* R, denoted by $\mu \succ^R \mu'$, if there exists $S \subseteq N$ such that $\mu \succ_S^R \mu'$. We omit the superscript when R is unambiguous and write $\succ_S$ and $\succ$.

The core of a matching problem contains all matchings which are not blocked by some coalition. Given a profile R, let $C(R)$ denote the core of R,[5] i.e.

$$C(R) = \{\mu \in \mathcal{M} | \text{ for all } \emptyset \neq S \subseteq N \text{ and all } \mu' \in \mathcal{M} \text{ we have } \mu' \not\succ_S^R \mu\}.$$

The core of a matching problem is always non-empty (Gale and Shapley, 1962) and the set of unmatched agents is identical for all matchings in the core (McVitie and Wilson, 1970). We also consider the core where blocking is only allowed by a certain set of coalitions (instead of all coalitions). Given a set of coalitions $\mathcal{T}$, let $C^{\mathcal{T}}(R)$ denote the $\mathcal{T}$-core of R (Kalai, Postlewaite, and Roberts, 1979), i.e.

$$C^{\mathcal{T}}(R) = \{\mu \in \mathcal{M} | \text{ for all } S \in \mathcal{T} \text{ and all } \mu' \in \mathcal{M} \text{ we have } \mu' \not\succ_S^R \mu\}.$$

It is well-known that the core of a matching problem is a complete lattice (Knuth (1976) attributes this result to John Conway).[6] Therefore, the core contains two

[5] The core of a one-to-one matching problem is often referred to as the set of stable matchings. In avoiding any confusion with stable sets we will not use this terminology.

[6] Many papers study the lattice structure of the core and the set of stable matchings in matching

matchings, called the M-optimal matching and the W-optimal matching (the two extremes of the lattice), such that the M-optimal matching is the matching which is both most preferred by the men and least preferred by the women in the core (similar for the W-optimal matching).

Given a profile R, let μ_M denote the M-optimal matching and μ_W the W-optimal matching in $C(R)$. Given two matchings $\mu, \mu' \in \mathcal{M}$, let $\mu \vee \mu'$ denote the mapping $\mu \vee \mu' : N \to N$ such that (i) for all $m \in M$, $(\mu \vee \mu')(m) = \mu(m)$ if $\mu(m) R_m \mu'(m)$, and otherwise $(\mu \vee \mu')(m) = \mu'(m)$, and (ii) for all $w \in W$, $(\mu \vee \mu')(m) = \mu(w)$ if $\mu'(w) R_w \mu(w)$, and otherwise $(\mu \vee \mu')(w) = \mu'(w)$. Note that $\mu \vee \mu'$ does not need to be a matching. Similarly we define $\mu \wedge \mu'$. Given a profile R and $V \subseteq \mathcal{M}$, we say that V is a lattice (under R) if for all $\mu, \mu' \in V$ we have $\mu \vee \mu' \in V$ and $\mu \wedge \mu' \in V$.

3 Stable Sets

A set of matchings is a stable set for a matching problem if it satisfies the following two robustness conditions: (i) no matching inside the set is dominated by a matching belonging to the set; and (ii) any matching outside the set is dominated by a matching belonging to the set.

Definition 1 *Let $R \in \mathcal{R}$ and $V \subseteq \mathcal{M}$. Then V is called a stable set for R if the following two properties hold:*

(i) (Internal stability) For all $\mu, \mu' \in V$, $\mu \not\succ \mu'$.

(ii) (External stability) For all $\mu' \in \mathcal{M} \backslash V$ there exists $\mu \in V$ such that $\mu \succ \mu'$.

Since the core consists of all undominated matchings, the core is always contained in any stable set. However, the core is not necessarily a stable set. A sufficient condition for the core not to be a stable set is that at the M-optimal matching the men can gain by reallocating their partners (and thus, the M-optimal matching is

problems; see for example, Blair (1988), Alkan (2001), Alkan and Gale (2003), and Echenique and Oviedo (2004b).

not Pareto-optimal for the men). By symmetry, of course, the parallel result holds for the women and the W-optimal matching.

Proposition 1 *Let R be a profile. If there exists an individually rational matching μ which Pareto dominates for M the M-optimal matching μ_M, then the core of R is not a stable set for R.*

Proof. Let μ be an individually rational matching which Pareto dominates for M the matching μ_M. Then we have for all $m \in M$, $\mu(m) R_m \mu_M(m)$, with strict preference holding for at least one man. Obviously, by the individual rationality of μ_M, this implies that any man who is matched to a woman at μ_M must be also matched to a woman at μ, i.e. $\mu(m) \in W$ for all $m \in M$ such that $\mu_M(m) \neq m$. Thus, by $\mu_M \in C(R)$ and the individual rationality of μ, we must have $\mu(M) \cap W = \mu_M(M) \cap W$ and that the set of unmatched agents is identical for both μ and μ_M. Since the set of unmatched agents is the same at any two matchings belonging to the core and $U(\mu) = U(\mu_M)$, we have $U(\mu) = U(\mu')$ for all $\mu' \in C(R)$. Now if there were a $\mu' \in C(R)$ such that $\mu' \succ \mu$, then by $U(\mu) = U(\mu')$, $\mu' \succ_{\{m,w\}} \mu$ for some man-woman pair $\{m, w\}$. Thus, $\mu'(m) P_m \mu(m)$, which is impossible because $\mu(m) R_m \mu_M(m)$ and μ_M is the matching which is most preferred by all men in $C(R)$ (i.e. $\mu_M(m) R_m \mu'(m)$). Hence, there is no matching $\mu' \in C(R)$ such that $\mu' \succ \mu$ and $C(R)$ is not externally stable. □

The following example is a matching problem where the core is not a stable set even though its M-optimal matching is Pareto-optimal for the men and its W-optimal matching is Pareto-optimal for the women. Hence, the example shows that the reverse conclusion of Proposition 1 is not true, i.e. if the M-optimal matching is Pareto-optimal for the men and its W-optimal matching is Pareto-optimal for the women, then the core is a stable set.

Example 1 Let $M = \{m_1, m_2, m_3, m_4, m_5, m_6\}$ and $W = \{w_1, w_2, w_3, w_4, w_5, w_6\}$.

