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IN ENTRY-LEVEL MEDICAL MARKETS*

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Abstract

We consider entry-level medical markets for physicians in the United Kingdom. These markets experienced failures which led to the adoption of centralized market mechanisms in the 1960's. However, different regions introduced different centralized mechanisms. We advise physicians who do not have detailed information about the rank-order lists submitted by the other participants. We demonstrate that in each of these markets in a low information environment it is not beneficial to reverse the true ranking of any two acceptable hospital positions. We further show that (i) in the Edinburgh 1967 market, ranking unacceptable matches as acceptable is not profitable for any participant and (ii) in any other British entry-level medical market, it is possible that only strategies which rank unacceptable positions as acceptable are optimal for a physician.

Keywords : matching market, incomplete information

1 Introduction

We consider entry-level medical markets for physicians in the United Kingdom. Until the 1960's graduating students were themselves responsible for finding their first hospital positions and hospitals were responsible for filling their vacancies. As a result of the competition among students for desirable positions, hospitals contracted earlier and earlier promising students.¹ By the 1940's hospitals were appointing promising students two years before they were finishing their education. This often caused subsequent regret because a promising student was later offered a more desirable post or the hospital could have later hired a better student. Because of these failures these markets were reorganized by a centralized clearinghouse. Each year first the finishing students submit a ranking of the available hospital positions and the hospitals submit a ranking of the finishing students. Salaries are not negotiated but included in the job description of an entry-level physician. Therefore, each participant's preference relation is a ranking of the other side of the market. Then the centralized clearinghouse matches students and hospitals on the basis of the submitted lists. Since any rankings could be submitted, the clearinghouse needs to determine the matching through a mechanism.

Different regions in the U.K. introduced different centralized mechanisms in the late 1960's and the 1970's. The markets in Newcastle, Birmingham, and Edinburgh adopted priority mechanisms in 1967, and the market in Cambridge and the London Hospital Medical College adopted linear programming mechanisms in the 1970's. The market in Edinburgh changed the mechanism in 1969 and based from then on the matching on a stable mechanism called deferred acceptance (DA) algorithm.² In 1971 the market in Cardiff also adopted the DA-algorithm. We will later describe all those mechanisms in detail. In a centralized market a submitted ranking does not need to reflect the true preference of a participant. In fact students seek for advice of which list is optimal for them to submit to the clearinghouse.³ If a student is fully informed about the rankings the others submitted, then in each of these markets it might be suboptimal for her to submit the true list. Truthful revelation of preferences is not a dominant strategy if a student is completely informed. However, in an entry-level medical market typically a student is not fully

¹See Roth (1991).

²Roth (1984) showed that the American markets are based on the DA-algorithm.

³Roth and Rothblum (1999, p. 22).

informed about the rank-order lists submitted by the other participants. We will show that the possibilities of strategic manipulation are considerably limited in a low information environment. We will say that a student's information is symmetric for two hospitals if she does not deduce any difference between them from her information about the rankings submitted by the other participants, or equivalently, her belief treats the two hospitals equally. Our first result demonstrates that if a student's information is symmetric for two hospitals, then it is not profitable for her to misrepresent her true ranking of those two hospitals. When a student has very little information, then her belief does not allow her to distinguish between any two hospitals. Our second result shows that if a student's information is completely symmetric, then in each of these markets the possibilities of manipulation are very restricted. Then a student might only gain by truncating her true preference. Such a list leaves the true ranking of the positions unchanged but might drop some acceptable positions.

We also search for advice of students who do not have completely symmetric information. We will show that the Edinburgh 1967 mechanism is distinguished among the mechanisms introduced in the U.K. medical markets. In the Edinburgh 1967 market it is never profitable for a participant to rank unacceptable matches as acceptable in the submitted list whereas in any other British medical market it is possible that only strategies that rank an unacceptable position as acceptable are optimal for a student.

Our results tell us that we need to be careful when interpreting the conclusions by Roth and Rothblum (1999). Their results for markets using the DA-algorithm follow from our general analysis of *all* British entry-level medical markets.⁴ In a low information environment our advice for strategic behavior is identical in all these markets. Therefore, we cannot conclude from this result that markets using the DA-algorithm perform better than markets using a priority mechanism. It is even the case that in theory the Edinburgh priority mechanism performs better than the DA-algorithm because ranking unacceptable positions as acceptable in the submitted list is not profitable in the Edinburgh 1967 market but in the DA-algorithm it might be optimal to include unacceptable positions. However, contrary to our conclusions, the market in Edinburgh abandoned the priority mechanism to adopt the DA-

⁴Referring to the DA-algorithm, they write as a conclusion of their results in the Abstract of their paper "This gives some insight into the successful operation of these market mechanisms."

algorithm. Roth (1991) argued that one of the reasons might be that the DA-algorithm always produces a stable matching for the submitted rankings whereas a priority mechanism does not. However, linear programming mechanisms also might produce unstable matchings but they are still in use nowadays.⁵ Therefore, why in practice the DA-algorithm performs better than priority mechanisms still remains an open puzzle.

We proceed as follows. In Section 2 we introduce the centralized matching market. In Section 3 we present a unifying analysis for advising students with symmetric information in British entry-level medical markets. In Section 4 we advise students with asymmetric information. Section 5 concludes and the Appendix contains some definitions and all proofs.

2 The Matching Market

An entry-level medical market consists of a set of new physicians (or students or interns) who each seek one position at a hospital and a set of hospitals which would like to employ some number of new physicians. To facilitate the exposition we will assume that each hospital desires at most one intern. Our results remain unchanged if we would allow hospitals to hire several applicants. Each hospital's position specifies the salary for an intern. Since salaries are not negotiated between an intern and a hospital, an intern's preference is simply a ranking of the available positions and remaining unemployed. Given a fixed salary, similarly each hospital's preference is a ranking over the interns and having its position vacant. If the market is organized by a centralized clearinghouse, then each participant is asked to submit a rank-order list of the other side of the market (each intern submits a list of hospitals and each hospital a list of interns) and the clearinghouse determines a matching between the interns and the hospitals on the basis of the submitted rankings. Here the clearinghouse needs to be able to specify a matching for any possible submitted lists. This means that a centralized medical market is based on a mechanism. Below we formalize the matching market.

Let W denote the set of workers (or interns), F denote the set of firms (or hospitals), and $V \equiv F \cup W$ denote the set of participants. Each worker $w \in W$ is equipped with a strict preference relation P_w over $F \cup \{w\}$ and

⁵Indeed, a recent laboratory experiment by Ünver (2001) shows that in small labor markets stability need not be required for the success of a matching mechanism in the long run.

each firm $f \in F$ with a strict preference relation P_f over $W \cup \{f\}$. Let \mathcal{P}_w denote the set of all strict preference relations of worker w over $F \cup \{w\}$ and \mathcal{P}_f denote the set of all strict preference relations of firm f over $W \cup \{f\}$. Given $w \in W$, $P_w \in \mathcal{P}_w$, and $v, v' \in F \cup \{w\}$, we write $v >_{P_w} v'$ if v is strictly preferred to v' under P_w and $v \geq_{P_w} v'$ if v is weakly preferred to v' under P_w . Let $A(P_w)$ be the set of firms that are *acceptable* to w under P_w , i.e. $A(P_w) \equiv \{f \in F \mid f >_{P_w} w\}$. Given $V' \subseteq F \cup \{w\}$, let $P_w|V'$ denote the restriction of P_w to the set V' . Given $f \in F$ and $P_f \in \mathcal{P}_f$, we define $>_{P_f}$, \geq_{P_f} , and $A(P_f)$ analogously. Let $\mathcal{P} \equiv \times_{v \in V} \mathcal{P}_v$. Elements of \mathcal{P} are called (*preference*) *profiles*. Given $P \in \mathcal{P}$, a *matching market* is a triplet (F, W, P) . Because F and W remain fixed, a matching market is simply a profile $P \in \mathcal{P}$.

A *matching* is a function $\mu : V \rightarrow V$ satisfying the following: (i) for all $w \in W$, $\mu(w) \in F \cup \{w\}$; (ii) for all $f \in F$, $\mu(f) \in W \cup \{f\}$; and (iii) for all $v \in V$, $\mu(\mu(v)) = v$.⁶ Given $v \in V$, we say that v is *unmatched under μ* if $\mu(v) = v$. Let \mathcal{M} denote the set of all matchings. A matching is stable under a given profile if (i) each agent weakly prefers her assignment to being unmatched and (ii) no pair of a worker and a firm blocks the matching, i.e. they mutually prefer each other to their assigned partners. Formally, given $P \in \mathcal{P}$ and $\mu \in \mathcal{M}$, μ is *stable under P* if (i) for all $v \in V$, $\mu(v) \in A(P_v) \cup \{v\}$, and (ii) there exists no pair $(w, f) \in W \times F$ such that $f >_{P_w} \mu(w)$ and $w >_{P_f} \mu(f)$. A *mechanism* is a function $M : \mathcal{P} \rightarrow \mathcal{M}$. A mechanism M is *stable* if for all $P \in \mathcal{P}$, $M[P]$ is stable under P .

Given $v \in W$ and $P_v \in \mathcal{P}_v$, we call $P_v|A(P_v)$ a *rank-order list* of v . For a physician a rank-order list is a ranking of all positions which are acceptable for her and for a hospital a rank-order list is a ranking of all physicians which are acceptable for it. The U.K. markets ask participants to submit only rank-order lists, i.e. they do not need to specify the ranking of the unacceptable matches. This means that the mechanisms they use are independent of the ranking of the unacceptable matches.

⁶All our results remain unchanged if some firms have more than one position available.

3 Advising Physicians with Symmetric Information

3.1 Symmetric Information

If a physician is fully informed about the rankings which the other agents submitted to the clearinghouse, then in each of the U.K. markets she might gain by lying instead of submitting her true ranking. Typically in the matching markets considered here, a physician is unaware of the rankings submitted by the others. Therefore, it is important to consider physicians having incomplete information. A physician's uncertainty is expressed by a belief (or random preference profile) associating a probability with each possible submitted lists. This in turn induces a probability distribution over all matchings for any ranking the physician reports. The probability of a matching is simply given by the sum of the probabilities of the others' possible submitted lists for which the mechanism yields that matching. We compare these probability distributions through first-order stochastic dominance of a participant's true preference relation. That is, a physician prefers submitting list 1 to submitting list 2 if and only if the distribution induced from list 1 stochastically dominates in terms of her true ranking the distribution induced from list 2. The physician then prefers list 1 to list 2 for any utility function representing her true preference. Below we incorporate physicians with incomplete information.

Given $w \in W$, let $\mathcal{P}_{-w} \equiv \times_{v \in V \setminus \{w\}} \mathcal{P}_v$. A *random preference profile* is a probability distribution \tilde{P}_{-w} over \mathcal{P}_{-w} . We interpret \tilde{P}_{-w} as w 's belief (or w 's information) about the *stated* preferences of the other players, i.e. their rankings submitted to the clearinghouse.⁷ A *random matching* \tilde{m} is a probability distribution over the set of matchings \mathcal{M} . Let $\tilde{m}(w)$ denote the distribution which \tilde{m} induces over w 's set of assignments $F \cup \{w\}$. Given a mechanism M and $P_w \in \mathcal{P}_w$, each randomized preference profile \tilde{P}_{-w} induces a random matching $M[P_w, \tilde{P}_{-w}]$ in the following way: for all $\mu \in \mathcal{M}$, $\Pr\{M[P_w, \tilde{P}_{-w}] = \mu\} = \Pr\{\tilde{P}_{-w} = P_{-w} \ \& \ M[P_w, P_{-w}] = \mu\}$. Note that

⁷We could also interpret \tilde{P}_{-w} as w 's belief about the true preferences of the other players. Then w needs to take into account for any profile the strategy any participant is playing. Here we could also allow for mixed strategies. Now obviously \tilde{P}_{-w} and the (mixed) strategies of the other players induce a belief over the submitted rankings. Our results then hold for any belief about others' true preferences and any strategies which induce a symmetric information over the submitted rankings.

$M[P_w, \tilde{P}_{-w}](w)$ is the distribution which $M[P_w, \tilde{P}_{-w}]$ induces over w 's set of assignments. Given $w \in W$, $P_w, P'_w, P''_w \in \mathcal{P}_w$, and a random preference profile \tilde{P}_{-w} , we say that strategy P'_w *stochastically P_w -dominates* the strategy P''_w , denoted by $M[P'_w, \tilde{P}_{-w}](w) \succ_{P_w} M[P''_w, \tilde{P}_{-w}](w)$, if for all $v \in F \cup \{w\}$, $\Pr\{M[P'_w, \tilde{P}_{-w}](w) \geq_{P_w} v\} \geq \Pr\{M[P''_w, \tilde{P}_{-w}](w) \geq_{P_w} v\}$.

In a medical market a student is often not able to distinguish between two hospitals. In other words she regards them to be symmetric. Obviously this symmetry should be reflected in her belief. Following Roth and Rothblum (1999) we will say that her information is symmetric for two hospitals if she assigns the same probability to any submitted profile and its symmetric profile where the roles of those two hospitals are interchanged.

Given $w \in W$, $P_w \in \mathcal{P}_w$, and $f, f' \in F$, let $P_w^{f \leftrightarrow f'}$ denote the list which exchanges the positions of f and f' in P_w and leaves the other positions in P_w unchanged. For example, if $P_w : f_1 f_4 f_2 f_3$, then $P_w^{f_3 \leftrightarrow f_4} : f_1 f_3 f_2 f_4$. Given $P_{-w} \in \mathcal{P}_{-w}$, under $P_{-w}^{f \leftrightarrow f'}$ each worker in $W \setminus \{w\}$ exchanges the positions of f and f' in her preference list, firms f and f' exchange their preferences ($P_{f'}$ becomes the preference of f in $P_{-w}^{f \leftrightarrow f'}$ and P_f the preference of f' in $P_{-w}^{f \leftrightarrow f'}$), and the preferences of the other firms remain unchanged.⁸

Given $w \in W$ and $f, f' \in F$, worker w 's information \tilde{P}_{-w} is $\{f, f'\}$ -*symmetric* if the distributions of \tilde{P}_{-w} and $\tilde{P}_{-w}^{f \leftrightarrow f'}$ coincide, i.e. for every profile P_{-w} , $\Pr\{\tilde{P}_{-w} = P_{-w}\} = \Pr\{\tilde{P}_{-w}^{f \leftrightarrow f'} = P_{-w}\}$, or equivalently P_{-w} and $P_{-w}^{f \leftrightarrow f'}$ are equally probable under \tilde{P}_{-w} . Worker w 's information \tilde{P}_{-w} is F -*symmetric* if \tilde{P}_{-w} is symmetric for any two firms belonging to F , i.e. \tilde{P}_{-w} is $\{f, f'\}$ -symmetric for all $f, f' \in F$.

3.2 A Unifying Analysis

In this subsection we provide a unifying analysis in advising students with symmetric information in the U.K. markets. First we will identify several basic properties of mechanisms. Then we will give two general theorems which apply to any market using a mechanism satisfying these properties. Later we will show that any of the U.K. mechanisms satisfies these properties.

One of the common features of the U.K. markets is that they treat participants on each side of the market symmetrically. This is very natural because

⁸Formally, let $P_{-w}^{f \leftrightarrow f'}$ denote the profile $P'_{-w} \in \mathcal{P}_{-w}$ such that for all $w' \in W \setminus \{w\}$, $P'_{w'} = P_{w'}^{f \leftrightarrow f'}$, for all $f'' \in F \setminus \{f, f'\}$, $P'_{f''} = P_{f''}$, $P'_f|W = P_{f'}|W$ and $A(P'_f) = A(P_{f'})$, and $P'_{f'}|W = P_f|W$ and $A(P'_{f'}) = A(P_f)$.

we do not want to discriminate between two hospitals since they belong to the same side of the market and have a priori the same rights of hiring any student. As we will show, another common feature of the U.K. markets is the following: suppose that the participants submitted rank-order lists and an intern is matched to the hospital at Oxford University. Then it is natural to require that when she improves in her submitted list the ranking of the hospital at Oxford University by exchanging its position with the position of a hospital which she formally declared to be preferred to the hospital at Oxford University, then she should remain matched to the hospital at Oxford University. This requirement also excludes the possibility for her to gain by such a simple manipulation. We will call this property “positive association”.

We only require anonymity of the mechanism for the firms. Given a matching $\mu \in \mathcal{M}$ and $f, f' \in F$, let $\mu^{f \leftrightarrow f'}$ denote the matching such that for all $w \in W$, (i) if $\mu(w) \notin \{f, f'\}$, then $\mu^{f \leftrightarrow f'}(w) = \mu(w)$, (ii) if $\mu(w) = f$, then $\mu^{f \leftrightarrow f'}(w) = f'$, and (iii) if $\mu(w) = f'$, then $\mu^{f \leftrightarrow f'}(w) = f$.

Anonymity: For all $P \in \mathcal{P}$, all $\mu \in \mathcal{M}$, and all $f, f' \in F$, if $M[P] = \mu$, then $M[P^{f \leftrightarrow f'}] = \mu^{f \leftrightarrow f'}$.

The next requirement says that given a profile, if a worker improves the ranking of the firm she is matched to by switching it with a firm which is ranked above the firm she is matched to, then her assignment remains unchanged under the new profile.

Positive Association: For all $P \in \mathcal{P}$, all $w \in W$, and all $f, f' \in F$, if $M[P](w) = f$ and $f' >_{P_w} f$, then $M[P_w^{f \leftrightarrow f'}, P_{-w}](w) = f$.

These two common properties of the U.K. markets allow us to give a first advice to participants with symmetric information: if an intern’s information is symmetric for two positions, then it is never beneficial for her to reverse the true ranking of those two positions in her submitted rank-order list. Of course, it is possible that she could gain by another more complicated misrepresentation.

Theorem 1 *In a matching market which uses an anonymous mechanism satisfying positive association, any strategy which reverses the true ranking of f and f' is stochastically dominated by a strategy which preserves the true ranking of f and f' for a worker with $\{f, f'\}$ -symmetric information.*

The advice in Theorem 1 is restricted to two positions. In a low information environment an intern is not able to deduce which rank-order lists were submitted by which participant. Then an intern’s information is symmetric for any two hospitals. To be able to specify an advice we need to identify two other common features of the U.K. markets. In these markets we do not want an intern to be assigned a position which is less preferred for her than being unemployed. If this would be the case, then she would immediately withdraw from the position. We will call this weak “stability” condition individual rationality. Another property which is common to the centralized mechanisms in the U.K. markets is that the matching is independent of the position where being unemployed is ranked: suppose that an intern submitted a rank-order list and is matched to the hospital at Cambridge University. Then if she would have dropped some less preferred hospitals from her submitted rank-order list but kept the ranking of the hospitals unchanged, then she remains matched to the hospital at Cambridge University. We will call this condition “independence of truncations”.

Individual Rationality: For all $P \in \mathcal{P}$ and all $v \in V$, $M[P](v) \geq_{P_v} v$.

A truncation strategy is a preference relation which ranks the firms in the same way as the true preference relation and each firm which is acceptable under the truncation strategy is also acceptable under the true preference relation. Formally, given $w \in W$ and $P_w \in \mathcal{P}_w$, a strategy $P'_w \in \mathcal{P}_w$ is a *truncation strategy of P_w* if (i) $P'_w|F = P_w|F$ and (ii) $A(P'_w) \subseteq A(P_w)$.⁹

If a worker truncates her preference in a way such that the firm she is matched to remains acceptable under the truncated preference, then independence of truncations requires that her assignment is the same under both profiles.