Let $R \in \mathcal{R}$ be such that (for each agent $i \in N$ we specify $R_i|A(R_i)$ only)

R_{m_1}	R_{m_2}	R_{m_3}	R_{m_4}	R_{m_5}	R_{m_6}	R_{w_1}	R_{w_2}	R_{w_3}	R_{w_4}	R_{w_5}	R_{w_6}
w_4	w_5	w_6	w_1	w_2	w_3	m_1	m_3	m_1	m_4	m_5	m_6
w_2	w_6	w_1	w_4	w_5	w_6	m_2	m_1	m_2	m_1	m_2	m_2
w_1	w_1	w_2				m_3	m_2	m_3			m_3
w_3	w_2	w_3				m_4	m_5	m_6			
	w_3										

Then

$$\mu_M = \begin{pmatrix} m_1 & m_2 & m_3 & m_4 & m_5 & m_6 \\ w_4 & w_5 & w_6 & w_1 & w_2 & w_3 \end{pmatrix},$$

and

$$\mu_W = \begin{pmatrix} m_1 & m_2 & m_3 & m_4 & m_5 & m_6 \\ w_1 & w_3 & w_2 & w_4 & w_5 & w_6 \end{pmatrix}.$$

Obviously, μ_M is Pareto-optimal for M and μ_W is Pareto-optimal for W. Let

$$\hat{\mu} = \begin{pmatrix} m_1 & m_2 & m_3 & m_4 & m_5 & m_6 \\ w_2 & w_3 & w_1 & w_4 & w_5 & w_6 \end{pmatrix}.$$

Note that $\hat{\mu}$ is obtained from μ_W when m_1 and m_3 exchange their assigned women w_1 and w_2. Then $\hat{\mu} \notin C(R)$ because (m_2, w_1) blocks $\hat{\mu}$. Note that (m_2, w_1) is the only man-woman pair blocking $\hat{\mu}$. Thus, if there is some $\mu \in C(R)$ such that $\mu \succ \hat{\mu}$, then we must have $\mu \succ_{\{m_2, w_1\}} \hat{\mu}$ and $\mu(m_2) = w_1$. However, this is impossible, i.e. for all $\mu \in C(R)$ we have $\mu(m_2) \neq w_1$.[7] Therefore, for all $\mu \in C(R)$ we have $\mu \not\succ \hat{\mu}$

[7] To see this, suppose $\mu(m_2) = w_1$ for some $\mu \in C(R)$. Then (m_2, w_6) cannot block μ, which implies $\mu(m_6) = w_6$. Since $\mu_M(w_5) = m_2$, $\mu_W(w_5) = m_5$, μ_M is the worst stable matching for the women in $C(R)$, and μ_W is the best stable matching for the women in $C(R)$, we must have $\mu(w_5) \in \{m_2, m_5\}$. Thus, by $\mu(m_2) = w_1$, we obtain $\mu(w_5) = m_5$ and $\mu(m_5) = w_5$. Similarly, $\mu_M(m_4) = w_1$ and $\mu_W(m_4) = w_4$ imply $\mu(m_4) \in \{w_1, w_4\}$. Since $\mu(m_2) = w_1$, we must have $\mu(m_4) = w_4$. Hence, $\mu(\{m_1, m_3\}) = \{w_2, w_3\}$. From $w_2 P_{m_1} w_3$, $w_2 P_{m_3} w_3$, and $\mu \in C(R)$ we obtain $\mu(m_1) = w_3$ and $\mu(m_3) = w_2$. But then (m_1, w_1) blocks μ, which contradicts $\mu \in C(R)$.

and $C(R)$ is not stable for R even though μ_M is Pareto-optimal for M and μ_W is Pareto-optimal for W. $\circ$

Furthermore, stable sets are not necessarily individually rational (the same is true for other cooperative games).

Example 2 Let $M = \{m_1, m_2, m_3\}$ and $W = \{w_1, w_2, w_3\}$. Let $R \in \mathcal{R}$ be such that

R_{m_1}	R_{m_2}	R_{m_3}	R_{w_1}	R_{w_2}	R_{w_3}
w_1	w_2	w_3	m_2	m_3	m_1
w_2	w_3	w_1	m_3	m_1	m_2
m_1	m_2	m_3	w_1	w_2	w_3
w_3	w_1	w_2	m_1	m_2	m_3

Let $\mu = \begin{pmatrix} m_1 & m_2 & m_3 \\ w_1 & w_2 & w_3 \end{pmatrix}$, $\mu' = \begin{pmatrix} m_1 & m_2 & m_3 \\ w_2 & w_3 & w_1 \end{pmatrix}$, and $\mu'' = \begin{pmatrix} m_1 & m_2 & m_3 \\ w_3 & w_1 & w_2 \end{pmatrix}$. Then $C(R) = \{\mu'\}$. It is a direct consequence of our main result (we will state Theorem 2 in the next section) that $V = \{\mu, \mu', \mu''\}$ is the unique stable set for R. $\circ$

Because of the bilateral structure of one-to-one matching problems the essential blocking coalitions are man-woman pairs and individuals. Therefore, with any set of matchings we may associate the man-woman pairs which are matched by some element belonging to this set and the individuals who are unmatched under some element belonging to this set. Given $V \subseteq \mathcal{M}$, let

$$\mathcal{T}(V) \equiv \{\{i, \mu(i)\} \mid i \in N \text{ and } \mu \in V\}.$$

The following is a simple and useful characterization of stable sets (This was already noted by von Neumann and Morgenstern (1944)).

Theorem 1 *Let R be a profile and $V \subseteq \mathcal{M}$. Then V is a stable set for R if and only if $V = C^{\mathcal{T}(V)}(R)$.*

Proof. (Only if) Let V be a stable set for R. By internal stability of V we have $V \subseteq C^{\mathcal{T}(V)}(R)$. Suppose $V \not\supseteq C^{\mathcal{T}(V)}(R)$. Let $\tilde{\mu} \in C^{\mathcal{T}(V)}(R) \backslash V$. But then by definition of $\mathcal{T}(V)$, there is no $\mu \in V$ such that $\mu \succ \tilde{\mu}$, which contradicts external stability of V.

(If) Let $V = C^{\mathcal{T}(V)}(R)$. By definition of $\mathcal{T}(V)$, V is internally stable. Let $\tilde{\mu} \in \mathcal{M} \backslash V$. Then there is $\mu \in V$ such that $\mu \succ \tilde{\mu}$ and V is externally stable. □

4 The Main Result

First, we show the following useful insight: if a matching is not dominated by any matching belonging to the core, then the set of unmatched agents is identical for this matching and any matching belonging to the core. Because any stable set contains the core and is internally stable, Proposition 2 implies that the set of unmatched agents is identical for any two matchings belonging to a stable set.

Proposition 2 *Let R be a profile and $\tilde{\mu} \in \mathcal{M} \backslash C(R)$. If for all $\mu \in C(R)$, $\mu \not\succ \tilde{\mu}$, then the set of unmatched agents is identical for $\tilde{\mu}$ and for all matchings in $C(R)$.*

Proof. Since the set of unmatched agents is identical for any two matchings belonging to $C(R)$, it suffices to show $U(\tilde{\mu}) = U(\mu_W)$.

First, suppose that there is $m \in M$ such that $\tilde{\mu}(m) = m$ and $\mu_W(m) \neq m$. Let $\mu_W(m) = w$. Since μ_W is individually rational, we have $\mu_W(m) P_m m$. Thus, from $\mu_W \not\succ_{\{m,w\}} \tilde{\mu}$ we obtain $\tilde{\mu}(w) P_w \mu_W(w)$. Hence, $\tilde{\mu}(w) \neq w$. Let $\tilde{\mu}(w) = m'$. If $\mu_W(m') = m'$, then by $\mu_W \not\succ_{\{m'\}} \tilde{\mu}$ we must have $\tilde{\mu}(m') P_{m'} m'$. But then (m', w) blocks μ_W, i.e. $\tilde{\mu} \succ_{\{m',w\}} \mu_W$, which is a contradiction to $\mu_W \in C(R)$. Therefore, we must have $\mu_W(m') \neq m'$. Thus, by $m' P_w \mu_W(w)$, $\tilde{\mu}(w) = m'$, and $\mu_W \in C(R)$, we have $\mu_W(m') P_{m'} \tilde{\mu}(m')$. Let $\mu_W(m') = w'$. Then again by $\mu_W \not\succ_{\{m',w'\}} \tilde{\mu}$ we must have $\tilde{\mu}(w') P_{w'} \mu_W(w')$. Continuing this way we find an infinite sequence of men and women which contradicts the finiteness of $M \cup W$. Hence, we have shown that if a man is unmatched at $\tilde{\mu}$, then he is also unmatched at all matchings belonging to $C(R)$. Since the same argumentation is also valid for women, we obtain $U(\tilde{\mu}) \subseteq U(\mu_W)$.

Second, suppose that there is $m \in M$ such that $\tilde{\mu}(m) \neq m$ and $\mu_W(m) = m$. Because $\mu_W \not\succ_{\{m\}} \tilde{\mu}$ we must have $\tilde{\mu}(m) P_m m$. Let $\tilde{\mu}(m) = w$. Then by $\mu_W \in C(R)$ we have $\tilde{\mu} \not\succ_{\{m,w\}} \mu_W$. Thus, by $\tilde{\mu}(m) P_m m$, we have $\mu_W(w) P_w m$. Because $\mu_W \not\succ_{\{w\}} \tilde{\mu}$, w cannot be unmatched at μ_W. Thus, $\mu_W(w) \neq w$. Let $\mu_W(w) = m'$. Again from $\mu_W \not\succ_{\{m',w\}} \tilde{\mu}$ and $\mu_W(w) P_w \tilde{\mu}(w)$ we obtain $\tilde{\mu}(m') P_{m'} \mu_W(m')$. Thus, $\tilde{\mu}(m') \neq m'$. Let $\tilde{\mu}(m') = w'$. Then by $\mu_W \in C(R)$ and $\tilde{\mu}(m') P_{m'} \mu_W(m')$, we have $\mu_W(w') P_{w'} \tilde{\mu}(w')$. Continuing this way we find an infinite sequence of men and women which contradicts the finiteness of $M \cup W$. Hence, we have shown that if a man is unmatched at all matchings belonging to $C(R)$, then he is also unmatched at $\tilde{\mu}$. Since the same argumentation is also valid for women, we obtain $U(\tilde{\mu}) \supseteq U(\mu_W)$. $\square$

Next we show that if V is a stable set for R, then it is also a stable set for the profile where all agents in the opposite set become acceptable (without changing any preferences between them) for any agent who is matched under the core and no agent is acceptable for all agents who are unmatched under the core. Therefore, the individual rationality constraint is irrelevant for the matched agents and when investigating stable sets we may constrain ourselves to one-to-one matching problems which contain the same number of men and women and any agent ranks all members belonging to the opposite set acceptable.

Proposition 3 *Let R be a profile, $\mu \in C(R)$, and $V \subseteq \mathcal{M}$. Let $\bar{R}$ be such that (i) for all $i \in U(\mu)$, $A(\bar{R}_i) = \emptyset$, (ii) for all $m \in M \backslash U(\mu)$, $A(\bar{R}_m) = W$ and $\bar{R}_m | W = R_m | W$, and (iii) for all $w \in W \backslash U(\mu)$, $A(\bar{R}_w) = M$ and $\bar{R}_w | M = R_w | M$. Then V is a stable set for R if and only if V is a stable set for $\bar{R}$.*

Proof. (Only if) Let V be a stable set for R. By Theorem 1, it suffices to show $V = C^{\mathcal{T}(V)}(\bar{R})$.

Since V is a stable set for R, we have $C(R) \subseteq V$. Thus, by internal stability of V under R and Proposition 2, we have for all $\mu' \in V$, $U(\mu') = U(\mu)$. Note that the construction of $\bar{R}$ does not change any preferences of R_i between any partners for any agent $i \in N \backslash U(\mu)$. Hence, by internal stability of V under R and $U(\mu') = U(\mu)$ for all $\mu' \in V$, we have $V \subseteq C^{\mathcal{T}(V)}(\bar{R})$. Suppose $V \not\supseteq C^{\mathcal{T}(V)}(\bar{R})$. Let $\bar{\mu} \in C^{\mathcal{T}(V)}(\bar{R}) \backslash V$. Then for all $\mu' \in V$, $\mu' \not\succ^{\bar{R}} \bar{\mu}$. Thus, by $C(R) \subseteq V$ and Proposition 2, $U(\bar{\mu}) = U(\mu)$. Hence, by construction of $\bar{R}$, we have for all $\mu' \in V$, $\mu' \not\succ^{R} \bar{\mu}$, and V is not externally stable under R, a contradiction.

(If) Let V be a stable set for $\bar{R}$. By Theorem 1, it suffices to show $V = C^{\mathcal{T}(V)}(R)$.

By the stability of V under $\bar{R}$, $C(\bar{R}) \subseteq V$. Let $\bar{\mu} \in C(\bar{R})$. By construction of $\bar{R}$, we have $C(R) \subseteq C(\bar{R})$. Since the set of unmatched agents is identical for all matchings belonging to $C(\bar{R})$ and $\mu \in C(R) \subseteq C(\bar{R})$, we have $U(\bar{\mu}) = U(\mu)$. By internal stability of V under $\bar{R}$ and Proposition 2, we have for all $\mu' \in V$, $U(\mu') = U(\bar{\mu}) = U(\mu)$. Hence, by construction of $\bar{R}$ from R and internal stability of V under $\bar{R}$, $V \subseteq C^{\mathcal{T}(V)}(R)$. Suppose $V \not\supseteq C^{\mathcal{T}(V)}(R)$. Let $\hat{\mu} \in C^{\mathcal{T}(V)}(R) \backslash V$. Since $C(R) \subseteq V$, we then have for all $\mu' \in C(R)$, $\mu' \not\succ^{R} \hat{\mu}$, and by Proposition 2, $U(\hat{\mu}) = U(\mu)$. But then by construction of $\bar{R}$ from R, $\mu' \not\succ^{\bar{R}} \hat{\mu}$ for all $\mu' \in V$, and V is not externally stable under $\bar{R}$, a contradiction. $\square$

Proposition 3 is a strategic equivalence result in the sense that any stable set for a profile R is also a stable set for the profile $\bar{R}$ where all core-unmatched agents rank all partners unacceptable and all core-matched agents rank all possible partners acceptable. This fact also implies that the core of $\bar{R}$ must be contained in any stable set for R.[8]

Our main result is the following characterization of stable sets.