Independence of Truncations: For all $P \in \mathcal{P}$, all $w \in W$, and all $P'_w \in \mathcal{P}_w$, if P'_w is a truncation of P_w and $M[P](w)$ is acceptable under P'_w , then $M[P'_w, P_{-w}](w) = M[P](w)$.

⁹Roth and Rothblum (1999) allow a truncation strategy to rank its unacceptable firms differently than the true ranking. This difference in our definition of a truncation strategy and Roth and Rothblum’s definition is irrelevant because in the U.K. markets the participants do not need to specify the ranking over the unacceptable matches. They are asked to submit rank-order lists only.

Our main theorem advises students with completely symmetric information. The possibilities of manipulation of such a student are considerably reduced. We show that any strategy which changes the true ranking of any two positions and/or adds unacceptable positions is suboptimal for her. The optimal strategic behavior for a student with F -symmetric information is to submit a list which leaves the true ranking of the hospitals unchanged and which does not rank any unacceptable hospital as acceptable.

Theorem 2 *In a matching market which uses an anonymous mechanism satisfying positive association, individual rationality, and independence of truncations, any non-truncation strategy is stochastically dominated by a truncation of the true preferences for a worker with F -symmetric information.*

3.3 The Mechanisms in the U.K. Markets

3.3.1 Priority Mechanisms

Priority mechanisms were adopted by the entry-level medical markets in Birmingham in 1966, in Edinburgh in 1967, and in Newcastle in 1967. Such a mechanism orders all possible student-hospital matches according to the rank of a hospital in a student's preference relation and the rank of a student in a hospital's preference relation. First priority is always given to a (1,1) match, then to (1,2) or (2,1) and so on. If two matches have one rank in common, then the pair for which the other rank is smaller has higher priority. For example, a (3,5) match always has higher priority than a (3,6) match and a (4,5) match.

A priority mechanism matches for each profile sequentially all student-hospital pairs which have the highest priority among all remaining pairs. First the priority mechanism checks whether there are (1,1) matches. If there are any (1,1) matches (i.e. a student-hospital pair who mutually rank each other first), then they are realized. Second the priority mechanism checks whether there are matches who have the second highest priority ((1,2) or (2,1)) and matches any such pair. And so on. All these matches are made only if the student-hospital pair mutually prefers each other to being unmatched (i.e. subject to individual rationality).

The markets using priority mechanisms differ in how they ordered matches. Both the Newcastle and the Birmingham market used as a basis the product

of the hospital's rank of the student and the student's rank of the hospital. If two matches yield the same product, then in Birmingham the tie is broken in favor of the hospital, so that a (2,1) match would have higher priority than a (1,2) match and a (6,1) match would have higher priority than a (2,3) match. In Newcastle ties are broken in favor of the student.

In Edinburgh matches were ordered lexicographically in hospitals' preferences. That is, (1,1) matches have first priority, followed by (2,1), (3,1), and so on. Only when all hospitals' first choices are exhausted, other matches ((1,2), (2,2), (3,2),...) are considered. We will call the Edinburgh mechanism a *lexicographic priority mechanism*.

Any priority mechanism is unstable, i.e. for some profiles it chooses a matching which is not stable under the announced profile (Roth, 1991, Proposition 10).

As we will show, priority mechanisms satisfy anonymity, positive association, individual rationality, and independence of truncations. Therefore, by Theorem 2, in these medical markets a physician with completely symmetric information cannot do better than submitting a truncation of her true list.

Corollary 1 *In a matching market which uses a priority mechanism, any non-truncation strategy is stochastically dominated by a truncation of the true preferences for a worker with F -symmetric information.*

In any of the U.K. markets using a priority mechanism it might be optimal for a physician with completely symmetric information to submit a truncation of her true preference instead of her true list. The example also establishes that if a worker is completely informed, then truthful revelation might be suboptimal.¹⁰

Example 1 Let $W = \{w_1, w_2, w_3\}$ and $F = \{f_1, f_2\}$. Let M be a priority mechanism giving higher priority to (2,1) matches than to (1,3) matches. Let P_{-w_1} be such that (for each agent we only specify the ranking over the acceptable matches)

P_{w_2}	P_{w_3}	P_{f_1}	P_{f_2}		$P_{w_2}^{f_1 \leftrightarrow f_2}$	$P_{w_3}^{f_1 \leftrightarrow f_2}$	$P_{f_1}^{f_1 \leftrightarrow f_2}$	$P_{f_2}^{f_1 \leftrightarrow f_2}$
f_2	f_2	w_2	w_1		f_1	f_1	w_1	w_2
		w_3						w_3
		w_1						w_1

¹⁰This is also shown in Proposition 5 by Roth (1991).

Let w_1 's information \tilde{P}_{-w_1} be such that $\Pr\{\tilde{P}_{-w_1} = P_{-w_1}\} = \Pr\{\tilde{P}_{-w_1} = P_{-w_1}^{f_1 \leftrightarrow f_2}\} = \frac{1}{2}$. Then worker w_1 's information is F -symmetric. Let $P_{w_1} : f_1, f_2, w_1$ be w_1 's true preference. We show that it is optimal for w_1 to submit $P'_{w_1} : f_1, w_1, f_2$. Under P , there are no (1,1) and (1,2) matches. Then (w_1, f_2) is a (2,1) match and (w_1, f_1) is a (1,3) match. Because (2,1) matches have higher priority than (1,3) matches, we have $M[P](w_1) = f_2$. Under $[P_{w_1}, P_{-w_1}^{f_1 \leftrightarrow f_2}]$, (w_1, f_1) is a (1,1) match and we have $M[P_{w_1}, P_{-w_1}^{f_1 \leftrightarrow f_2}](w_1) = f_1$. Thus, if w_1 submits P_{w_1} , then w_1 is matched with probability $\frac{1}{2}$ to f_1 and with probability $\frac{1}{2}$ to f_2 .

Under $[P'_{w_1}, P_{-w_1}]$, there are no (1,1), (1,2) and (2,1) matches. Then (w_1, f_1) is a (1,3) match and $M[P_{w_1}, P_{-w_1}](w_1) = f_1$. Similarly as above, $M[P'_{w_1}, P_{-w_1}^{f_1 \leftrightarrow f_2}](w_1) = f_1$. Thus, if w_1 submits P'_{w_1} , then w_1 is matched with probability 1 to f_1 – an improvement over truthful revelation. \triangleleft

3.3.2 Linear Programming Mechanisms

Linear programming mechanisms were introduced in the entry level medical markets in the London Hospital Medical College in 1973 and in Cambridge in 1978. Such a mechanism assigns to each student-hospital match a weight with respect to the submitted rank-order lists. These weights are decreasing, i.e. (1,1) matches have greatest weight, then (1,2) and (2,1) matches, and so on. For any submitted rankings, the mechanism calculates a score through these weights for each individually rational matching and chooses one with maximal score. When there are several individually rational matchings which maximize the score, then tie breaking is necessary. We will assume that in such a case, the linear programming mechanism chooses each matching with maximal score with equal probability.¹¹ If for any submitted lists there is a unique matching with maximal score, then the problem of tie breaking can be ignored.

In the London Hospital Medical College choices 1, 2, 3, and 4 are given weights 20, 14, 9, and 5, respectively (Shah and Farrow, 1974). Then (1,1) matches have weight $20 + 20 = 40$, (1,2) and (2,1) matches weight $14 + 20 = 34$, (1,3) and (3,1) matches weight $9 + 20 = 29$, and (1,4) and (4,1) matches weight $5 + 20 = 25$. The linear programming mechanism used in Cambridge assigns to matches (1,1), (2,1), (3,1), (1,2), (2,2), (3,2), and (1,3) the weights 8, 7, 6, 5, 4, 3, and 2, respectively.

¹¹Due to this problem of tie breaking, Appendix C will extend our analysis to mechanisms choosing a lottery for each profile.

Any linear programming mechanism is unstable, i.e. for some profiles it chooses a matching which is not stable under the announced profile (Roth, 1991, Proposition 8). Such a mechanism even may fail to make (1,1)-matches, i.e. there are situations in which a student-hospital pair mutually prefers each other most but they are not matched to each other by the linear programming mechanism. Nevertheless linear programming mechanisms are still in use nowadays.

Using Theorem 2 we show that in these markets a student with completely symmetric information cannot do better than by submitting a truncation of her true preference list.

Corollary 2 *In a matching market which uses a linear programming mechanism, any non-truncation strategy is stochastically dominated by a truncation of the true preferences for a worker with F -symmetric information.*

Using Example 1 it can also be shown that for a student with completely symmetric information submitting a truncation might be better than submitting the true list in the London market and the Cambridge market.

3.3.3 The DA-algorithm

In entry-level medical markets interns and hospitals can dissolve a match and recontract. Therefore, it has been argued that mechanisms which produce unstable matchings may fail and should be replaced by stable ones. Indeed, in Edinburgh the lexicographic priority mechanism was replaced by a stable mechanism in 1969. The market in Cardiff also adopted a stable mechanism in 1971. Both markets introduced the most prominent stable mechanism which is used in centralized entry-level labor markets: the hospital-proposing deferred-acceptance algorithm (Gale and Shapley, 1962). We refer to the algorithm as the *DA-algorithm*.

The (hospital-proposing) DA-algorithm, denoted by DA.

STEP 1: Each hospital proposes to its favorite student. Each student rejects the proposal of any hospital that is unacceptable to her, and each student who receives more than one acceptable proposal rejects all but her most preferred among these, which she holds.

STEP k : Each hospital whose proposal was rejected at the previous step proposes to its favorite student among those who are acceptable for it and

who did not reject it at a previous step. Each student who receives proposals rejects all but her most preferred acceptable offer among those who propose to it and the proposal she held (if any).