Theorem 2 *Let R be a profile and $V \subseteq \mathcal{M}$. Then V is a stable set for R only if V is a maximal set satisfying the following properties:*

[8] However, the core of $\bar{R}$ is not necessarily a stable set for R.

(a) $C(R) \subseteq V$.

(b) V is a lattice.

(c) The set of unmatched agents is identical for all matchings belonging to V.

Furthermore, V is a stable set for R if V is the unique maximal set satisfying properties (a), (b), and (c).

Proof. (Only if) Let V be a stable set for R. First, we show that V satisfies (a), (b), and (c). By external stability of V, we have $C(R) \subseteq V$ and V satisfies (a). Let $\mathcal{T} = \{\{i, \mu(i)\} \mid i \in N \text{ and } \mu \in V\}$. By Theorem 1, $V = C^{\mathcal{T}}(R)$. For all $i \in N$, let $\mathcal{T}(i) = \{\mu(i) \mid \mu \in V\} \backslash \{i\}$. Let $\bar{R} \in \mathcal{R}$ be such that (i) for all $m \in M$, $\bar{R}_m|\mathcal{T}(m) = R_m|\mathcal{T}(m)$, and for all $w \in \mathcal{T}(m)$ and all $w' \in W \backslash \mathcal{T}(m)$, $w\bar{P}_m m \bar{P}_m w'$, and (ii) for all $w \in W$, $\bar{R}_w|\mathcal{T}(w) = R_w|\mathcal{T}(w)$, and for all $m \in \mathcal{T}(w)$ and all $m' \in M \backslash \mathcal{T}(w)$, $m\bar{P}_w w \bar{P}_w m'$.

We show that from the construction of $\bar{R}$ it follows that $C(\bar{R}) = C^{\mathcal{T}}(R)$. Let $\bar{\mu} \in C(\bar{R})$. If $\bar{\mu} \notin C^{\mathcal{T}}(R)$, then there exists some $S \in \mathcal{T}$ and $\mu \in \mathcal{M}$ such that $\mu \succ_S^R \bar{\mu}$. Since $\mathcal{T} = \mathcal{T}(V)$ and V is stable for R, we may assume $\mu \in V = C^{\mathcal{T}}(R)$. Because $\bar{\mu}$ is individually rational under $\bar{R}$, $\bar{\mu}$ is also individually rational under R. Thus, if $\mu \succ_S^R \bar{\mu}$, then $S = \{m, w\}$ for some man-woman pair. Then, by the construction of $\bar{R}$ and $\mu \in V$, we also have $\mu \succ_S^{\bar{R}} \bar{\mu}$ which contradicts $\mu \in C(\bar{R})$. Hence, we have $C(\bar{R}) \subseteq C^{\mathcal{T}}(R)$. In showing the reverse inclusion relation, let $\mu \in C^{\mathcal{T}}(R)$. If $\mu \notin C(\bar{R})$, then there exists some $\emptyset \neq S \subseteq N$ and $\bar{\mu} \in \mathcal{M}$ such that $\bar{\mu} \succ_S^{\bar{R}} \mu$. Since the matching problem is one-to-one, the essential blocking coalitions are only individuals and man-woman pairs. Thus, we may assume that S is a singleton or a man-woman pair. By the construction of $\bar{R}$, there exists a matching $\hat{\mu} \in V$ such that $\hat{\mu}(S) = \bar{\mu}(S)$. From the construction of $\bar{R}$, then we also have $\hat{\mu} \succ_S^R \mu$. This contradicts the internal stability of V because $\mu, \hat{\mu} \in V$ and $C^{\mathcal{T}}(R) = V$. Hence, we have $C(\bar{R}) \supseteq C^{\mathcal{T}}(R)$.

We know that $C(\bar{R})$ is a lattice. Because the preferences restricted to $C(\bar{R})$ are identical under R and $\bar{R}$ and $V = C(\bar{R})$, we have that V is a lattice under R and V satisfies (b). Furthermore, the set of unmatched agents is identical for all matchings belonging to $C(\bar{R})$. Since $V = C(\bar{R})$, V satisfies (c).

Second, we show that V is a maximal set satisfying (a), (b), and (c). Suppose not. Because V satisfies (a), (b), and (c), then there exists a set $V' \subseteq \mathcal{M}$ satisfying (a), (b), and (c) such that $V \subseteq V'$ and $V' \neq V$. Let $\tilde{\mu} \in V' \backslash V$. By external stability of V there exists $\mu \in V$ such that $\mu \succ \tilde{\mu}$. Because $V \subseteq V'$ and V' satisfies (c), the set of unmatched agents is identical for μ and $\tilde{\mu}$. Thus, by $\mu \succ \tilde{\mu}$, we must have $\mu \succ_{\{m,w\}} \tilde{\mu}$ for some man-woman pair (m,w). Hence, $\mu(m) = w$, $\tilde{\mu}(m) \neq m$, $\tilde{\mu}(w) \neq w$, $wP_m\tilde{\mu}(m)$, and $mP_w\tilde{\mu}(w)$. Now when calculating $\mu \vee \tilde{\mu}$ we obtain $(\mu \vee \tilde{\mu})(m) = w$ and $(\mu \vee \tilde{\mu})(w) = \tilde{\mu}(w)$. By $\tilde{\mu}(w) \neq m$, $\mu \vee \tilde{\mu}$ is not a matching which is a contradiction to $V' \subseteq \mathcal{M}$ and V' being a lattice.

(If) Let V be the unique maximal set satisfying (a), (b), and (c). We prove that V is a stable set for R. First, we show that V is internally stable. Let $\mu, \tilde{\mu} \in V$. By (c), the set of unmatched agents is identical for μ and $\tilde{\mu}$. Thus, if $\mu \succ \tilde{\mu}$, then $\mu \succ_{\{m,w\}} \tilde{\mu}$ for some man-woman pair (m,w), i.e. $\mu(m) = w$, $wP_m\tilde{\mu}(m)$ and $mP_w\tilde{\mu}(w)$. Then similarly as above it follows that $\mu \vee \tilde{\mu}$ is not a matching, a contradiction to V being a lattice.

Second, we show that V is externally stable. Suppose not. Then there is some $\tilde{\mu} \in \mathcal{M} \backslash V$ such that for all $\mu \in V$, $\mu \not\succ \tilde{\mu}$. By $C(R) \subseteq V$, (c) and Proposition 2, the set of unmatched agents is identical for $\tilde{\mu}$ and all matchings belonging to V.

Let $\mathcal{T} = \{\{i, \mu(i)\} \mid i \in N \text{ and } \mu \in C(R) \cup \{\tilde{\mu}\}\}$. For all $i \in N$, let $\mathcal{T}(i) = \{\mu(i) \mid \mu \in C(R) \cup \{\tilde{\mu}\}\} \backslash \{i\}$. Let $\bar{R} \in \mathcal{R}$ be such that (i) for all $m \in M$, $\bar{R}_m|\mathcal{T}(m) = R_m|\mathcal{T}(m)$, and for all $w \in \mathcal{T}(m)$ and all $w' \in W \backslash \mathcal{T}(m)$, $w\bar{P}_m m \bar{P}_m w'$, and (ii) for all $w \in W$, $\bar{R}_w|\mathcal{T}(w) = R_w|\mathcal{T}(w)$, and for all $m \in \mathcal{T}(w)$ and all $m' \in M \backslash \mathcal{T}(w)$, $m\bar{P}_w w \bar{P}_w m'$. By construction, $C(R) \cup \{\tilde{\mu}\} \subseteq C(\bar{R})$. We know that $C(\bar{R})$ is a lattice under $\bar{R}$. Furthermore, for all $\mu \in C(\bar{R})$ and all $i \in N$, $\mu(i) \in \mathcal{T}(i) \cup \{i\}$. Because the preferences restricted to $C(\bar{R})$ are identical under $\bar{R}$ and R, it follows that $C(\bar{R})$ is a lattice for R. By construction, $C(\bar{R}) \supseteq C(R)$. Furthermore, the set of unmatched agents is identical for all matchings belonging to $C(\bar{R})$. Hence, $C(\bar{R})$ is a set of matchings satisfying (a), (b), and (c).

Because $\mathcal{M}$ is finite, there exists a maximal set $V' \supseteq C(\bar{R})$ such that V' satisfies (a), (b), and (c). Then V' is a maximal set satisfying (a), (b), and (c) and $\tilde{\mu} \in V' \backslash V$, which contradicts the fact that V is the unique maximal set satisfying (a), (b), and (c). Hence, V must be externally stable. $\square$

Remark 1 It is straightforward to check that properties (a), (b), and (c) are mutually independent in Theorem 2.[9] Let $M = \{m_1, m_2\}$ and $W = \{w_1, w_2\}$.

- (b) & (c) $\not\Rightarrow$ (a): Let R be a profile and μ^I denote the matching such that $\mu^I(i) = i$ for all $i \in N$. Then $\{\mu^I\}$ is a (maximal) set satisfying (b) and (c). Whenever $C(R) \neq \{\mu^I\}$, the set $\{\mu^I\}$ violates (a).

- (a) & (c) $\not\Rightarrow$ (b): Let R be the profile such that $w_1 P_{m_1} w_2 P_{m_1} m_1$, $w_2 P_{m_2} w_1 P_{m_2} m_2$, $m_1 P_{w_1} m_2 P_{w_1} w_1$, and $m_2 P_{w_2} m_1 P_{w_2} w_2$. Let $\mu = \begin{pmatrix} m_1 & m_2 \\ w_1 & w_2 \end{pmatrix}$ and $\mu' = \begin{pmatrix} m_1 & m_2 \\ w_2 & w_1 \end{pmatrix}$. Then $C(R) = \{\mu\}$ and $\{\mu, \mu'\}$ is a (maximal) set satisfying (a) and (c). Since $\mu \vee \mu'$ is not a matching, the set $\{\mu, \mu'\}$ violates (b).

- (a) & (b) $\not\Rightarrow$ (c): Let R be the profile such that $m_1 P_{m_1} w_1 P_{m_1} w_2$, $m_2 P_{m_2} w_1 P_{m_2} w_2$, $m_1 P_{w_1} w_1 P_{w_1} m_2$, and $w_2 P_{w_2} m_2 P_{w_2} m_1$. Let $\mu'' = \begin{pmatrix} m_1 & m_2 \\ w_1 & m_2 \end{pmatrix}$. Then $C(R) = \{\mu^I\}$ and $\{\mu^I, \mu''\}$ is a (maximal) set satisfying (a) and (b) (where μ^I is defined as above). Obviously, the set $\{\mu^I, \mu\}$ violates (c).

An important consequence of Theorem 2 is that any stable set contains a matching which is both most preferred by the men and least preferred by the women in the stable set. This is due to the fact that by (b), any stable set is a lattice, i.e. the preferences of men and women are opposed for the matchings belonging to a stable set. Furthermore, the stability of a set implies that the matching, which is most

[9]Note that the core of R satisfies (a), (b), and (c) but it may not be a maximal set satisfying these properties.

preferred by the men in the stable set, is not Pareto dominated for the men by any individually rational matching. Therefore, this matching is Pareto-optimal for the men if all agents in the opposite set are acceptable for any agent.

Corollary 1 *Let R be a profile. Then any stable set V for R contains a matching which is both most preferred by the men and least preferred by the women in V, namely $\vee_{\mu \in V} \mu$, and V contains a matching which is both least preferred by the men and most preferred by the women in V, namely $\wedge_{\mu \in V} \mu$.*

Note that if V is the unique maximal set satisfying (a), (b), and (c) of Theorem 2, then V is the unique stable set for R. An immediate corollary of our main result is the answer to the question when the core is the unique stable set for a one-to-one matching problem.

Corollary 2 *Let R be a profile. The core $C(R)$ is the unique stable set for R if and only if $C(R)$ is a maximal set satisfying (b) the set is a lattice and (c) the set of unmatched agents is identical for all matchings belonging to the set.*

Proof. (Only if) If $C(R)$ is a stable set for R, then by Theorem 2, $C(R)$ is a maximal set satisfying (a), (b), and (c). Hence, $C(R)$ is a maximal set satisfying (b) and (c).

(If) If $C(R)$ is a maximal set satisfying (b) and (c) of Theorem 2, then $C(R)$ is the unique maximal set satisfying (a), (b), and (c) of Theorem 2. Hence, by Theorem 2, $C(R)$ is the unique stable set for R. $\square$

The following example shows that for the stability of a set V it is not *sufficient* for V to be a maximal set satisfying properties (a), (b), and (c) in Theorem 2. Furthermore, for the stability of a set V it is not *necessary* for V to be the unique maximal set satisfying properties (a), (b), and (c) in Theorem 2.