STOP: The DA-algorithm terminates at a step at when each hospital which was rejected at the previous step has proposed to all its acceptable students. The outcome of the algorithm is the matching at which each student is matched to the hospital she is holding when the algorithm stops. Workers who did not receive any proposal and hospitals which were rejected by all students acceptable to them, remain unmatched. We denote by $DA[P]$ the matching that the DA-algorithm yields when applied to profile P . \circ

Under complete information truthful revelation of preferences is a dominant strategy for hospitals whereas a student might profit by misrepresenting her true ranking (Roth, 1982). Under incomplete information each hospital can still do no better than submitting its true ranking to the DA-algorithm (Roth, 1989). Until Roth and Rothblum (1999) there was no advice when participants are not fully informed. Their main results follow from our unifying analysis of the U.K. markets.¹² We will show that the DA-algorithm is just another mechanism satisfying the properties of Theorem 2.

Corollary 3 (Roth and Rothblum, 1999, Theorem 2) *In a matching market which uses the DA-algorithm, any non-truncation strategy is stochastically dominated by a truncation of the true preferences for a worker with F -symmetric information.*

Again it might be better for a student with completely symmetric information to submit a truncation instead of her true list.¹³

4 Advising Physicians with Asymmetric Information

A student's information does not need to be symmetric for all hospitals. If this is the case, then Theorem 2 does not give any advice to such a par-

¹²Note that Roth and Rothblum (1999, Theorem 1) follows from Theorem 1.

¹³This can also be seen from Example 5 below when w_1 believes each P_{-w_1} and $P_{-w_1}^{f_1 \leftrightarrow f_2}$ with probability $\frac{1}{2}$ and her true preference is $P_{w_1} : f_1, f_2, w_1$. When submitting P_{w_1} , she is matched with probability $\frac{1}{2}$ to f_1 and with probability $\frac{1}{2}$ to f_2 . When she submits $P'_{w_1} : f_1, w_1, f_2$, she is matched with probability 1 to f_1 , an improvement.

ticipant. We investigate whether a participant is able to gain by ranking unacceptable matches as acceptable. If the clearinghouse is not able to elicit true preferences (as it is the case in any British medical market), then we would like to design at least a mechanism under which each participant submits a list ranking only matches as acceptable which are also acceptable under the true preference. Of course, by Example 1, it might be optimal to drop acceptable matches.

In the Edinburgh 1967 market we advise each participant not to include any unacceptable match in the submitted rank-order list. For a lexicographic priority mechanism it is suboptimal for a physician to add unacceptable positions to her submitted rank-order list and for a hospital to add unacceptable students to its submitted rank-order list.

Theorem 3 *In a matching market using a lexicographic priority mechanism,*

- (i) any strategy which lists an unacceptable firm as acceptable is stochastically dominated by a strategy which does not list any unacceptable firm as acceptable for any worker; and*
- (ii) any strategy which lists an unacceptable worker as acceptable is stochastically dominated by a strategy which does not list any unacceptable worker as acceptable for any firm.*

Any further advice would depend on “how” symmetric a physician’s information is. If her information is symmetric for a subset of hospitals, then through Theorem 1 it follows that she should truthfully reveal her preference restricted to this set.

Unfortunately Theorem 3 does not carry over to any other entry-level medical market in the U.K.. In any other market (i.e. the Birmingham 1966 market, the Newcastle 1967 market, the London 1973 market, the Cambridge 1978 market, and the markets using the DA-algorithm) it might happen that only strategies which rank unacceptable positions as acceptable are optimal for a physician. The Appendix provides examples establishing this claim.

5 Conclusion

Our advice of strategic behavior for students with symmetric information is identical in all British entry-level medical markets (Theorem 1 and Theorem

2). If a student's information is completely symmetric, then it is optimal for her to submit a truncation of her true preference. Such a list leaves the true ranking of the positions unchanged and does not rank any unacceptable position as acceptable. In each market it is possible that a student profits from dropping acceptable positions from her true rank-order list.

When considering participants with asymmetric information, the Edinburgh 1967 market is distinguished among the British medical markets: it is the only market in which for *both sides* (students and hospitals) it is sub-optimal to include unacceptable matches in their rank-order lists. This is surprising because the Edinburgh market abandoned the priority mechanism and adopted the DA-algorithm in 1969. In the Birmingham 1966 market and the Newcastle 1967 market students and hospitals were again early contracting and submitted a list containing just the pre-arranged partner.¹⁴¹⁵ However, any mechanism discussed in this paper realizes the match of a student and a hospital who submitted rank-order lists containing mutually each other as the only acceptable match. Therefore, this kind of pre-arrangement is possible in any British entry-level medical market. It is still puzzling why in the markets using linear programming mechanisms or the DA-algorithm pre-contracting is less observed.

¹⁴This behavior has been duplicated by Kagel and Roth (2000) in a laboratory experiment for the Newcastle 1967 market. However, to my knowledge it is not clear why the Edinburgh market abandoned its priority mechanism.

¹⁵When a hospital is allowed to hire several students and information is complete, Sönmez (1999) shows that no stable mechanism is immune to pre-arranged matches.

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APPENDIX.

The Appendix contains all the proofs.

A Proofs of Subsection 3.2

Theorem 1 is a consequence of the following result.

Theorem 4 *Let M be an anonymous mechanism satisfying positive association. Let $w \in W$, $f, f' \in F$, and \tilde{P}_{-w} be a $\{f, f'\}$ -symmetric random preference profile. Let $P_w, \bar{P}_w \in \mathcal{P}_w$ be such that $f >_{P_w} f'$ and $f >_{\bar{P}_w} f'$. Then we have $M[\bar{P}_w, \tilde{P}_{-w}](w) \succ_{P_w} M[\bar{P}_w^{f \leftrightarrow f'}, \tilde{P}_{-w}](w)$.*

Proof. Let $v \in F \cup \{w\}$ be such that $v \notin \{f, f'\}$. For all $\bar{P}_{-w} \in \mathcal{P}_{-w}$, by anonymity, $M[\bar{P}](w) = v$ if and only if $M[\bar{P}^{f \leftrightarrow f'}](w) = v$. Because \bar{P}_{-w} and $\bar{P}_{-w}^{f \leftrightarrow f'}$ are equally probable under \tilde{P}_{-w} , it follows that

$$\Pr\{M[\bar{P}_w, \tilde{P}_{-w}](w) = v\} = \Pr\{M[\bar{P}_w^{f \leftrightarrow f'}, \tilde{P}_{-w}](w) = v\}. \quad (1)$$

Let $\bar{P}_{-w} \in \mathcal{P}_{-w}$. If $M[\bar{P}](w) = f'$, then by $f >_{\bar{P}_w} f'$ and positive association, $M[\bar{P}_w^{f \leftrightarrow f'}, \bar{P}_{-w}](w) = f'$. Thus,

$$\Pr\{M[\bar{P}_w, \tilde{P}_{-w}](w) = f'\} \leq \Pr\{M[\bar{P}_w^{f \leftrightarrow f'}, \tilde{P}_{-w}](w) = f'\}. \quad (2)$$

Hence, by (1) and (2) and since $f >_{P_w} f'$ and $M[P_w, \tilde{P}_{-w}](w)$ is a probability distribution, we have $M[\bar{P}_w, \tilde{P}_{-w}](w) \succ_{P_w} M[\bar{P}_w^{f \leftrightarrow f'}, \tilde{P}_{-w}](w)$, the desired conclusion. \square

Proof of Theorem 2. Let M be an anonymous mechanism satisfying positive association, individual rationality, and independence of truncations. Let $w \in W$, $P_w \in \mathcal{P}_w$, and \tilde{P}_{-w} be a F -symmetric random preference profile. Without loss of generality, suppose that $F = \{f_1, \dots, f_{|F|}\}$ and $f_1 >_{P_w} f_2 >_{P_w} \dots >_{P_w} f_{|F|}$. Let $\bar{P}_w \in \mathcal{P}_w$ be a strategy.

Let $\hat{P}_w \in \mathcal{P}_w$ be such that $\hat{P}_w|F = P_w|F$ and $|A(\hat{P}_w)| = |A(\bar{P}_w)|$. First we show that \hat{P}_w stochastically P_w -dominates \bar{P}_w .

Let $\sigma : \{1, \dots, |F|\} \rightarrow \{1, \dots, |F|\}$ be the mapping such that $f_{\sigma(1)} >_{\bar{P}_w} f_{\sigma(2)} >_{\bar{P}_w} \dots >_{\bar{P}_w} f_{\sigma(|F|)}$. If $\bar{P}_w|F \neq P_w|F$, then there exist $i, i+j \in \{1, \dots, |F|\}$ such that $\sigma(i) \geq i+j$, $\sigma(i+j) \leq i$, and for all $l \in \{i+1, \dots, i+j-$

1}, $\sigma(l) = l$. Therefore, $f_{\sigma(i+j)} >_{P_w} f_{\sigma(i)}$ and $f_{\sigma(i)} >_{\bar{P}_w} f_{\sigma(i+j)}$. Then it follows from (1) in the proof of Theorem 1 that for all $v \in (F \cup \{w\}) \setminus \{f_{\sigma(i)}, f_{\sigma(i+j)}\}$, $\Pr\{M[\bar{P}_w^{f_{\sigma(i)} \leftrightarrow f_{\sigma(i+j)}}, \tilde{P}_{-w}](w) = v\} = \Pr\{M[\bar{P}_w, \tilde{P}_{-w}](w) = v\}$ and from (2) in the proof of Theorem 1 that $\Pr\{M[\bar{P}_w^{f_{\sigma(i)} \leftrightarrow f_{\sigma(i+j)}}, \tilde{P}_{-w}](w) = f_{\sigma(i+j)}\} \geq \Pr\{M[\bar{P}_w, \tilde{P}_{-w}](w) = f_{\sigma(i+j)}\}$. Hence, by $f_{\sigma(i+j)} >_{P_w} f_{\sigma(i)}$,

$$M[\bar{P}_w^{f_{\sigma(i)} \leftrightarrow f_{\sigma(i+j)}}, \tilde{P}_{-w}](w) \succ_{P_w} M[\bar{P}_w, \tilde{P}_{-w}](w).$$