Example 3 Let $M = \{m_1, m_2, m_3, m_4\}$ and $W = \{w_1, w_2, w_3, w_4\}$. Let $R \in \mathcal{R}$ be such that

R_{m_1}	R_{m_2}	R_{m_3}	R_{m_4}	R_{w_1}	R_{w_2}	R_{w_3}	R_{w_4}
w_2	w_1	w_1	w_1	m_1	m_2	m_3	m_4
w_3	w_2	w_3	w_4	m_4	m_1	m_1	m_1
w_1	w_3	w_2	w_2	m_2	m_3	m_2	m_2
w_4	w_4	w_4	w_3	m_3	m_4	m_4	m_3
m_1	m_2	m_3	m_4	w_1	w_2	w_3	w_4

Let $\mu = \begin{pmatrix} m_1 & m_2 & m_3 & m_4 \\ w_1 & w_2 & w_3 & w_4 \end{pmatrix}$, $\mu' = \begin{pmatrix} m_1 & m_2 & m_3 & m_4 \\ w_2 & w_1 & w_3 & w_4 \end{pmatrix}$, and $\mu'' = \begin{pmatrix} m_1 & m_2 & m_3 & m_4 \\ w_3 & w_2 & w_1 & w_4 \end{pmatrix}$. Then $C(R) = \{\mu\}$. Let $V \equiv \{\mu, \mu'\}$ and $V' \equiv \{\mu, \mu''\}$. Note that μ is Pareto dominated for the men via both μ' and μ'' and that no other matching Pareto dominates for the men μ. Thus, any matching $\mu''' \in \mathcal{M}\backslash\{\mu, \mu', \mu''\}$ is dominated by μ, i.e. $\mu \succ \mu'''$.[10] Furthermore, $\mu' \succ_{\{m_2,w_1\}} \mu''$ and $\mu'' \not\succ \mu'$. Hence, we have (i) V is a stable set for R because $\mu' \succ \mu''$ and for all $\mu''' \in \mathcal{M}\backslash\{\mu, \mu', \mu''\}$, $\mu \succ \mu'''$ and (ii) V' is a maximal set satisfying properties (a), (b), and (c) in Theorem 2 but V' is not a stable set for R because $\mu \not\succ \mu'$ and $\mu'' \not\succ \mu'$. $\circ$

Remark 2 The if-part of Theorem 2 is one of very few results saying that if a set possesses certain properties, then it is a stable set or the core. Characterizations of the core as a solution for all problems have been obtained via properties relating different problems. For example, "consistency" plays the important role in the characterizations of the core of Sasaki and Toda (1992) for one-to-one matching problems and of Peleg (1986b) for cooperative games.[11] In Theorem 2 all properties apply only to a single problem.

[10]Since $\mu''' \neq \mu'$ and $\mu''' \neq \mu''$, μ''' cannot Pareto dominate μ for the men. Thus, by $\mu''' \neq \mu$, there is some $m_i \in M$ such that $\mu'''(m_i) \neq w_i$ and $w_i P_{m_i} \mu'''(m_i)$. Then μ dominates μ''' via $\{m_i, w_i\}$.

[11]In these contexts, Demange (1986) finds a certain "strong stability" condition of the core which is sufficient for the core to be non-manipulable by agents who evaluate any set of outcomes in terms of its most preferred element.

Remark 3 Some literature studies only stable sets which are individually rational. Then the definition of stable sets needs to be adjusted by requiring external stability for individually rational matchings only. It can be checked that Theorem 2 remains unchanged if we restrict ourselves to individually rational matchings.

5 Many-To-One Matching Problems

It is a typical feature that results for one-to-one matching problems do not extend to many-to-one matching problems.[12] We will show that this also applies to most of our results. Instead of introducing the formal many-to-one matching model we will use the reverse version of the ingenious trick by Gale and Sotomayor (1985) and only consider one-to-one matching problems and associate with it (if possible) a many-to-one matching problem with responsive preferences. For all our examples it suffices to consider the possibility of merging two men, say m_1 and m_2, to one agent. Given a one-to-one matching problem (M, W, R), we say that (M, W, R) corresponds to a many-to-one matching problem where m_1 and m_2 are merged to $\{m_1, m_2\}$ if (i) $R_{m_1}|W = R_{m_2}|W$ and $A(R_{m_1}) = A(R_{m_2})$ (the preferences of m_1 and m_2 are identical) and (ii) for all $w \in W$, $m_1 P_w m_2$ and there is no $v \in M \cup \{w\}$ such that $m_1 P_w v P_w m_2$ (each woman ranks m_1 above m_2 and the positions of m_1 and m_2 in the woman's ranking are adjacent to each other). In the corresponding problem, $\{m_1, m_2\}$ can be matched with up to two women and their preference $R^*_{\{m_1,m_2\}}$ is responsive to R_{m_1} over the sets containing fewer than or equal to two women, i.e. for all distinct $w, w', w'' \in W$,

$$\{w, w'\} P^*_{\{m_1,m_2\}} \{w, w''\} \Leftrightarrow w' P_{m_1} w''.$$

[12]This has been shown already for manipulation issues and that with substitutable preferences the set of unmatched agents may change for matchings in the core (Martinez, Massó, Neme, and Oviedo, 2000).

It is easy to see that Theorem 1 remains true for many-to-one matching problems and that in general the following implications hold: (i) V is internally stable in the one-to-one matching problem $\Rightarrow$ V is internally stable in the corresponding many-to-one matching problem and (ii) V is externally stable in the corresponding many-to-one matching problem $\Rightarrow$ V is externally stable in the one-to-one matching problem. However, the reverse directions of these statements are not true in general. There does not need to be any relationship between the stable sets of the one-to-one matching problem and its corresponding many-to-one matching problem, i.e. (i) V is a stable set in the one-to-one matching problem $\not\Rightarrow$ V is a stable set in the corresponding many-to-one matching problem and (ii) V is a stable set in the corresponding many-to-one matching problem $\not\Rightarrow$ V is a stable set in the one-to-one matching problem. Furthermore, in the corresponding many-to-one matching problem a stable set may not be a lattice and the set of unmatched agents may not be identical for all matchings belonging to a stable set. Thus, Proposition 2 and Theorem 2 do not carry over to many-to-one matching problems. The following example establishes these facts.