Applying the above argument inductively (note that the above construction (i) does not switch firms which are already at the “right” position under \hat{P}_w and \bar{P}_w , i.e. if $\sigma(l) = l$, then f_l is not involved in any exchange of positions and (ii) switches two firms only if the preferences are reversed under P_w and \bar{P}_w) it follows from transitivity of \succ_{P_w} that

$$M[\hat{P}_w, \tilde{P}_{-w}](w) \succ_{P_w} M[\bar{P}_w, \tilde{P}_{-w}](w). \quad (3)$$

If $A(\hat{P}_w) \subseteq A(P_w)$, then \hat{P}_w is a truncation of P_w which stochastically P_w -dominates \bar{P}_w , the desired conclusion. Otherwise, if $A(\hat{P}_w) \not\subseteq A(P_w)$, then by $\hat{P}_w|F = P_w|F$ and independence of truncations, for all $f \in A(P_w)$ and all $P_{-w} \in \mathcal{P}_{-w}$, if $M[\hat{P}_w, P_{-w}](w) = f$, then $M[P_w](w) = f$. Thus, for all $f \in A(P_w)$, $\Pr\{M[P_w, \tilde{P}_{-w}](w) = f\} \geq \Pr\{M[\hat{P}_w, \tilde{P}_{-w}](w) = f\}$. Because M is individually rational, it follows that $M[P_w, \tilde{P}_{-w}](w) \succ_{P_w} M[\hat{P}_w, \tilde{P}_{-w}](w)$. Hence, by transitivity of \succ_{P_w} and (3), $M[P_w, \tilde{P}_{-w}](w) \succ_{P_w} M[\bar{P}_w, \tilde{P}_{-w}](w)$. Thus, P_w is a truncation of P_w which stochastically P_w -dominates \bar{P}_w , the desired conclusion. \square

B Priority Mechanisms

First we define priority mechanisms formally. Second we establish Corollary 1 by showing that any priority mechanism satisfies the properties of Theorem 2. Third we prove Theorem 3 by showing that in a lexicographic priority mechanism no participant benefits from including unacceptable matches in her submitted rank-order list. Fourth we provide an example to demonstrate that this conclusion is false for the priority mechanisms which were used in Birmingham and Newcastle.

Given $P_w \in \mathcal{P}_w$ and $f \in F$, we will denote by $rank(f, P_w)$ the rank of f in P_w . Similarly we define $rank(w, P_f)$. Given a profile P , we call (w, f) a (a, b) match if $rank(f, P_w) = a$ and $rank(w, P_f) = b$.

A *priority function* is a one-to-one mapping $g : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ such that (i) for all $(a, b), (a', b) \in \mathbb{N} \times \mathbb{N}$, if $a < a'$, then $g(a, b) < g(a', b)$, and (ii) for all $(a, b), (a, b') \in \mathbb{N} \times \mathbb{N}$, if $b < b'$, then $g(a, b) < g(a, b')$. For each profile the priority mechanism based on g matches sequentially all worker-firm pairs which have the highest priority according to g subject to individual rationality. Let M^g denote the priority mechanism based on g .

Proof of Corollary 1. Let M^g be a priority mechanism with priority function g . It suffices to show that M^g is a mechanism satisfying the properties of Theorem 2. It is obvious that M^g is anonymous and individually rational.

In showing positive association, let $w \in W$ and $P \in \mathcal{P}$. Let $M^g[P](w) = f$ and $f' \in F$ be such that $f' >_{P_w} f$. Under $[P_w^{f \leftrightarrow f'}, P_{-w}]$, each pair (f'', w) with $f'' \in F \setminus \{f\}$ has at most the same priority as under P . Under $[P_w^{f \leftrightarrow f'}, P_{-w}]$ only the pair (f, w) has higher priority than under P . Thus, $M^g[P](w) = f$ implies $M^g[P_w^{f \leftrightarrow f'}, P_{-w}](w) = f$.

In showing independence of truncations, let $w \in W$ and $P \in \mathcal{P}$. Let $M^g[P](w) = f$ and $P'_w \in \mathcal{P}_w$ be such that $P'_w|F = P_w|F$ and $f \in A(P'_w) \subseteq A(P_w)$. Then under $[P'_w, P_{-w}]$ each pair (f'', w) with $f'' \in A(P'_w)$ has the same priority as under P . Thus, $M^g[P](w) = f$ implies $M^g[P_w^{f \leftrightarrow f'}, P_{-w}](w) = f$. \square

Proof of Theorem 3. Let M^g be a lexicographic priority mechanism. Because of the symmetry of the matching market, it suffices to show (i) for the F -lexicographic priority mechanism and the W -lexicographic priority mechanism.

(i) for F -lexicographic priority function: Let g be the F -lexicographic priority function, i.e. for all $(a, b), (a', b') \in \mathbb{N} \times \mathbb{N}$, if $b < b'$, then $g(a, b) < g(a', b')$.

Let $w \in W$ and \tilde{P}_{-w} be w 's belief about the rank-order lists submitted by the other participants. Let $\bar{P}_w \in \mathcal{P}_w$ be w 's true preference and $P_w \in \mathcal{P}_w$ be a strategy. Let $P'_w \in \mathcal{P}_w$ be such that $A(P'_w) = A(\bar{P}_w) \cap A(P_w)$ and $P'_w|A(P'_w) = P_w|A(P'_w)$. We show that P'_w stochastically \bar{P}_w -dominates P_w .

Let $f \in A(P'_w)$ and $P_{-w} \in \mathcal{P}_{-w}$ be such that $M^g[P](w) = f$. Since g is F -lexicographic and the preferences of the others remain unchanged, we have $M^g[P'_w, P_{-w}](w) \neq w$. Let $M^g[P'_w, P_{-w}](w) = f'$. If (w, f) is an (a, b) match under P , then by $P'_w|A(P'_w) = P_w|A(P'_w)$ and $A(P'_w) \subseteq A(P_w)$, (w, f) has at least priority $g(a, b)$ under $[P'_w, P_{-w}]$. Consider the sequence of matches realized by M^g at P and realized by M^g at $[P'_w, P_{-w}]$. Because the preferences of the other participants are identical, the sequence of matches is identical

until w is matched at P or at $[P'_w, P_{-w}]$.

If $f' \neq f$, then by $M^g[P](w) = f$ and because g is F -lexicographic and (w, f) has at least priority $g(a, b)$ under $[P'_w, P_{-w}]$, we must have that w is matched earlier at $[P'_w, P_{-w}]$ than at P . This is only possible if $\text{rank}(w, P_{f'}) < \text{rank}(w, P_f) = b$. But then the match (w, f') has also higher priority than (w, f) under P , which contradicts $M^g[P](w) = f$. Thus, $M^g[P'_w, P_{-w}](w) = f$ and $\Pr\{M^g[P'_w, P_{-w}](w) = f\} \geq \Pr\{M^g[P](w) = f\}$. Because $A(P'_w) \subseteq A(\bar{P}_w)$ and M^g is individually rational, we have $M^g[P'_w, \tilde{P}_{-w}](w) \succ_{\bar{P}_w} M^g[P_w, \tilde{P}_{-w}](w)$, the desired conclusion.

(i) for W -lexicographic priority function: Let g be the W -lexicographic priority function, i.e. for all $(a, b), (a', b') \in \mathbb{N} \times \mathbb{N}$, if $a < a'$, then $g(a, b) < g(a', b')$.

Let $w \in W$ and \tilde{P}_{-w} be w 's belief about the rank-order lists submitted by the other participants. Let $\bar{P}_w \in \mathcal{P}_w$ be w 's true preference and $P_w \in \mathcal{P}_w$ be a strategy. Suppose P_w ranks an unacceptable firm above an acceptable firm in $A(P_w)$. Let $f \in A(P'_w) \cap A(\bar{P}_w)$ and $f' \in A(P'_w) \setminus A(\bar{P}_w)$ be such that $f' >_{P_w} f$. Without loss of generality, we suppose that f' is the P_w most preferred unacceptable firm and f is the \bar{P}_w most preferred acceptable firm which is ranked acceptable and below f' under P_w .

Let $f'' \in F$ be such that $f'' >_{P_w} f'$. Because g is W -lexicographic, we have for all $P_{-w} \in \mathcal{P}_{-w}$, $M^g[P](w) = f'' \Leftrightarrow M^g[P_w^{f \leftrightarrow f'}, P_{-w}](w) = f''$. Thus, for all $f'' \in F$ such that $f'' >_{P_w} f'$,

$$\Pr\{M^g[P_w^{f \leftrightarrow f'}, \tilde{P}_{-w}](w) = f''\} = \Pr\{M^g[P_w, \tilde{P}_{-w}](w) = f''\}. \quad (4)$$

Let $f'' \in F \setminus \{f\}$ be such that $f' >_{P_w} f''$. Because g is W -lexicographic, we have for all $P_{-w} \in \mathcal{P}_{-w}$, if $M^g[P](w) = f''$, then the match (w, f'') has the same priority under $[P_w^{f \leftrightarrow f'}, P_{-w}]$. Because (w, f) is the only match which has higher priority under $[P_w^{f \leftrightarrow f'}, P_{-w}]$ than under P , we have $M^g[P_w^{f \leftrightarrow f'}, P_{-w}](w) \in \{f'', f\}$. Thus, for all $f'' \in F \setminus \{f\}$ such that $f' >_{P_w} f''$,

$$\begin{aligned} & \{P_{-w} \in \mathcal{P}_{-w} \mid M^g[P](w) = f''\} \subseteq \\ & \{P_{-w} \in \mathcal{P}_{-w} \mid M^g[P_w^{f \leftrightarrow f'}, P_{-w}](w) = f'' \text{ or } M^g[P_w^{f \leftrightarrow f'}, P_{-w}](w) = f\}. \end{aligned} \quad (5)$$

Because priority mechanisms satisfy positive association, we also have

$$\{P_{-w} \in \mathcal{P}_{-w} \mid M^g[P](w) = f\} \subseteq \{P_{-w} \in \mathcal{P}_{-w} \mid M^g[P_w^{f \leftrightarrow f'}, P_{-w}](w) = f\}. \quad (6)$$

Now observe from (4), (5), and (6) that for each firm which is acceptable under \bar{P}_w and P_w the probability does not decrease when submitting $P_w^{f \leftrightarrow f'}$ instead of P_w or if it decreases for f'' , then by our choice of f , $f >_{\bar{P}_w} f''$, and by (5), all the lost probability is put on the more preferred firm f .