Example 4 Let $M = \{m_1, m_2, m_3\}$ and $W = \{w_1, w_2, w_3, w_4\}$. Let $R \in \mathcal{R}$ be such that

R_{m_1}	R_{m_2}	R_{m_3}	R_{w_1}	R_{w_2}	R_{w_3}	R_{w_4}
w_1	w_1	w_2	m_3	m_1	m_1	m_1
w_2	w_2	w_1	m_1	m_2	m_2	m_2
w_3	w_3	w_3	m_2	m_3	m_3	m_3
w_4	w_4	w_4	w_1	w_2	w_3	w_4
m_1	m_2	m_3				

Let $\mu = \begin{pmatrix} m_1 & m_2 & m_3 & w_4 \\ w_2 & w_3 & w_1 & w_4 \end{pmatrix}$ and $\mu' = \begin{pmatrix} m_1 & m_2 & m_3 & w_4 \\ w_1 & w_3 & w_2 & w_4 \end{pmatrix}$. Then $C(R) = \{\mu\}$ and $\{\mu, \mu'\}$ is the unique maximal set satisfying properties (a), (b), and (c) of Theorem 2. Hence, $\{\mu, \mu'\}$ is the unique stable set for the one-to-one matching problem.

Now consider the corresponding many-to-one matching problem where we merge m_1 and m_2 to one agent, denoted by $\{m_1, m_2\}$ (note that this is possible since R_{m_1} and R_{m_2} agree over the set of women and each woman ranks m_1 and m_2 adjacent and in the same order). Let $\tilde{\mu} = \begin{pmatrix} m_1 & m_2 & m_3 & w_3 \\ w_2 & w_4 & w_1 & w_3 \end{pmatrix}$, $\tilde{\mu}' = \begin{pmatrix} m_1 & m_2 & m_3 & w_3 \\ w_1 & w_4 & w_2 & w_3 \end{pmatrix}$, and $\tilde{\mu}'' = \begin{pmatrix} m_1 & m_2 & m_3 & w_1 \\ w_3 & w_4 & w_2 & w_1 \end{pmatrix}$. Then in the corresponding many-to-one matching problem, $\mu \not\succ_{\{\{m_1,m_2\},w_2,w_3\}} \tilde{\mu}$ (since m_1 and m_2 are merged to one agent and w_2 is indifferent between μ and $\tilde{\mu}$) and $\mu' \not\succ_{\{\{m_1,m_2\},w_1,w_3\}} \tilde{\mu}$ (since w_1 prefers m_3 to m_1). Thus, $\mu \not\succ \tilde{\mu}$ and $\mu' \not\succ \tilde{\mu}$, which implies that $\{\mu, \mu'\}$ is not externally stable in the corresponding many-to-one matching problem. Let $V = \{\mu, \mu', \tilde{\mu}, \tilde{\mu}', \tilde{\mu}''\}$. Without loss of generality, let $\{w_3, w_4\} P^*_{\{m_1,m_2\}} w_1 P^*_{\{m_1,m_2\}} w_2$. It is easy to check that V is a stable set for R in the corresponding many-to-one matching problem if $\{w_1, w_4\} R^*_{\{m_1,m_2\}} \{w_2, w_3\}$ (if $\{w_2, w_3\} P^*_{\{m_1,m_2\}} \{w_1, w_4\}$, then $\mu \succ_{\{\{m_1,m_2\},w_2,w_3\}} \tilde{\mu}'$ and $\{\mu, \mu', \tilde{\mu}, \tilde{\mu}''\}$ is a stable set for R).[13] Hence, we have established the following facts:

(i) In the one-to-one matching problem, $\{\mu, \mu'\}$ is a stable set for R and V is not a stable set for R because V is not internally stable.

(ii) In the corresponding many-to-one matching problem, V is a stable set for R and $\{\mu, \mu'\}$ is not a stable set for R because $\{\mu, \mu'\}$ is not externally stable.

(iii) In the corresponding many-to-one matching problem, V is a stable set for R. The set of unmatched agents is not identical for any two matchings belonging

[13]To see this, let $\hat{\mu} \in \mathcal{M}\backslash V$. If $\hat{\mu}(m_3) \in \{m_3, w_3, w_4\}$, then $\mu \succ_{\{m_3,w_1\}} \hat{\mu}$. If $\hat{\mu}(w_2) = w_2$, then $\mu' \succ_{\{m_3,w_2\}} \hat{\mu}$. If two or more women are unmatched under $\hat{\mu}$, then by $\hat{\mu}(w_2) \neq w_2$, two women out of $\{w_1, w_3, w_4\}$ are unmatched. Since $\hat{\mu}(m_3) \in \{w_1, w_2\}$, the merged agent $\{m_1, m_2\}$ is matched to at most one woman and by $\{w_3, w_4\} P^*_{\{m_1,m_2\}} w_1$, it follows that $\hat{\mu}$ is dominated by a matching in V (because $\mu'(\{m_1, m_2\}) = \{w_1, w_3\}$, $\tilde{\mu}'(\{m_1, m_2\}) = \{w_1, w_4\}$, and $\tilde{\mu}''(\{m_1, m_2\}) = \{w_3, w_4\}$). Now it follows that (i) if $\hat{\mu}(m_3) = w_2$, then $\hat{\mu} \in \{\mu', \tilde{\mu}', \tilde{\mu}''\} \subseteq V$; and (ii) if $\hat{\mu}(m_3) = w_1$, then $\hat{\mu} \in \{\mu, \tilde{\mu}\} \subseteq V$ or $\hat{\mu}(\{m_1, m_2\}) = \{w_3, w_4\}$ (which is not possible because we cannot have $\hat{\mu}(w_2) = w_2$). Hence, V is externally stable. It is straightforward that V is internally stable.

to V since $U(\mu) = \{w_4\} \neq \{w_3\} = U(\tilde{\mu})$. Furthermore, all women strictly prefer being matched with any man to being unmatched. Since under $\mu \wedge \tilde{\mu}$ all women choose their most preferred partner from μ and $\tilde{\mu}$ and $U(\mu) \cap U(\tilde{\mu}) = \emptyset$, under $\mu \wedge \tilde{\mu}$ no woman can be unmatched, which is impossible because there are only three man. Hence, $\mu \wedge \tilde{\mu}$ is not a matching and V is not a lattice. Thus, Proposition 2 and Theorem 2 do not carry over to many-to-one matching problems. ∘

It is straightforward to check that the conclusions of Example 4 are independent of which responsive extension we choose for $\{m_1, m_2\}$.

Recall that a matching is dominated by another matching via a coalition only if all members of the coalition strictly prefer the other matching to the initial matching. The literature also refers to $\succ$ as the strong dominance relation among matchings. The weak dominance relation allows some members of the blocking coalition to be indifferent between the initial and the new matching. It is well known that results for many-to-one matching problems change when considering weak dominance instead of strong dominance. For one-to-one matching problems this distinction is irrelevant since agents' preferences are strict and the problem is one-to-one. Therefore, all results remain identical under either dominance relation. The same is true for most cooperative games like games with transferable utility or with non-transferable utility.

Since for one-to-one matching problems it is irrelevant which dominance relation we use, one may wonder whether the conclusions of Example 4 remain true when considering weak dominance. To be more precise, we introduce the weak dominance relation. Let R be a profile. Given two matchings μ, μ' and a coalition $S \subseteq N$, we say that μ *weakly dominates* μ' *via* S *(under* R*)*, denoted by $\mu \succ_S^w \mu'$, if (i) $\mu(S) = S$, (ii) for all $i \in S$, $\mu(i) R_i \mu'(i)$, and (iii) for some $i \in S$, $\mu(i) P_i \mu'(i)$. We say that μ *weakly dominates* μ' *(under* R*)*, denoted by $\mu \succ^w \mu'$, if there exists $S \subseteq N$ such that $\mu \succ_S^w \mu'$. We say that a set V is a *strongly stable set for* R if it satisfies conditions (i) and (ii) of Definition 1 when $\succ$ is replaced by $\succ^w$. We will refer to (i) as internal strong stability and to (ii) as external strong stability. It is easy to see that in Example 4, the set $\{\mu, \mu'\}$ is a strongly stable set for R in the corresponding many-to-one matching problem. Nevertheless, as the following example shows, there does not need to be any relationship between the strongly stable sets of the one-to-one matching problem and its associated many-to-one matching problem.

Example 5 Let $M = \{m_1, m_2, m_3, m_4\}$ and $W = \{w_1, w_2, w_3\}$. Let $R \in \mathcal{R}$ be such that

R_{m_1}	R_{m_2}	R_{m_3}	R_{m_4}	R_{w_1}	R_{w_2}	R_{w_3}
w_1	w_1	w_1	w_1	m_1	m_3	m_3
w_2	w_2	w_2	w_2	m_2	m_4	m_4
w_3	w_3	w_3	w_3	m_3	m_1	m_1
m_1	m_2	m_3	m_4	m_4	m_2	m_2
				w_1	w_2	w_3

Let $\mu = \begin{pmatrix} m_1 & m_2 & m_3 & m_4 \\ w_1 & m_2 & w_2 & w_3 \end{pmatrix}$. Then $C(R) = \{\mu\}$ and $\{\mu\}$ is the unique maximal set satisfying properties (a), (b), and (c) of Theorem 2. Hence, $C(R)$ is the unique strongly stable set for the one-to-one matching problem.

Now consider the corresponding many-to-one matching problem where we merge m_1 and m_2 to one agent $\{m_1, m_2\}$ and m_3 and m_4 to one agent $\{m_3, m_4\}$ (note that this is possible since the men's preferences agree over the set of women and each woman ranks m_1 and m_2 adjacent and in the same order and m_3 and m_4 adjacent and in the same order).

Let $\{w_2, w_3\} P^*_{\{m_1,m_2\}} w_1$ and $w_1 P^*_{\{m_3,m_4\}} \{w_2, w_3\}$. Let $\mu' = \begin{pmatrix} m_1 & m_2 & m_3 & m_4 \\ w_2 & w_3 & w_1 & m_4 \end{pmatrix}$. Then in the corresponding many-to-one matching problem, $\mu \not\succ^w \mu'$, which implies that $\{\mu\}$ is not externally strongly stable in the corresponding many-to-one matching problem. Let $V = \{\mu, \mu'\}$. It is easy to check that V is a strongly stable set for R in the corresponding many-to-one matching problem.[14] Hence, we have established the following facts:

(i) In the one-to-one matching problem, $\{\mu\}$ is a strongly stable set for R and V is not a strongly stable set for R because V is not internally strongly stable.

(ii) In the corresponding many-to-one matching problem, V is a strongly stable set for R and $\{\mu\}$ is not a strongly stable set for R because $\{\mu\}$ is not externally strongly stable.[15]

(iii) In the corresponding many-to-one matching problem, V is a strongly stable set for R. The set of unmatched agents is not identical for any two matchings belonging to V since $U(\mu) = \{m_2\} \neq \{m_4\} = U(\mu')$. Thus, Proposition 2 and Theorem 2 do not carry over to many-to-one matching problems when considering the weak dominance relation $\succ^w$. $\circ$

6 Conclusion

In general both the core and stable sets may not exist for cooperative games (Lucas (1969) and Einy and Shitovitz (1996) for stable sets). Since the core of one-to-one matching problems is always non-empty, one may wonder why we should be interested in stable sets. Such a judgement would be based on properties of solution concepts, i.e. such reasoning is *a posteriori* after having defined a solution concept. However, more importantly any judgement of any solution concept should be *a priori* based

[14]To see this, let $\hat{\mu} \in \mathcal{M}\backslash V$. If $\hat{\mu}(w_1) = w_1$, then $\mu' \succ^w_{\{\{m_3,m_4\},w_1\}} \hat{\mu}$. Let $\hat{\mu}(w_1) \neq w_1$. If $w_1 \in \hat{\mu}(\{m_1, m_2\})$, then either $\hat{\mu} = \mu$ or $\mu \succ^w_{\{\{m_3,m_4\},w_2,w_3\}} \hat{\mu}$. Let $w_1 \in \hat{\mu}(\{m_3, m_4\})$. If $\{w_1\} = \hat{\mu}(\{m_3, m_4\})$, then either $\hat{\mu} = \mu'$ or $\mu' \succ^w_{\{\{m_1,m_2\},w_2,w_3\}} \hat{\mu}$. If $\{w_1\} \subsetneq \hat{\mu}(\{m_3, m_4\})$, then $\mu \succ^w_{\{\{m_1,m_2\},w_1\}} \hat{\mu}$. Hence, V is externally strongly stable. It is straightforward that V is internally strongly stable.

[15]Note that V is not a stable set for R since for $\tilde{\mu} = \begin{pmatrix} m_1 & m_2 & m_3 & m_4 \\ w_1 & w_2 & w_3 & m_4 \end{pmatrix}$ we have both $\mu \not\succ \tilde{\mu}$ and $\mu' \not\succ \tilde{\mu}$.

on the economic meaning of its definition. Any core matching is unblocked, i.e., no coalition has any incentive to deviate from it. However, the core as a set does not possess any additional appealing property other than the stability of any single core matching. This is not true for a stable set since as a whole set it satisfies internal stability and external stability. If we select a single matching from a stable set, then this matching is unlikely to be externally stable as a set (unless all agents unanimously agree which matching is most preferred among all matchings). Stable sets should be truly understood as a multi-valued solution concept. They are appealing for situations where agents agree to choose a set of possible outcomes and the finally chosen outcome is enforced. For example, (in school choice) agents may be prohibited from changing their partners chosen by the possible outcome. Then it may be questionable to rule out matchings which are not blocked by any possible enforceable outcome.

Of course, on a practical level the success of a solution concept depends on its properties and its applicability. For one-to-one matching problems we found that the core and stable sets share a number of well-known properties. Our main result did not impose any restriction on the matching problem under consideration (other than it is one-to-one).

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