If there is again an unacceptable firm which is ranked above an acceptable firm under $P_w^{f \leftrightarrow f'}$, then we derive the same conclusions. After a finite number of switches we will arrive at a strategy $P'_w \in \mathcal{P}_w$ such that $A(P'_w) = A(P_w)$ and for all $f \in A(P'_w) \cap A(\bar{P}_w)$ and all $f' \in A(P'_w) \setminus A(\bar{P}_w)$, $f >_{P'_w} f'$.

The following strategy drops the unacceptable firms from the list P'_w . Let $P''_w \in \mathcal{P}_w$ be such that $A(P''_w) = A(P'_w) \cap A(\bar{P}_w)$ and $P''_w|A(P''_w) = P'_w|A(P''_w)$. Because M^g satisfies individual rationality and independence of truncations, similarly to the proof of Theorem 2 we have $M^g[P''_w, \tilde{P}_{-w}](w) \succ_{\bar{P}_w} M^g[P'_w, \tilde{P}_{-w}](w)$.

Since (4), (5), and (6) hold at each switch from P_w to P'_w , it follows that P''_w stochastically \bar{P}_w -dominates P_w and P''_w does not rank any unacceptable firm as acceptable, the desired conclusion. \square

The following example shows that in both the Birmingham 1966 market and the Newcastle 1967 market it is possible that only strategies which rank unacceptable positions as acceptable are optimal for a physician.

Example 2 Let $W = \{w_1, w_2, w_3\}$ and $F = \{f_1, f_2, f_3, f_4\}$. Let M be a priority mechanism which gives higher priority to (2, 1) matches than to (1, 3) matches and which gives higher priority to (1, 3) matches than to (4, 1) matches. Let P_{-w_1} and \hat{P}_{-w_1} be such that (for each agent we only specify the ranking over the acceptable matches)

P_{w_2}	P_{w_3}	P_{f_1}	P_{f_2}	P_{f_3}	P_{f_4}	$P_{w_2}^{f_1 \leftrightarrow f_2}$	$P_{w_3}^{f_1 \leftrightarrow f_2}$	$P_{f_1}^{f_1 \leftrightarrow f_2}$	$P_{f_2}^{f_1 \leftrightarrow f_2}$	$P_{f_3}^{f_1 \leftrightarrow f_2}$	$P_{f_4}^{f_1 \leftrightarrow f_2}$
f_2	f_2	w_2	w_1	w_2	w_2	f_1	f_1	w_1	w_2	w_2	w_2
		w_3		w_3	w_3			w_3	w_3	w_3	w_3
		w_1						w_1	w_1		
\hat{P}_{w_2}	\hat{P}_{w_3}	\hat{P}_{f_1}	\hat{P}_{f_2}	\hat{P}_{f_3}	\hat{P}_{f_4}	$\hat{P}_{w_2}^{f_1 \leftrightarrow f_2}$	$\hat{P}_{w_3}^{f_1 \leftrightarrow f_2}$	$\hat{P}_{f_1}^{f_1 \leftrightarrow f_2}$	$\hat{P}_{f_2}^{f_1 \leftrightarrow f_2}$	$\hat{P}_{f_3}^{f_1 \leftrightarrow f_2}$	$\hat{P}_{f_4}^{f_1 \leftrightarrow f_2}$
f_2	f_2	w_1	w_2	w_2	w_2	f_1	f_1	w_2	w_1	w_2	w_2

Let w_1 's information \tilde{P}_{-w_1} be such that $\Pr\{\tilde{P}_{-w_1} = P_{-w_1}\} = \Pr\{\tilde{P}_{-w_1} = P_{w_2}^{f_1 \leftrightarrow f_2}\} = \Pr\{\tilde{P}_{-w_1} = \hat{P}_{-w_1}\} = \Pr\{\tilde{P}_{-w_1} = \hat{P}_{w_3}^{f_1 \leftrightarrow f_2}\} = \frac{1}{4}$. Then worker w_1 's

information is $\{f_1, f_2\}$ -symmetric and $\{f_3, f_4\}$ -symmetric.¹⁶

Let $P_{w_1} : f_1, f_2, w_1, f_3, f_4$ be w_1 's true preference. Note that \tilde{P}_{-w_1} is $A(P_{w_1})$ -symmetric and $F \setminus A(P_{w_1})$ -symmetric. We show that it is optimal for w_1 to submit $P'_{w_1} : f_1, f_3, f_4, f_2, w_1$.

It follows as in Example 1 that $M[P](w_1) = f_2$ and $M[P_{w_1}, P_{-w_1}^{f_1 \leftrightarrow f_2}](w_1) = f_1$. It is easy to check that $M[P_{w_1}, \hat{P}_{-w_1}](w_1) = f_1$ and $M[P_{w_1}, \hat{P}_{-w_1}^{f_1 \leftrightarrow f_2}](w_1) = f_2$. Thus, if w_1 submits P_{w_1} , then w_1 is matched with probability $\frac{1}{2}$ to f_1 and with probability $\frac{1}{2}$ to f_2 .

Under $[P'_{w_1}, P_{-w_1}]$, (w_1, f_2) is a $(4, 1)$ match whereas (w_1, f_1) is a $(1, 3)$ match. Thus, because $(1, 3)$ matches have higher priority than $(4, 1)$ matches and there are no $(1, 1)$, $(1, 2)$, and $(2, 1)$ matches, we have $M[P'_{w_1}, P_{-w_1}](w_1) = f_1$. It is easy to check that $M[P'_{w_1}, P_{-w_1}^{f_1 \leftrightarrow f_2}](w_1) = f_1$, $M[P'_{w_1}, \hat{P}_{-w_1}](w_1) = f_1$, and $M[P'_{w_1}, \hat{P}_{-w_1}^{f_1 \leftrightarrow f_2}](w_1) = f_2$. Thus, if w_1 submits P'_{w_1} , then w_1 is matched with probability $\frac{3}{4}$ to f_1 and with probability $\frac{1}{4}$ to f_2 – an improvement over truthful revelation.

Note that w_1 cannot be matched with higher probability to f_1 by submitting another list. If she does not include f_2 in her list, then under $\hat{P}_{-w_1}^{f_1 \leftrightarrow f_2}$ she remains unemployed. However, when her submitted list contains f_2 , then she needs to include at least one unacceptable position because otherwise she will not be matched under P_{-w_1} to f_1 . Therefore, for w_1 only strategies which rank unacceptable positions as acceptable are optimal. \triangleleft

C Linear Programming Mechanisms

First we define linear programming mechanisms formally. Second we extend our analysis to mechanisms choosing for each profile a lottery over all matchings. Third we establish Corollary 2 by showing that any linear programming mechanism satisfies the probabilistic analogues of the properties of Theorem 2. Fourth we provide two example to demonstrate that in the London market and the Cambridge market a participant may benefit from including unacceptable matches in her submitted rank-order list.

A linear programming mechanism assigns to each possible (a, b) match a positive weight. These weights are strictly decreasing in both components.

¹⁶Formally we would need to add another four profiles for switching the positions of f_3 and f_4 . However, at each of the above profiles, switching f_3 and f_4 does not change the rank-order list for each agent and a priority mechanism only takes the profile of rank-order lists into account.

Formally, a weighting function is a function $h : (\mathbb{N} \times \mathbb{N}) \cup \{(0, 0)\} \rightarrow \mathbb{R}_+$ such that (i) $h(0, 0) = 0$, (ii) for all $(a, b), (a', b) \in \mathbb{N} \times \mathbb{N}$, if $a < a'$, then $h(a, b) > h(a', b)$, and (iii) for all $(a, b), (a, b') \in \mathbb{N} \times \mathbb{N}$, if $b < b'$, then $h(a, b) > h(a, b')$. Recall that $\text{rank}(f, P_w)$ denotes the rank of f in P_w and $\text{rank}(w, P_f)$ the rank of w in P_f . If f is most preferred under P_w , then $\text{rank}(f, P_w) = 1$. We also set $\text{rank}(w, P_w) = 0$. Given a profile $P \in \mathcal{P}$ and an individually rational matching μ the score of μ is the sum of the weights of the matched worker-firm pairs, i.e.

$$s^h(\mu, P) = \sum_{w \in W} h(\text{rank}(\mu(w), P_w), \text{rank}(w, P_{\mu(w)})).$$

Given a weighting function h , the linear programming mechanism M^h chooses for each profile $P \in \mathcal{P}$ an individually rational matching with maximal score among all individually rational matchings. Let $\arg \max s^h(\cdot, P)$ denote the set of all individually rational matchings which maximize the weighted score s^h among all individually rational matchings. We will assume that if there are several matchings maximizing the score under a submitted profile, then the linear programming mechanism breaks the ties uniformly, i.e. each of these maximizers is chosen with equal probability. In such a case a linear programming mechanism chooses a lottery over matchings. We will see that this does not create any significant problem. Below we extend our analysis to mechanisms choosing a lottery for each profile.

For the remaining part of Appendix C let M denote a *probabilistic* mechanism choosing for each profile P a lottery over \mathcal{M} . Given a matching μ , let $\Pr\{M[P] = \mu\}$ denote the probability that M assigns to μ when the profile of submitted rankings is P . Similarly, given $w \in W$ and $f \in F$, let $\Pr\{M[P](w) = f\}$ denote the probability that w is assigned to f under the lottery $M[P]$.

For probabilistic mechanisms the properties anonymity, positive association, individual rationality, and independence of truncations are given below.

Anonymity: For all $P \in \mathcal{P}$, all $\mu \in \mathcal{M}$, and all $f, f' \in F$, $\Pr\{M[P] = \mu\} = \Pr\{M[P^{f \leftrightarrow f'}] = \mu^{f \leftrightarrow f'}\}$.

Positive Association: For all $P \in \mathcal{P}$, all $w \in W$, and all $f, f' \in F$, if $\Pr\{M[P](w) = f\} > 0$ and $f' >_{P_w} f$, then $\Pr\{M[P_w^{f \leftrightarrow f'}, P_{-w}](w) = f\} = 1$.

Individual Rationality: For all $P \in \mathcal{P}$, all $v \in V$, and all $\mu \in \mathcal{M}$, if $\Pr\{M[P] = \mu\} > 0$, then $\mu(v) \geq_{P_v} v$.

Independence of Truncations: For all $P \in \mathcal{P}$, all $w \in W$, all $P'_w \in \mathcal{P}_w$, and all $f \in F$, if P'_w is a truncation of P_w , $\Pr\{M[P](w) = f\} > 0$, and f is acceptable under P'_w , then for all $f' \in A(P'_w)$, $\Pr\{M[P'_w, P_{-w}](w) = f'\} \geq \Pr\{M[P](w) = f'\}$.

It is straightforward to check that Theorem 4 remains true for probabilistic mechanisms satisfying anonymity and positive association. Furthermore (1) in the proof of Theorem 4 is also valid. Then it is easy to see that Theorem 2 holds for probabilistic mechanisms.

The linear programming mechanism with weighting function h is formally defined as follows: for all $P \in \mathcal{P}$ and all individually rational matchings μ such that

$$s^h(\mu, R) = \max_{\mu' \text{ is individually rational}} s^h(\mu', R),$$

we have $\Pr\{M^h[P] = \mu\} = 1/|\arg \max s^h(\cdot, P)|$ (here $|S|$ denotes the cardinality of set S).

Proof of Corollary 2. Let M^h be a linear programming mechanism with weighting function h . By Theorem 2, it suffices to show that M^h is a probabilistic mechanism satisfying anonymity, positive association, individual rationality, and independence of truncations. By definition, M^h satisfies individual rationality.

In showing anonymity, let $P \in \mathcal{P}$, $\mu \in \mathcal{M}$, and $f, f' \in F$. Then for all $w \in W$, $\text{rank}(w, P_f) = \text{rank}(w, P_{f'}^{f \leftrightarrow f'})$, $\text{rank}(w, P_{f'}) = \text{rank}(w, P_f^{f \leftrightarrow f'})$, $\text{rank}(f, P_w) = \text{rank}(f', P_w^{f \leftrightarrow f'})$, and $\text{rank}(f', P_w) = \text{rank}(f, P_w^{f \leftrightarrow f'})$. Thus, $s^h(\mu, P) = s^h(\mu^{f \leftrightarrow f'}, P^{f \leftrightarrow f'})$. Because μ was arbitrary, we have $\Pr\{M^h[P] = \mu\} = \Pr\{M^h[P^{f \leftrightarrow f'}] = \mu^{f \leftrightarrow f'}\}$, the desired conclusion.

In showing positive association, let $P \in \mathcal{P}$, $w \in W$, and $f, f' \in F$ be such that $\Pr\{M^h[P](w) = f\} > 0$ and $f' >_{P_w} f$. Let $\mu \in \arg \max s^h(\cdot, P)$ be such that $\mu(w) = f$. Because h is strictly decreasing and $\text{rank}(f, P_w^{f \leftrightarrow f'}) < \text{rank}(f, P_w)$, we have $s^h(\mu, [P_w^{f \leftrightarrow f'}, P_{-w}]) > s^h(\mu, P)$. Furthermore, for all matchings μ' which are individually rational under P , if $\mu'(w) \neq f$, then $s^h(\mu', [P_w^{f \leftrightarrow f'}, P_{-w}]) \leq s^h(\mu', P)$. By $\Pr\{M^h[P](w) = f\} > 0$ and individual rationality of M^h , the set of individually rational matchings is identical under

P and $[P_w^{f \leftrightarrow f'}, P_{-w}]$. Hence, $\Pr\{M^h[P_w^{f \leftrightarrow f'}, P_{-w}](w) = f\} = 1$, the desired conclusion.

In showing independence of truncations, let $P \in \mathcal{P}$, $w \in W$, $P'_w \in \mathcal{P}_w$, and $f \in F$ be such that P'_w is a truncation of P_w , $\Pr\{M[P](w) = f\} > 0$, and f is acceptable under P'_w . Since P'_w is a truncation of P_w , we have for all matchings μ , (i) if μ is individually rational under $[P'_w, P_{-w}]$, then μ is individually rational under P , and (ii) if μ is individually rational under $[P'_w, P_{-w}]$, then $s^h(\mu, [P'_w, P_{-w}]) = s^h(\mu, P)$. Because $\Pr\{M[P](w) = f\} > 0$ and f is acceptable under P'_w , there exists $\mu \in \arg \max s^h(\cdot, P)$ such that $\mu(w) = f$ and μ is individually rational under $[P'_w, P_{-w}]$. Then it follows that $\arg \max s^h(\cdot, [P'_w, P_{-w}])$ is the set of matchings which maximize s^h under P and satisfy individual rationality under $[P'_w, P_{-w}]$. Hence, for all $f' \in A(P'_w)$, $\Pr\{M^h[P'_w, P_{-w}](w) = f'\} \geq \Pr\{M^h[P](w) = f'\}$, the desired conclusion. \square

The following two examples show that the linear programming mechanisms used in the U.K. also have the flaw that it is possible that only strategies which rank unacceptable positions as acceptable are optimal for a student.

Example 3 (London 1973 Market) In the London Hospital Medical College choices 1, 2, 3, and 4 are given weights 20, 14, 9, and 5, respectively (Shah and Farrow, 1974). Thus, (1,1) matches have weight $20 + 20 = 40$, (1,2) and (2,1) matches weight $14 + 20 = 34$, (1,3) and (3,1) matches weight $9 + 20 = 29$, and (1,4) and (4,1) matches weight $5 + 20 = 25$.

Consider Example 2. Note that under all profiles which w_1 believes with positive probability under \tilde{P}_{-w_1} , there is no pair (w, f) such that $w \in \{w_2, w_3\}$ and (w, f) are mutually acceptable. Since a linear programming mechanism only chooses individually rational matchings, for each of these profiles the linear programming mechanism matches w_1 to the firm f with maximal weight.

Given the above weights, the weight of a (2,1) match is greater than the weight of a (1,3) match and the weight of a (1,3) match is greater than the weight of a (4,1) match. Then the same conclusions hold for the London mechanism as in Example 2: If $P_{w_1} : f_1, f_2, w_1, f_3, f_4$ is w_1 's true preference, then it is optimal for w_1 to submit $P'_{w_1} : f_1, f_3, f_4, f_2, w_1$.

Note that it is suboptimal for w_1 to include only one acceptable firm in her list. If w_1 would submit $P''_{w_1} : f_1, f_3, f_2, w_1$, then under $[P''_{w_1}, P_{-w_1}]$, (w_1, f_1) is a (1,3) match and (w_1, f_2) is a (3,1) match and both result in the same weight

29. If the linear programming mechanism breaks ties with equal probability, then by submitting P''_{w_1} , w_1 is matched with probability $\frac{5}{8}$ to f_1 and with probability $\frac{3}{8}$ to f_2 . This is suboptimal for w_1 because by submitting P'_{w_1} , w_1 is matched with probability $\frac{3}{4}$ to f_1 and with probability $\frac{1}{4}$ to f_2 . Therefore, for w_1 only strategies which rank both unacceptable positions as acceptable are optimal. \triangleleft

Example 4 (Cambridge 1978 Market) Let $W = \{w_1, w_2, w_3\}$ and $F = \{f_1, f_2, f_3\}$. We use the same weights for the Cambridge linear programming mechanism as Roth (1991). Let M be the linear programming mechanism which assigns to matches (1,1), (2,1), (3,1), (1,2), (2,2), (3,2), and (1,3) weights 8, 7, 6, 5, 4, 3, and 2, respectively. Let P_{-w_1} and \hat{P}_{-w_1} be such that

$$\begin{array}{c|c|c|c|c} P_{w_2} & P_{w_3} & P_{f_1} & P_{f_2} & P_{f_3} \\ \hline f_1 & f_3 & w_1 & w_2 & w_3 \\ & & w_3 & w_1 & \\ & & w_2 & & \end{array} \parallel \begin{array}{c|c|c|c|c} P_{w_2}^{f_1 \leftrightarrow f_2} & P_{w_3}^{f_1 \leftrightarrow f_2} & P_{f_1}^{f_1 \leftrightarrow f_2} & P_{f_2}^{f_1 \leftrightarrow f_2} & P_{f_3}^{f_1 \leftrightarrow f_2} \\ \hline f_2 & f_3 & w_2 & w_1 & w_3 \\ & & w_1 & w_3 & \\ & & & w_2 & \end{array}$$

$$\begin{array}{c|c|c|c|c} \hat{P}_{w_2} & \hat{P}_{w_3} & \hat{P}_{f_1} & \hat{P}_{f_2} & \hat{P}_{f_3} \\ \hline f_2 & f_2 & w_1 & w_2 & w_2 \end{array} \parallel \begin{array}{c|c|c|c|c} \hat{P}_{w_2}^{f_1 \leftrightarrow f_2} & \hat{P}_{w_3}^{f_1 \leftrightarrow f_2} & \hat{P}_{f_1}^{f_1 \leftrightarrow f_2} & \hat{P}_{f_2}^{f_1 \leftrightarrow f_2} & \hat{P}_{f_3}^{f_1 \leftrightarrow f_2} \\ \hline f_1 & f_1 & w_2 & w_1 & w_2 \end{array}$$

Let w_1 's information \tilde{P}_{-w_1} be such that $\Pr\{\tilde{P}_{-w_1} = P_{-w_1}\} = \Pr\{\tilde{P}_{-w_1} = P_{-w_1}^{f_1 \leftrightarrow f_2}\} = \Pr\{\tilde{P}_{-w_1} = \hat{P}_{-w_1}\} = \Pr\{\tilde{P}_{-w_1} = \hat{P}_{-w_1}^{f_1 \leftrightarrow f_2}\} = \frac{1}{4}$. Then worker w_1 's information is $\{f_1, f_2\}$ -symmetric.

Let $P_{w_1} : f_1, f_2, w_1$ be w_1 's true preference. We show that it is optimal for w_1 to submit $P'_{w_1} : f_1, f_3, f_2, w_1$.

Let μ be such that $\mu(w_i) = f_i$ for $i \in \{1, 2, 3\}$. Under P , (w_1, f_1) is a (1,2) match, (w_2, f_2) is a (1,3) match, and (w_3, f_3) is a (1,1) match. Thus, under P the score of μ is $5 + 2 + 8 = 15$. Let μ' be such that $\mu'(w_1) = f_2$, $\mu'(w_2) = w_2$, and $\mu'(w_3) = f_3$. Under P , (w_1, f_2) is a (2,1) match and (w_3, f_3) is a (1,1) match. Thus, under P the score of μ' is $7 + 8 = 15$. It is easy to check that μ and μ' are the two individually rational matchings with maximal score under P . Because M breaks the tie with equal probabilities, μ is chosen with probability $\frac{1}{2}$ and μ' is chosen with probability $\frac{1}{2}$ under P .

It is left to the reader to check that $M[P_{w_1}, P_{-w_1}^{f_1 \leftrightarrow f_2}](w_1) = f_1$, $M[P_{w_1}, \hat{P}_{-w_1}](w_1) = f_1$, and $M[P_{w_1}, \hat{P}_{-w_1}^{f_1 \leftrightarrow f_2}](w_1) = f_2$. Hence, if w_1 submits P_{w_1} , then she is matched with probability $\frac{5}{8}$ to f_1 and with probability $\frac{3}{8}$ to f_2 .

Under $[P'_{w_1}, P_{-w_1}]$, (w_1, f_2) becomes a $(3, 1)$ match. Thus, the score of μ' decreases by 1 whereas the score of μ remains unchanged. Hence, $M[P'_{w_1}, P_{-w_1}](w_1) = f_1$. The other choices are identical when submitting P'_{w_1} or P_{w_1} . Hence, if w_1 submits P'_{w_1} , then she is matched with probability $\frac{3}{4}$ to f_1 and with probability $\frac{1}{4}$ to f_2 – an improvement over truthful revelation.

Note that w_1 cannot be matched with higher probability to f_1 by submitting another list. If she does not include f_2 in her list, then under $\hat{P}_{-w_1}^{f_1 \leftrightarrow f_2}$ she remains unemployed. However, when her submitted list contains f_2 , then she needs to include at least one unacceptable position because otherwise she will be matched with positive probability to f_2 under P_{-w_1} . Therefore, for w_1 only strategies which rank unacceptable positions as acceptable are optimal. \triangleleft

D The DA-Algorithm

First we establish Corollary 3 by showing that the DA-algorithm satisfies the properties of Theorem 2. Second we provide the example by Roth and Rothblum (1999) to demonstrate that only strategies which rank unacceptable matches as acceptable may be optimal for a participant.

Proof of Corollary 3. By Theorem 2 it suffices to show that the DA-algorithm satisfies anonymity, positive association, individual rationality, and independence of truncations. It is straightforward that the DA-algorithm is anonymous and individually rational.

In showing positive association, let $P \in \mathcal{P}$, $w \in W$, and $f, f' \in F$ be such that $DA[P](w) = f$ and $f' \succ_{P_w} f$. Let $V' = \{v \in F \cup \{w\} \mid f \geq_{P_w} v\}$. Then $P_w|V' = P_w^{f \leftrightarrow f'}|V'$. When applying the DA-algorithm to P , w only receives offers from firms belonging to V' . Thus, by $P_w|V' = P_w^{f \leftrightarrow f'}|V'$, when applying the DA-algorithm to $[P_w^{f \leftrightarrow f'}, P_{-w}]$, w makes the same rejections and acceptances and only receives offers from firms belonging to V' . Hence, $DA[P_w^{f \leftrightarrow f'}, P_{-w}](w) = f$, the desired conclusion.

In showing independence of truncations, let $P \in \mathcal{P}$, $w \in W$, and $P'_w \in \mathcal{P}_w$ be such that P'_w is a truncation of P_w and $M[P](w)$ is acceptable under P'_w . Let $\mu = M[P]$. Since $\mu(w) \in A(P'_w)$, μ is stable under $[P'_w, P_{-w}]$. Let μ' denote the firm-optimal matching under $[P'_w, P_{-w}]$. If μ is not the firm-optimal matching under $[P'_w, P_{-w}]$, then $\mu(w) \geq_{P'_w} \mu'(w)$ and for all $w' \in W \setminus \{w\}$, $\mu(w') \geq_{P_{w'}} \mu'(w')$ (Gale and Shapley, 1962). Because the set of unmatched

workers is the same under any two stable matchings (Roth and Sotomayor, 1990, Theorem 2.22) and $\mu(w) = f$, we have $\mu'(w) \neq w$. Thus, $\mu'(w) >_{P_w} w$ and $\mu(w) \geq_{P_w} \mu'(w)$. Hence, by the stability of μ' under $[P'_w, P_{-w}]$, μ' is stable under P . Because $\mu \neq \mu'$ and for all $w' \in W$, $\mu(w') \geq_{P_{w'}} \mu'(w')$, this contradicts the fact that $DA[P] = \mu$ and the DA-algorithm computes for each profile its firm-optimal stable matching (Gale and Shapley, 1962). \square

Matching markets using the DA-algorithm also have the drawback that it is possible that only rank-order lists including unacceptable positions are optimal for a physician. For the sake of completeness we provide the example given by Roth and Rothblum (1999).

Example 5 (Roth and Rothblum, 1999, Example 3) Let $W \equiv \{w_1, w_2\}$ and $F \equiv \{f_1, f_2, f_3\}$. Let $P_{-w_1}^1, P_{-w_1}^2, P_{-w_1}^3, P_{-w_1}^4 \in \mathcal{P}_{-w_1}$ be such that

P_{w_2}	P_{f_1}	P_{f_2}	P_{f_3}	$P_{w_2}^{f_1 \leftrightarrow f_2}$	$P_{f_1}^{f_1 \leftrightarrow f_2}$	$P_{f_2}^{f_1 \leftrightarrow f_2}$	$P_{f_3}^{f_1 \leftrightarrow f_2}$
f_2	w_2	w_1	w_2	f_2	w_2	w_1	w_2
f_1	w_1	w_2	w_1	f_3	w_1	w_2	w_1
f_3				f_1			
\hat{P}_{w_2}	\hat{P}_{f_1}	\hat{P}_{f_2}	\hat{P}_{f_3}	$\hat{P}_{w_2}^{f_1 \leftrightarrow f_2}$	$\hat{P}_{f_1}^{f_1 \leftrightarrow f_2}$	$\hat{P}_{f_2}^{f_1 \leftrightarrow f_2}$	$\hat{P}_{f_3}^{f_1 \leftrightarrow f_2}$
f_2	w_2	w_1	w_2	f_2	w_2	w_1	w_2
f_1		w_2		f_3		w_2	
f_3				f_1			

Under \tilde{P}_{-w_1} worker w_1 believes each of the profiles $P_{-w_1}, P_{-w_1}^{f_1 \leftrightarrow f_2}, \hat{P}_{-w_1}$, and $\hat{P}_{-w_1}^{f_1 \leftrightarrow f_2}$ with probability $\frac{1}{4}$. Then \tilde{P}_{-w_1} is $\{f_2, f_3\}$ -symmetric.

Let $P_{w_1} : f_1, f_2, w_1, f_3$ be w_1 's true preference. If she submits P_{w_1} , then she is matched with probability $\frac{1}{4}$ to f_1 (by $DA[P_{w_1}, P_{-w_1}^{f_1 \leftrightarrow f_2}](w_1) = f_1$) and with probability $\frac{3}{4}$ to f_2 (by $DA[P](w_1) = DA[P_{w_1}, \hat{P}_{-w_1}](w_1) = DA[P_{w_1}, \hat{P}_{-w_1}^{f_1 \leftrightarrow f_2}](w_1) = f_2$).

If w_1 submits $P'_{w_1} : f_1, f_3, f_2, w_1$, then under P_{-w_1} she is matched to f_1 and for the other profiles the DA-algorithm assigns her the same position. Thus, she is matched with probability $\frac{1}{2}$ to f_1 and with probability $\frac{1}{2}$ to f_2 . It is now easy to see that w_1 's unique optimal strategy is to submit P'_{w_1} , i.e. ranking an unacceptable firm as acceptable is optimal. \triangleleft